

NEW COUNTING FORMULA FOR DOMINATING SETS IN PATH AND CYCLE GRAPHS

Leomarich F. Casinillo

Department of Mathematics and Physics, Visayas State University, Visca, Baybay City, Leyte, Philippines
Email: leomarich_casinillo@yahoo.com

Abstract. Let $G = (V(G), E(G))$ be a path or cycle graph. A subset D of $V(G)$ is a dominating set of G if for every $u \in (V(G) \setminus D)$, there exists $v \in D$ such that $uv \in E(G)$, that is, $N[D] = V(G)$. The domination number of G , denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G . A set $D_1 \subseteq V(G)$ is a set containing dominating vertices of degree 2, that is, each vertex is internally stable. A set $D_2 \subseteq V(G)$ is a set containing dominating vertices where one of the element say $a \in D_2$, $\deg(a) = 1$ and the rest are of degree 2. A set $D_3 \subseteq V(G)$ is a set containing dominating vertices in which two of the elements say $b, c \in D_3$, $\deg(b) = \deg(c) = 1$. This paper developed a new combinatorial formula that determines the number of ways of putting a dominating set in a path and cycle graphs of order $n \geq 1$ and $n \geq 3$, respectively. Further, a combinatorial function $P_G^1(n)$, $P_G^2(n)$, and $P_G^3(n)$ that determines the probability of getting the set D_1 , D_2 , and D_3 , respectively in graph G of order n were constructed.

Keywords: Dominating set, path, cycle, probability

I. INTRODUCTION

A *connected graph* G is a pair of set $V(G)$ and $E(G)$, where $V(G)$ is a finite non-empty set called the *vertex-set* of G and $E(G)$ is a set of unordered pair $\{a, b\}$ or simply ab of distinct elements from $V(G)$ called the *edge-set* of G where a and b are distinct elements in $V(G)$. Let $v \in V(G)$. The *degree* of vertex $v \in V(G)$, denoted by $\deg(v)$ is the number of edges incident with v in G . Then *neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $X \subseteq V(G)$, then the *open neighborhood* of X is the set $N_G(X) = N(X) = \bigcup_{v \in X} N_G(v)$. The *closed neighborhood* of X is $N_G[X] = N[X] = X \cup N(X)$. A *walk* is a sequence u_1, u_2, \dots, u_n of vertices of graph G such that $\{u_i, u_{i+1}\} \in E(G)$ for each $i = 1, 2, \dots, n-1$. Vertices u_1 and u_n are the endpoints of the walk while the vertices u_2, u_3, \dots, u_{n-1} are *internal vertices* of the walk. The length of walk is the number of edges on the walk, i.e., the walk u_1, u_2, \dots, u_n has length $n-1$. A *path* is a walk that does not repeat edges and does not end where it starts, i.e., $u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_n$, $u_1 \neq u_n$. A path of order n and length $n-1$ is denoted by P_n where $n \geq 1$. A *cycle* is a walk that does not repeat edges and does end where it starts, i.e., $u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_n \rightarrow u_1$. A cycle graph of order n and length n is denoted by C_n where $n \geq 3$. For other types of graphs, readers may refer to [6, 10]. Let a and b be two distinct vertices in graph G . The *distance* $d_G(a, b)$ between two vertices a and b of a graph G is defined as the length of the shortest walk between a and b in G . If there is no walk between a and b then we declare $d_G(a, b) = \infty$. The *eccentricity* $e(a)$ is the distance to a vertex farthest from a , that is, $e(a) = \max\{d_G(a, b)\}$. For more information on the concepts of graphs, please refer to [5, 6, 10].

A subset D of $V(G)$ is a *dominating set* of G if for every $a \in V(G) \setminus D$, there exists $b \in D$ such that $ab \in E(G)$, i.e., $N[D] = V(G)$ where the elements of D is called *dominating vertices*. The domination number of G , denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G . Let $I \subseteq G$. Then, I is an *independent dominating set* if it is a dominating set in G and no two vertices in set I are adjacent. The *independent domination number* of G , denoted by $i(G) = \gamma_i(G)$, is the smallest cardinality of an independent dominating set of G . The concepts of domination in graphs is investigated in [1, 4, 7, 9, 11, 12]. In the study of Casinillo [3], the *sum* of two paths P_n and P_m is defined as connecting the last vertex of P_n to the first vertex of P_m and it is denoted as $P_n \Phi P_m = P_{n+m}$. Also, it is introduced that path of order n with *one dominating vertex* at i^{th} vertex is denoted by P_n^i . A set $D_1 \subseteq V(G)$ is a set containing dominating vertices of degree 2. A set $D_2 \subseteq V(G)$ is a set containing dominating vertices where one of the element say $u \in D_2, \deg(u) = 1$. A set $D_3 \subseteq V(G)$ is a set containing dominating vertices in which two of the elements say $c, d \in D_3, \deg(c) = \deg(d) = 1$. Hence, $D = D_1 \cup D_2 \cup D_3$. Let S_G be the set of all possible ways of getting dominating set D in a graph G and let $\varphi_G(n) = |S_G|$ be a function that determines the number of ways of putting dominating set in a graph G . In addition, let $f_G(n)$, $g_G(n)$ and $h_G(n)$ be functions that determines the number of ways of putting set D_1 , D_2 and D_3 , respectively in a graph G , that is, $f_G(n) = f(D_1)$, $g_G(n) = g(D_2)$, and $h_G(n) = h(D_3)$. The probability of getting D_1 , D_2 and D_3 in a graph G are denoted by $P_G^1(n)$, $P_G^2(n)$ and $P_G^3(n)$, respectively and defined as follows:

- i.) $P_G^1(n) = \text{Prob}(D_1) = \frac{f_G(n)}{\varphi_G(n)}$;
- ii.) $P_G^2(n) = \text{Prob}(D_2) = \frac{g_G(n)}{\varphi_G(n)}$; and
- iii.) $P_G^3(n) = \text{Prob}(D_3) = \frac{h_G(n)}{\varphi_G(n)}$

where $P_G^1(n) + P_G^2(n) + P_G^3(n) = 1$.

Currently, in the paper of Casinillo [3], the number of independent domination in path is determined using the Fibonacci binary string and in the paper of Samanmoo et al. [13], the γ -independent dominating graphs of paths and cycles were presented. Furthermore, Arocha and Llano [2] computed the number of dominating k -sets of paths, cycles and wheels using the concept of domination polynomials. Hence, the purpose of this paper is to present new counting formula for the number of dominating sets of path and cycle graphs that depends on the order of graph. In particular, a new combinatorial formula that determines the number of ways of putting a dominating set in a path and cycle graphs of order $n \geq 1$ and $n \geq 3$, respectively were developed. Furthermore, the combinatorial functions $P_G^1(n)$, $P_G^2(n)$, and $P_G^3(n)$ that determines the probability of getting the set D_1 , D_2 , and D_3 , respectively in graph G were constructed.

II. RESULTS

We begin this section by considering a remark which is immediate from above definition of dominating vertex in graph G . This result determines the minimum domination number of path and cycle graphs.

Remark 2.1. [2, 8] Let P_n be a path of order $n \geq 1$ and C_n be a cycle graph of order $n \geq 3$. Then,

$$\gamma(P_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0(\text{mod}3) \\ \frac{n+2}{3} & \text{if } n \equiv 1(\text{mod}3) \\ \frac{n+1}{3} & \text{if } n \equiv 2(\text{mod}3) \end{cases}$$

For path P_1 and P_2 , it is clear that the dominating number is 1. The next Theorem is a direct consequence of *Remark 2.1*. The Theorem is different meaning to the result of Casinillo [2] that deals with Fibonacci number of domination in path which only generates independent dominating sets. We have to note that in path and cycle graphs, $|D| = |I|$, however, the number of ways of getting set D and I are different. In particular, this Theorem determines the number of all possible ways of putting a dominating sets (connected and independent) in path graph P_n which depends on the order n .

Theorem 2.1. Let $G = P_n$ where $n \in \mathbb{N}$. Then,

$$|S_G| = \varphi_G(n) = \begin{cases} 1 & \text{if } n \equiv 0(\text{mod}3) \\ \frac{1}{18}(n^2 + 13n + 4) & \text{if } n \equiv 1(\text{mod}3) \\ \frac{1}{3}(n + 4) & \text{if } n \equiv 2(\text{mod}3) \end{cases}$$

Proof: To prove Theorem 2.1, we consider the following cases below:

Case 1. Let $G = P_{3j}$ where $j \in \mathbb{N}$. By *Remark 2.1*, $\gamma(G) = j$ for all $j \in \mathbb{N}$ and implies that for all $u_i \in D$, $\bigcap_{all i} N(u_i) = \emptyset$ and $\deg(u_i) = 2$. Thus, there is a unique configuration that for any $u_j \in D$, $\exists a, b \in V(G) \setminus D$ such that $N(u_i) = \{a, b\}$ and $d_G(u_i, u_{i+1}) = 3$. Hence, it clearly follows that $|S_G| = \varphi_G(n = 3j) = 1$.

Case 2. Let $G = P_{3j+1}$ where $j \in \{0, 1, 2, \dots\}$. Obviously, if $j = 0$, then $n = 1$, $|D| = |I| = 1$ and $|S_G| = 1$. By *Remark 2.1*, it follows that $\gamma(G) = j + 1$ and we obtained the following subcases below:

Subcase 1. Firstly, there is a unique configuration of independent dominating set considering that $\exists u, v \in I = D$ such that $\deg(u) = \deg(v) = 1$.

Subcase 2. Considering that there exists $a, b \in V(G) \setminus D$ and $\deg(a) = \deg(b) = 1$ such that $N(a) \cup N(b) \subset D$. When $n = 7$, there is 1 possibility that the arrangement of the independent dominating and non-dominating vertices are alternate. On the other hand, if $n \geq 10$, then there exists edge/s such that it is incident of two non-dominating vertices. Let E_i be the edges. Note that P_2^2 and P_2^1 are in the first and last position of path G , respectively. Hence, $|V(G) \setminus V(P_2^2 \cup P_2^1)| = n - 4$ and implies that the dominating number remaining is $\frac{n-4}{3}$. This

follows that there are $\frac{n-7}{3}$ edges incident of two non-dominating vertices, that is, $E_1, E_2, \dots, E_{\frac{n-7}{3}}$. Combinatorically speaking, the $\frac{n-4}{3}$ independent dominating vertices and edges

$E_1, E_2, \dots, E_{\frac{n-7}{3}}$ can be arranged in $\left(\frac{n-4}{3}\right)^2$ ways. However, the $\frac{n-4}{3}$ independent dominating vertices and edges $E_1, E_2, \dots, E_{\frac{n-7}{3}}$ are not distinct, then the possible arrangement is given by

$$\sum_{i=1}^{\frac{n-4}{3}} i = \frac{n^2 - 5n + 4}{18}.$$

Subcase 3. Suppose that there exists $u \in D$ and $v \in V(G) \setminus D$ such that $\deg(u) = \deg(v) = 1$ and $N(v) \subset D$, then there are $\frac{2(n-1)}{3}$ distinct arrangement can be form for $\gamma(G) = j + 1$ by *Remark 2.1*.

Subcase 4. Lastly, if no vertex in D is an end vertex in G , then $\exists u, v \in D$ such that $uv \in E(G)$. This implies that the position of vertex u is in P_n^{3i-1} and vertex v is in P_n^{3i} where $i \in \{1, 2, \dots, \frac{n-1}{3}\}$. Clearly, there are $\frac{n-1}{3}$ configuration of this form.

Combining the 4 subcases, then it follows that

$$\varphi_G(n = 3j + 1) = 1 + \frac{n^2 - 5n + 4}{18} + \frac{2(n-1)}{3} + \frac{n-1}{3} = \frac{n^2 + 13n + 4}{18}.$$

Case 3. Let $G = P_{3j+2}$ where $j \in \{0, 1, 2, \dots\}$. Clearly, if $n = 2$, then there are 2 possible ways of generating a dominating set. Now we consider $n \geq 5$. By *Remark 2.1*, clearly, we have $\gamma(G) = j + 1$, and again, we consider the following subcases below:

Subcase 1. If one of the vertex in D has degree 1, i.e., there exists $u \in V(G) \setminus D$ such that $\deg(u) = 1$, then it follows that for all $u_{j+1} \in D$, $\bigcap_{all j} N(u_{j+1}) = \emptyset$. And it is concluded that there are 2 possible arrangement of this form in graph G .

Subcase 2. On the other hand, if for all $v_j \in D$, $\deg(v_j) = 2$, then there exists edge/s $E_i \in E(G)$ such that it is incident with non-dominating vertices. Obviously, if $n = 5$, then there is only 1 possible way to generate a dominating set in G . Now consider $n \geq 8$. Since P_2^2 and P_2^1 are in the first and last position of G , respectively, then $|V(G) \setminus V(P_2^2 \cup P_2^1)| = n - 4$ and the dominating vertices remaining is $\frac{n-5}{3}$. So, there are $\frac{n-5}{3}$ edges incident of two non-dominating vertices, that is, $E_1, E_2, \dots, E_{\frac{n-5}{3}}$. So, it follows that there are $\frac{n-5}{3} + 1 = \frac{n-2}{3}$ distinct configuration of dominating sets of this form in graph G . Combining the 2 subcases, then we end up with

$$\varphi_G(n = 3j + 2) = 2 + \frac{n-2}{3} = \frac{n+4}{3}.$$

Hence, if we combine all three cases, then this completes the proof. \square

The following corollaries are immediate results of Theorem 2.1 above. The result shows the combinatorial formula of getting the number of ways of generating sets D_1 , D_2 , and D_3 in graph G . For a trivial graph $G = P_1$, sets D_1 , D_2 , and D_3 are not applicable.

Corollary 2.1. *Let G be a path of order $n \geq 3$. Then,*

$$f_G(n) = f(D_1) = \begin{cases} 1 & \text{if } n \equiv 0(\text{mod } 3) \\ \frac{1}{18}(n^2 + n - 2) & \text{if } n \equiv 1(\text{mod } 3) \\ \frac{1}{3}(n - 2) & \text{if } n \equiv 2(\text{mod } 3) \end{cases}$$

Proof: Obvious from Theorem 2.1. \square

Corollary 2.2. *Let G be a path of order $n \geq 1$. Then,*

$$g_G(n) = g(D_2) = \begin{cases} 0 & \text{if } n \equiv 0(\text{mod } 3) \\ \frac{2}{3}(n - 1) & \text{if } n \equiv 1(\text{mod } 3) \\ 2 & \text{if } n \equiv 2(\text{mod } 3) \end{cases}$$

Proof: Obvious from Theorem 2.1. \square

Corollary 2.3. Let G be a path of order $n \geq 4$. Then,

$$h_G(n) = h(D_3) = \begin{cases} 0 & \text{if } n \equiv 0 \text{ or } 2 \pmod{3} \\ 1 & \text{if } n \equiv 1 \pmod{3} \end{cases}$$

Proof: Obvious from Theorem 2.1. □

The following Remarks are direct consequence of the Corollaries above. We have to note that if $G = P_n$, then $\varphi_G(n) = f_G(n) + g_G(n) + h_G(n)$.

Remark 2.2. Let $G = P_n$. If $n \equiv 0 \pmod{3}$, then $h_G(n) = g_G(n) < f_G(n) = \varphi_G(n)$.

Remark 2.3. Let $G = P_n$. If $n \geq 8$ and $n \equiv 1$ or $2 \pmod{3}$, then
 $h_G(n) < g_G(n) \leq f_G(n) < \varphi_G(n)$.

The following results determine the probability of getting sets D_1 , D_2 , and D_3 in graph $G = P_n$ where $n \geq 1$ with certain restrictions.

Theorem 2.2. Let G be a path of order $n \geq 3$. Then,

$$P_G^1(n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3} \\ \frac{n^2+n-2}{n^2+13n+4} & \text{if } n \equiv 1 \pmod{3} \\ \frac{n-2}{n+4} & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Proof: Let $G = P_n$ where $n \in \mathbb{N} \setminus \{1, 2\}$. By getting the ratio of Corollary 2.1 and Theorem 2.1, that is, $f_G(n)/\varphi_G(n)$, then this completes the proof. □

Remark 2.4. Let $G = P_n$. If $n \leq 2$, then $P_G^1(n) = 0$.

Theorem 2.3. Let G be a path of order $n \geq 1$. Then,

$$P_G^2(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{3} \\ \frac{12n-12}{n^2+13n+4} & \text{if } n \equiv 1 \pmod{3} \\ \frac{6}{n+4} & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Proof: Let $G = P_n$ where $n \in \mathbb{N}$. Then, by getting the ratio of Corollary 2.2 and Theorem 2.1, that is, $g_G(n)/\varphi_G(n)$. And this completes the proof. □

Theorem 2.4. Let G be a path of order $n \geq 4$. Then,

$$P_G^3(n) = \begin{cases} 0 & \text{if } n \equiv 0 \text{ or } 2 \pmod{3} \\ \frac{18}{n^2+13n+4} & \text{if } n \equiv 1 \pmod{3} \end{cases}$$

Proof: Let $G = P_n$ where $n \in \mathbb{N} \setminus \{1\}$. Hence, by getting the ratio of Corollary 2.3 and Theorem 2.1, that is, $h_G(n)/\varphi_G(n)$, then this completes the proof. □

Remark 2.5. Let $G = P_n$. If $n = 1$, then $P_G^3(n) = 0$.

By the definition of independent domination in graphs above, we obtained the following remark.

Remark 2.6. Let $G = P_n$. Then, the following holds:

- i.) $\sum_{v \in D_1} \deg_G(v) = 2\gamma_i(G)$;
- ii.) $\sum_{v \in D_2} \deg_G(v) = 2\gamma_i(G) - 1$; and
- iii.) $\sum_{v \in D_3} \deg_G(v) = 2(\gamma_i(G) - 1)$.

The following result determines the number of all possible ways of generating a dominating set in cycle graph C_n where $n \geq 3$.

Theorem 2.5. Let $G = C_n$ where $n \in \mathbb{N} \setminus \{1, 2\}$. Then,

$$|S_G| = \varphi_G(n) = \begin{cases} 6 & \text{if } n = 4 \\ 3 & \text{if } n \equiv 0 \pmod{3} \\ \frac{1}{18}(n^3 - 5n^2 + 22n) & \text{if } 7 \leq n \equiv 1 \pmod{3} \\ \frac{1}{3}(n^2 - 2n) & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Proof. Suppose that $G = C_n$ where $n \in \mathbb{N} \setminus \{1, 2\}$. Then, we consider that following 4 cases below:

Case 1. If $n = 4$, then we have $|V(G) \setminus D| = |D| = 2$. When the elements of D are alternate with the elements of $V(G) \setminus D$, this implies that there are 2 possible arrangement of this form in graph G . On the other hand, if the elements of D are connected in G , then there are 4 possible ways since the eccentricity of $u \in D$ to $v \in D$ is $e(u) = 3$. Thus, $|S_G| = \varphi_G(n = 4) = 6$.

Case 2. Considering $n \equiv 0 \pmod{3}$, then $\forall u_j \in D, \exists a, b \in V(G) \setminus D$ such that $u_j \in N(a) \cap N(b)$ and $\bigcap_{all j} N(u_j) = \emptyset$. Since G is a cycle and $d_G(u_j, u_{j+1}) = 3$, then this follows that $|S_G| = \varphi_G(n) = 3$.

Case 3. Let $7 \leq n \equiv 1 \pmod{3}$. Consider first a graph $H = P_n$. Then, we reflect on the following subcases below:

Subcase 1. By Remark 2.1 and Theorem 2.1, there is only one configuration of independent dominating set considering that there exists $u, v \in D = I$ such that $\deg(u) = \deg(v) = 1$. Let u_1, u_2, \dots, u_n be the vertices of graph H . Then, u_1 and u_n are dominating vertex. Now, to form a cycle graph we connect the two vertices u_1 and u_n by adding an edge. This implies that $e(u_1)$ from u_n is $n - 1$ and it obviously follows that there are n possible ways of generating dominating vertex of this form in graph G .

Subcase 2. Again, by Remark 2.1 and Theorem 2.1, there are $\frac{n^2 - 5n + 4}{18}$ arrangement of dominating set given that two vertices in D are in the second vertex and second to last vertex of H . Let v_1, v_2, \dots, v_n be the vertices of path H . Connecting v_1 and v_n by adding an edge, we form a cycle graph G . Since $e(v_1) = n - 1$ from vertex v_n , then this implies that there are $n \left(\frac{n^2 - 5n + 4}{18} \right)$ ways to generate dominating vertices of this form. If we combine the two sub

cases, then we obtain, $|S_G| = \varphi_G(n) = n \left(\frac{n^2 - 5n + 4}{18} \right) + n$. Simplifying the equation, we end up with,

$$|S_G| = \varphi_G(n) = \frac{n^3 - 5n^2 + 22n}{18}.$$

Case 4. Lastly, we let $n \equiv 2 \pmod{3}$ and $H = P_n$. Then, by *Remark 2.1* and *Theorem 2.1*, there are $\frac{n-2}{3}$ configuration in graph H considering that $\forall u_j \in D, \deg(u_j) = 2$. Let u_1, u_2, \dots, u_n be the vertices of graph H . By connecting u_1 and u_n by adding an edge, we obtain a cycle graph G . Since $e(u_1) = n - 1$ from vertex u_n , then it follows that $|S_G| = \varphi_G(n) = n \left(\frac{n-2}{3} \right) = \frac{n^2 - 2n}{3}$. Hence, combining all cases and this completes the proof. \square

By the *Theorem 2.5* above, we obtain the following remarks.

Remark 2.7. Let $G = C_n$ where $n \geq 3$. Then, $\varphi_G(n) = f_G(n)$ and $g_G(n) = h_G(n) = 0$.

Remark 2.8. Let $G = C_n$ where $n \geq 3$. Then, $P_G^1(n) = 1$ and $P_G^2(n) = P_G^3(n) = 0$

III. CONCLUSION

This paper constructed a new combinatorial formulae that counts the number of ways of putting dominating set in path and cycle graphs. These formulae are function of the order of the said two graphs.

REFERENCES

- [1] Allan, R. B. & Laskar, R., "On domination and independent domination number of a graph," *Discrete Mathematics*, vol. 23, pp. 73-76, 1978.
- [2] Arocha, J. L. & Llano, B., "The number of dominating k-sets of paths, cycles and wheels", *arXiv preprint:1601.01268*, 2018.
- [3] Casinillo, L. F., "A note on Fibonacci and Lucas number of domination in path", *Electronic Journal of Graph Theory and Applications*, vol. 6, no. 2, pp. 317-325, 2018.
- [4] Casinillo, L. F., "Odd and even repetition sequences of independent domination number", *Notes on Number Theory and Discrete Mathematics*, vol. 26, no. 1, pp. 8-20, 2020.
- [5] Casinillo, L. F., Lagumbay, E. T. & Abad, H. R. F., "A note on connected interior domination in join and corona of two graphs," *IOSR Journal of Mathematics*, vol. 13, no. 2, pp. 66-69, 2017.
- [6] Chartrand, G. and Zhang, P., *A First Course in Graph Theory*, New York: Dover Publication, Inc., 2012.
- [7] Cockayne, E. J., & Hedetniemi, S. T., "Towards a theory of domination in graph," *Networks Advanced Topics*, vol. 7, pp. 247-261, 1977.
- [8] Goddard, W. & Henning, M. A., "Independent domination in graph: A survey and recent results," *Discrete Mathematics*, vol. 313, pp. 839-854, 2013.
- [9] Haynes, T. W., Hedetniemi, S. T., & Slater, P. J., *Fundamentals of Domination in Graphs*, New York: Marcel Dekker, 1998.
- [10] Ore, O., *Theory of Graphs*, American Mathematical Society Providence, R. I., 1962.
- [11] Perderon, A. S., & Vestergaard, P. D., "The number of independent sets in unicyclic graphs," *Discrete Applied Mathematics*, vol. 152, pp. 246-256, 2005.

- [12] Tarr, J. M., *Domination in Graphs*. Graduate Theses and Dissertations. Retrieved from <https://scholarcommons.usf.edu/etd/1786>, 2010.
- [13] Samanmoo, R., Trakultraipruk, N., & Ananchuen, N., “ γ -Independent dominating graphs of paths and cycles,” *Maejo International Journal of Science and Technology*, vol. 13, no. 3, pp. 245-256, 2019.