SOME RESULT OF NON-COPRIME GRAPH OF INTEGERS MODULO \( n \) GROUP FOR \( n \) A PRIME POWER

Masriani\(^1\), Rina Juliana\(^2\), Abdul Gazir Syarifudin\(^3\), I Gede Adhitya Wisnu Wardhana\(^4\), Irwansyah\(^5\) and Ni Wayan Switrayni\(^6\)

\(^{1,2,3,4,5,6}\)Universitas Mataram, Mataram, Indonesia

Email: \(^1\)masriani@unram.ac.id, \(^2\)rinajuliana@unram.ac.id, \(^3\)abdulgazirs@gmail.com, \(^4\)adhitya.wardhana@unram.ac.id, \(^5\)irw@unram.ac.id, \(^6\)niwayan.switrayni@unram.ac.id

Penulis Korespondensi

Abstract. One interesting topic in algebra and graph theory is a graph representation of a group, especially the representation of a group using a non-coprime graph. In this paper, we describe the non-coprime graph of integers modulo \( n \) group and its subgroups, for \( n \) is a prime power or \( n \) is a product of two distinct primes.

Keywords: group, integer modulo, non-coprime.

I. INTRODUCTION

The non-coprime graph of a finite group was introduced by Mansoori et al. \cite{1}. In \cite{1}, the authors determined some numerical invariants of the non-coprime graph of a finite group, such as its diameter, girth, dominating number, independence, and chromatic number. Moreover, they characterize the planar non-coprime graph of a group and the regular non-coprime graph of a nilpotent group. Furthermore, they also stated a connection between the non-coprime graphs and some prime graphs.

Aghababaei-Beni and Jafarzadeh \cite{2} investigated the properties of Cartesian and tensor products of non-coprime graphs of finite groups such as the dihedral and semi-dihedral groups. They considered the properties such as the independence, clique, chromatic number, covering number, diameter, connectedness, and the existence of the Eulerian spanning subgraph. They also gave a characterization for such graphs to be an Eulerian graph and to be a planar graph. Recently, Aghababaei et al. \cite{3} extended some results in \cite{2}. They studied the non-coprime of a finite group with respect to a subgroup and investigated some properties of such a graph, including its diameter, chromatic number, clique, and the number of connected components. They also investigated some properties of the non-coprime graph of the nilpotent group.

Some authors give some properties of the non-coprime graph and the coprime graph to more specific groups. Rilwan et al. give some properties of the non-coprime graph of integer \cite{4}, Juliana et al. give some properties of the non-coprime graph of an integer modulo \cite{7}, and Syarifudin et al. give some properties of the non-coprime graph of dihedral groups \cite{8}.

In this paper, we describe the non-coprime graph of the group \( \mathbb{Z}_n \) and that of its subgroups, where \( n \) is a prime power or \( n \) is a product of two distinct primes. We used the result of the coprime graph of the group \( \mathbb{Z}_n \) as the non-coprime graph is the duality of the coprime graph \cite{6}. This paper is organized as follows. Section 2 (Some Basic Notions) collects some basic
notions related to group and graph. We give our main results in Section 3 (Main Results). Some concluding remarks are collected in Section 4 (Conclusions). Finally, we give some related references in the References section.

II. SOME BASIC NOTION

Let $G$ be a finite group and $|G|$ be the number of elements in $G$ or the order of $G$. The definition of the order of an element in $G$ is as follows.

**Definition 1.** Let $G$ be a finite group with the identity element $e$. The order of $g \in G$, denoted by $|g|$, is the smallest positive integer $n$ such that $g^n = e$.

Let $H$ be any subgroup of $G$. In the rest of the paper, if $H$ is a subgroup of $G$, then we denote it by $H \leq G$. Also, let $a$ be an element in $G$. A subgroup $\langle a \rangle = \{a^n | n \in \mathbb{Z}\}$ is called a cyclic subgroup of $G$ generated by an element $a$. The following theorem states a relation between $|H|$ and $|G|$.

**Theorem 1.** (Lagrange’s Theorem [4]). If $G$ is a finite group and $H \leq G$, then $|H|$ is a divisor of $|G|$.

As a consequence of Theorem 1, we have that $|\langle g \rangle|$ divides $|G|$.

A graph is one crucial object in mathematics, especially in discrete mathematics and its applications. The definition of a graph is as follows.

**Definition 2.** [5]. A graph is a pair $\Gamma = (V, E)$, where $V$ is a non-empty set of vertices, and $E \subseteq V \times V$ is a set of edges.

We have to note that, in the rest of the paper, we only use a simple undirected graph, i.e., we assume that $(v_i, v_j) = (v_j, v_i)$ for all $(v_i, v_j) \in E$.

**Definition 3.** An undirected graph $\Gamma$ is complete if for any $v_i, v_j \in V$, we have that $(v_i, v_j) \in E$. If $|V| = m$, then we denote an undirected complete graph $\Gamma$ as $K_m$.

Let $a$ and $b$ be two integers. The greatest common divisor of $a$ and $b$ usually denoted by $(a, b)$. The following definition defines the non-coprime graph of a finite group.

**Definition 4.** [1]. Let $G$ be a finite group. The non-coprime graph of $G$ denoted by $\overline{G}$, is a graph whose vertices are all elements of $G \setminus \{0\}$. Two different vertices $x$ and $y$ in $\overline{G}$ are adjacent if $(|x|, |y|) \neq 1$.

III. MAIN RESULT

Let $\mathbb{Z}_n = \{0,1,\ldots,n-1\}$ be the group of integers modulo $n$ with addition (mod $n$) operation. The following proposition gives the non-coprime graph of $\mathbb{Z}_n$ when $n$ is a prime number.
Proposition 1. If $n$ is a prime number, then the non-coprime graph of $\mathbb{Z}_n$ is a complete graph.

Proof. Since $n$ is a prime number, we have that $|i| = n$, for all $i = 1, 2, ..., n - 1$. So, $|i| = n$, for all $i = 1, 2, ..., n - 1$. Therefore, the non-coprime graph of $\mathbb{Z}_n$ is a complete graph $K_{n-1}$. 

Here is an example of Proposition 1.

Example 1. Let $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$. As we can see, $|1| = 7, |2| = 7, |3| = 7, |4| = 7, |5| = 7, |6| = 7$. So, we have that $|i| = n$, for all $i = 1, 2, ..., n - 1$. Therefore, the non-coprime graph of $\mathbb{Z}_7$ is a complete graph.

Let $n = p^s$ for some prime number $p$ and a natural number $s \geq 2$. The following theorem describes the non-coprime graph of $\mathbb{Z}_n$, when $n = p^s$.

Theorem 2. If $n = p^s$, for some prime number $p$ and natural number $s \geq 2$, then the non-coprime graph of $\mathbb{Z}_n$ is a complete graph.

Proof. Let $a$ be an element in $\mathbb{Z}_{p^s}$ with $(p^s, a) \neq 1$. The element $a$ can be written as $a = p^k q$, for some $1 \leq k < s$ and an integer $q$, where $(p, q) = 1$. As a consequence, we have that $|a| = p^{s-k}$. Also, for any $b \in \mathbb{Z}_{p^s}$ with $(p^s, b) = 1$, we have that $|b| = p^s$. These imply $|a| = n$, for all $a \in \mathbb{Z}_{p^s} - \{0\}$. Therefore, the non-coprime graph of $\mathbb{Z}_{p^s}$ is a complete graph $K_{p^s-1}$.

Here is an example of Theorem 2.

Example 2. Let $\mathbb{Z}_{3^2} = \{0, 1, 2, 3, 4, 5, 6\}$. As we can see, $|1| = 9, |2| = 9, |3| = 3, |4| = 9, |5| = 9, |6| = 3, |7| = 9, |8| = 9$. Consequently, we have that $a$ and $b$ are adjacent in $\Gamma_{\mathbb{Z}_{3^2}}$ for all $a, b \in \mathbb{Z}_{3^2} - \{0\}$. The non-coprime graph of $\mathbb{Z}_{3^2}$ is shown in Figure 2.
Let \( n \) be a product of two distinct primes. The following theorem describes the non-coprime graph of \( \mathbb{Z}_n \), when \( n \) is a product of two distinct primes.

**Theorem 3.** Let \( n = p_1 p_2 \), where \( p_1, p_2 \) are two distinct primes. If \( H \) is a proper subgroup of \( \mathbb{Z}_n \), then the non-coprime graph of \( H \) is complete.

**Proof.** Let \( H \) be any proper subgroup of \( \mathbb{Z}_n \). By Theorem 1 (Lagrange’s Theorem), we have that \( |H| = p_1 \) or \( |H| = p_2 \). Therefore, by Proposition 1, we have that \( \Gamma_H \) is a complete graph.

Here is an example of Theorem 3.

**Example 3.** Let \( \mathbb{Z}_{15} = \{0, 1, 2, \ldots, 14\} \). We can check that non-trivial subgroups of \( \mathbb{Z}_{15} \) are \( \langle 3 \rangle \) and \( \langle 5 \rangle \). Moreover, we can see that \( \langle 3 \rangle = \{0, 3, 6, 9, 12\} \) and \( \langle 5 \rangle = \{0, 5, 10\} \). The non-coprime graphs of \( \langle 3 \rangle \) and \( \langle 5 \rangle \) are shown in Figure 3.

![Figure 3. Non-coprime graph of subgroups in \( \mathbb{Z}_{15} \)](image)

**IV. CONCLUSIONS**

We have shown that the non-coprime graph of \( \mathbb{Z}_n \), when \( n \) is a prime power, is a complete graph \( K_{n-1} \). Moreover, when \( n \) is a product of two distinct primes, the non-coprime graphs of its non-trivial subgroups are complete graphs.

**REFERENCE**


