New Bound For Absolute Value of All Complex Roots of A Polynomial

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Abstract. Upper bounds on the absolute values of polynomial roots in $\mathbb{R}[X]$, such as

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

(where $a_n > 0$), are typically represented as constraints within a certain region in the complex plane. Lagrange and Cauchy were among the first to establish upper bounds for all complex roots of such polynomials. Notably, Lagrange's bound is sharper than Cauchy's bound when $1 > \sum |\frac{a_i}{a_n}|$ for all i except the largest term.

In this paper, we introduce a general sequence of polynomials associated with a positive real sequence $\{a_n\}_{n=1}^{\infty}$. Using the Lagrange and Cauchy bounds for polynomial roots, we prove that the sum of the first and second-largest elements in the set

$$\left\{ \sqrt[i+1]{|a_i - a_{i+1}|}, a_1 + 1 \mid 1 \leq i < n \right\}$$

serves as an upper bound for the roots of $f(x)$.
Keywords: Roots of polynomials, Lagrange bound, Cauchy bound.

Introduction

The problem of approximately locating the roots of $f(x)$ using simple operations on its coefficients is a well-established issue that has generated a substantial amount of research, as highlighted in the comprehensive surveys $[6, 8]$ and the references cited within them. These simple location methods are employed for various theoretical purposes, such as providing sufficient conditions to ensure the stability of $f(x)$ or that all its roots lie within the unit circle. Additionally, they are often utilized in iterative algorithms to compute the roots of $f(x)$, particularly to generate initial approximations that initiate the iterations $[5, 7]$. Recently, there has been growing interest in polynomial eigenvalue problems, leading to the development of simple criteria for estimating the eigenvalues of matrix polynomials $[4]$. However, to maintain brevity, matrix polynomials will not be addressed in this study.
A number T is said to be an upper bound for the roots of a polynomial $p(x)$ with real coefficients if T is greater than or equal to the absolute value of all roots of $p(x)$. Root bounds play a critical role in understanding the behavior of polynomials and have numerous applications in numerical analysis, algebraic equations, and approximation theory.

For a polynomial $f(x)$ of degree n,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

the roots are bounded above by the unique positive root of the associated Cauchy polynomial:

$$|a_n|x^n - |a_{n-1}|x^{n-1} - \cdots - |a_0| = 0.$$

According to the Cauchy formula, this upper bound is given by:

$$\max\left\{1, \sum_{i=1}^{n-1}\left|\frac{a_i}{a_n}\right|\right\} [2].$$

We now present several important theorems related to upper bounds of polynomial roots.

Theorem 1 *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with real coefficients. Then, the number of positive real roots of $f(x)$ is equal to the number of sign changes in its coefficients, or less than this by an even integer [4].*

Theorem 2 *Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \in \mathbb{R}[X]$. Then, an upper bound for the absolute value of $f(x)$'s roots is given by $R + \rho$, where $R \geq \rho$ are the largest numbers in the set $\left\{\sqrt[i]{|a_i|}\,;\, i \in \mathbb{N}\right\}$ [3].*

Theorem 3 *Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \in \mathbb{R}[X]$, and let α be the unique positive root of the polynomial:*

$$g(x) = x^n - |a_1|x^{n-1} - \cdots - |a_{n-1}|x - |a_n|.$$

Then, any $\alpha < b$ is a valid bound for the absolute values of the roots of $f(x)$ [1].

Theorem 4 *Let $p_m(x)$ be the characteristic polynomial of the m-th order linear recursive sequence $u(n)$, related to a_n. Let α_m be the unique, positive real root of $p_m(x)$. Then $\alpha_m < \alpha_{m+1}$.*

Proof. The polynomial $p_m(x)$ has a positive coefficient for x^{m+1}, and α_m is the unique positive real root. For any $x \in (\alpha_m, \infty)$, we have $p_m(x) > 0$. Consider α_{m+1}, the root of $p_{m+1}(x)$:

$$\begin{aligned} p_{m+1}(\alpha_{m+1}) &= 0, \\ \Rightarrow \alpha_{m+1}p_m(\alpha_{m+1}) - a_{m+1} &= 0, \\ \Rightarrow p_m(\alpha_{m+1}) &= \frac{a_{m+1}}{\alpha_{m+1}} > 0. \end{aligned}$$

Since $p_m(\alpha_{m+1}) > 0$, and $p_m(x) > 0$ for $x > \alpha_m$, it follows that $\alpha_m < \alpha_{m+1}$. □

Theorem 5 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^+$ such that $\lim_{n\to\infty} \sqrt[n]{a_n} < \infty$. Then, for any $n \in \mathbb{N}$, $\mu(n) = R(n) + \rho(n)$ is an upper bound for the absolute value of all $p(a_n, n; x)$ complex roots, where $R(n) \geq \rho(n)$ are the largest two elements of the set:*

$$\{\sqrt[i+1]{|a_i - a_{i+1}|}, a_1 + 1\}.$$

Proof. Since $\lim_{n\to\infty} \sqrt[n]{a_n} < \infty$, Lagrange's root bound ensures that $2\max_{1\leq i\leq n}\{\sqrt[i]{a_i}\}$ is an upper bound for the roots of $p(a_n, n; x)$. Therefore, $\{\alpha_n\}_{n=1}^{\infty}$ is bounded and monotonic increasing. Consequently, it converges to a limit α such that $p(a_n, n; \alpha) = p(a_n, n+1; \alpha)$ as $n \to \infty$.

Define a sequence b_n as:

$$b(n) = \begin{cases} a(n), & n \leq m, \\ a(m+1), & n > m. \end{cases}$$

Clearly, $b(n)$ is $(m+1)$-finally stable. To estimate α, solve the equation:

$$p(b_n, n+1; x) = p(b_n, n; x),$$

which expands to:

$$x^{n+1} - \sum_{i=1}^{n+1} b_i x^{n-i+1} = x^n - \sum_{i=1}^{n} b_i x^{n-i}.$$

After simplification:

$$x^{n-m}\Big(x^{m+1} - (a_1+1)x^m + \sum_{i=1}^{m}(a_i - a_{i+1})x^{m-i}\Big) = 0.$$

The positive real root α of this polynomial satisfies Lagrange's root bound. Let γ_n denote the sum of the first and second-largest numbers in:

$$\{\sqrt[i+1]{|a_i - a_{i+1}|}, a_1 + 1\}.$$

Thus, $\alpha \leq \gamma_n$, and by Cauchy's root bound, γ_n is an upper bound for all complex roots of $p(a_n, n; x)$ for $n \leq m$. □

Example 1 Let a, b, c be positive real numbers, and define the sequence:

$$a(n) = \begin{cases} a, & \text{if } n = 1, \\ b, & \text{if } n = 2, \\ c, & \text{if } n \geq 3. \end{cases}$$

For $m \geq 3$:

$$g_m(x) = x^3 - (a+1)x^2 + (a-b)x + (b-c).$$

The sequence $a(n)$ is 3-finally stable, and α is the positive real root of $g_m(x)$, which bounds the absolute values of all roots of $p(a_n, n; x)$.

In the special case where $b = c$, we find:

$$\alpha = \frac{1 + a + \sqrt{1 - 2a + a^2 + 4b}}{2}.$$

If $a = b = c$, then $\alpha = a + 1$. For example, if $a = 1$, $\alpha = 2$.

Example 2 Consider $p(x) = x^5 - 2x^4 - 10x^3 - 65x^2 - 15x - 9$. Using Cauchy's root bound, the sum of the first and second-largest elements in the set:

$$\{2, \sqrt[2]{10}, \sqrt[3]{65}, \sqrt[4]{15}, \sqrt[5]{9}\}$$

yields 7.183.

Using the proposed bound, the sum of the first and second-largest numbers in:

$$\{3, \sqrt[2]{8}, \sqrt[3]{55}, \sqrt[4]{50}, \sqrt[5]{6}\}$$

yields 6.8021.

Hence, the new bound is sharper than Cauchy's bound.

Theorem 6 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^+$ such that $\lim_{n\to\infty} \sqrt[n]{a_n} < \infty$. For $n \in \mathbb{N}$, $\mu(n) = R(n) + \rho(n)$ is an upper bound for the absolute value of all roots of $p(a_n, n; x)$, where $R(n) \geq \rho(n)$ are the largest two elements in:*

$$\{\sqrt[i+1]{|a_i - 2a_{i+1} + a_{i+2}|}, a_1 + 2\}.$$

Proof. This follows directly from Cauchy's bound and Theorem 2. □

I. Application

In this section, we present some convergence of real unique positive sequences relevant to certain elementary and useful mathematical sequences that appear in applicable branches of science. These sequences exhibit specific properties, such as being monotonic (increasing or decreasing), less than one, symmetric, or constant.

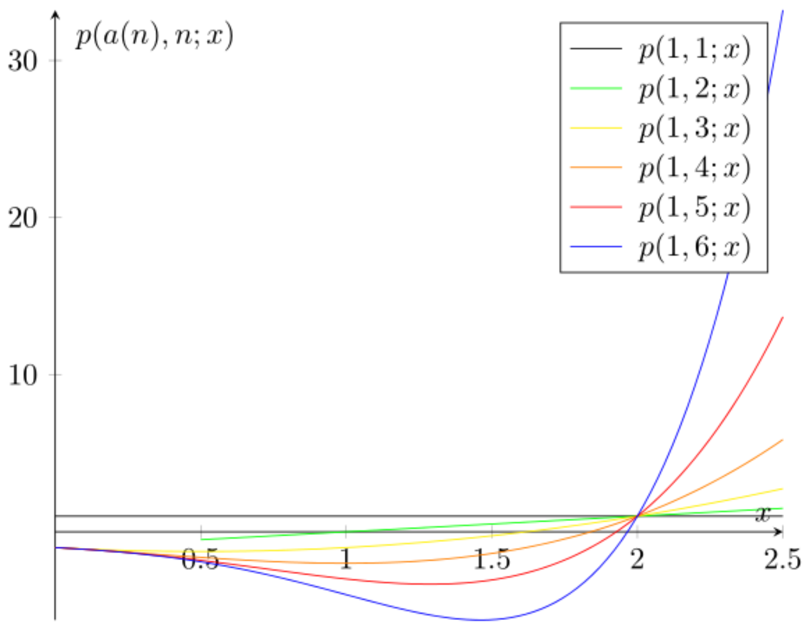

Figure 1. Convergence of $p(1, i; x)$ zeros for $1 \leq i \leq 6$ to exact $\alpha = 2$, relevant to the constant sequence $a(n) = 1$.

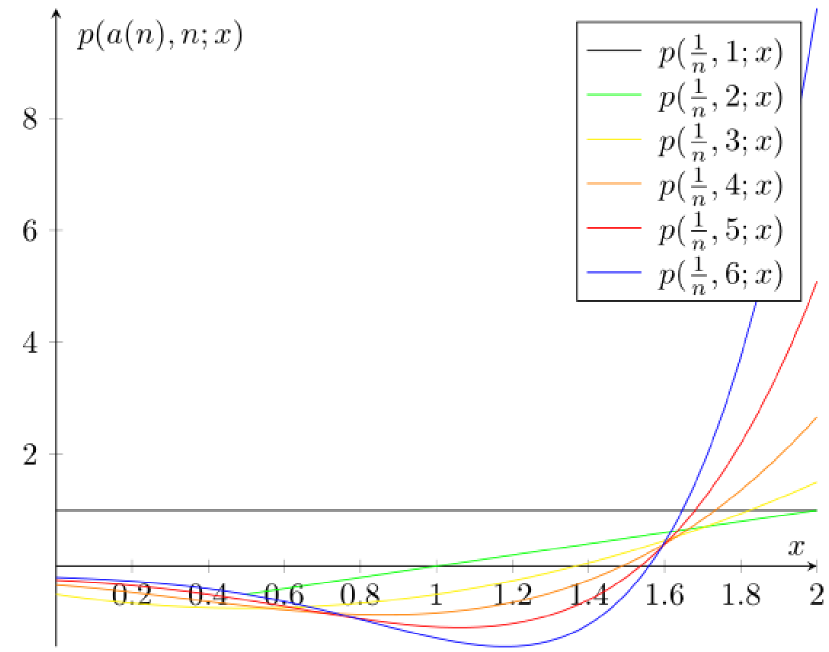

Figure 2. Convergence of $p(1/n, i; x)$ zeros for $1 \leq i \leq 6$ to exact $\alpha = 1.58122$, relevant to the decreasing sequence $a(n) = 1/n$, which belongs to $(0, 1]$.

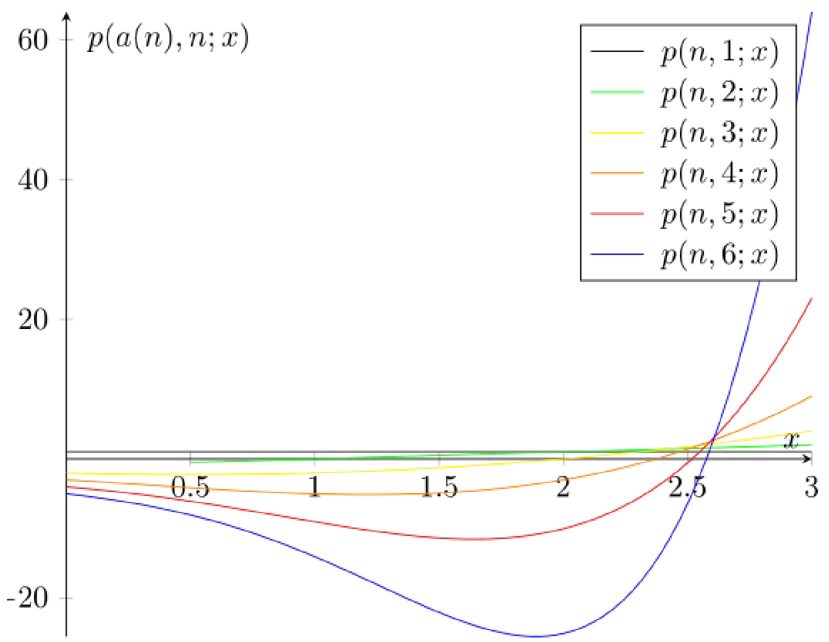

Figure 3. Convergence of $p(n, i; x)$ zeros for $1 \leq i \leq 6$ to exact $\alpha = 2.6178$, relevant to the increasing sequence $a(n) = n$.

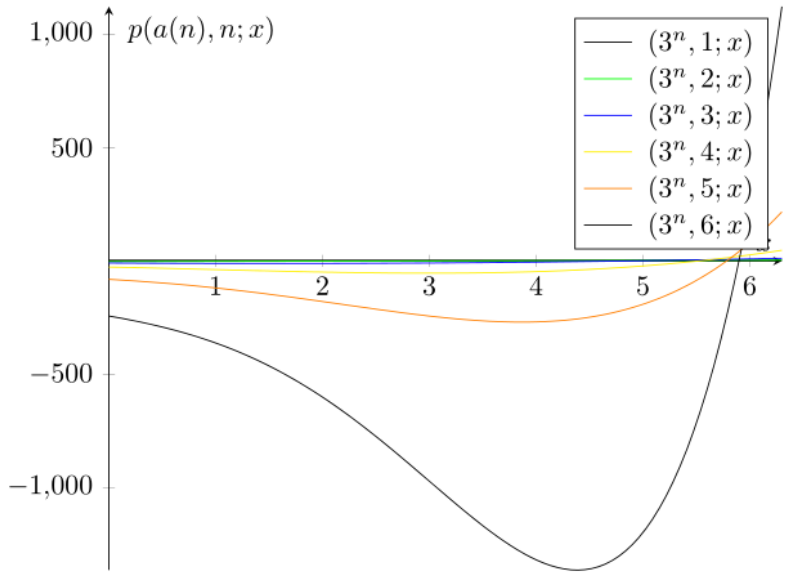

Figure 4. Convergence of $p(3^n, i; x)$ zeros for $1 \leq i \leq 6$ to exact $\alpha = 6$, relevant to the exponentially increasing sequence $a(n) = 3^n$.

Based on the plots corresponding to different sequences, the table below shows how the positive unique zeros of $p(a(n), n; x)$, denoted as α_n, converge to $\alpha_\infty = \alpha$.

Sequence $a(n)$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n \to \infty$
1	1	1.618	1.839	1.927	1.965	2
$1/n$	1	1.366	1.487	1.535	1.558	1.5812
n	1	2	2.374	2.518	2.576	2.6178
3^n	3	4.854	5.517	5.782	5.897	6

As we can observe from all the above examples, $\lim_{n\to\infty} \sqrt[n]{a(n)} < \infty$ holds true.

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