JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 3 NO. 2 (NOV 2020)
Available online at www.jfma.math.fsm.undip.ac.id

PREDICTIVE CONTROL FOR MAX-PLUS LINEAR SYSTEMS IN PRODUCTION SYSTEMS

Lathifatul Aulia[1], Widowati[2*], R. Heru Tjahjana[3], Sutrisno[4]

[1,2 3,4]Department of Mathematics, Diponegoro University, Semarang, Indonesia
Email: [1]lathifatulaulia213@yahoo.com, [2]widowati@lecturer.undip.ac.id,
[3]redemtusherutjahjana@lecturer.undip.ac.id, [4]s.sutrisno@live.undip.ac.id
[*] Corresponding author

Abstract. Discrete event systems, also known as DES, are class of system that can be applied to systems having an event that occurred instantaneously and may change the state. It can also be said that a discrete event system occurs under certain conditions for a certain period because of the network that describes the process flow or sequence of events. Discrete event systems belong to class of nonlinear systems in classical algebra. Based on this situation, it is necessary to do some treatments, one of which is linearization process. In the other hand, a Max-Plus Linear system is known as a system that produces linear models. This system is a development of a discrete event system that contains synchronization when it is modeled in Max-Plus Algebra. This paper discusses the production system model in manufacturing industries where the model pays the attention into the process flow or sequence of events at each time step. In particular, Model Predictive Control (MPC) is a popular control design method used in many fields including manufacturing systems. MPC for Max-Plus-Linear Systems is used here as the approach that can be used to model the optimal input and output sequences of discrete event systems. The main advantage of MPC is its ability to provide certain constraints on the input and output control signals. While deciding the optimal control value, a cost criterion is minimized by determining the optimal time in the production system that modeled as a Max-Plus Linear (MPL) system. A numerical experiment is performed in the end of this paper for tracking control purposes of a production system. The results were good that is the controlled system showed a good performance.

Keywords: Discrete Event systems, Model Predictive Control, Max-plus Algebra, Max-Plus Linear Systems.

I. INTRODUCTION

The theory of DES is divided into three main approaches consisting of logical approach that considers the occurrence of events, a quantitative approach which addresses the problem of evaluating and optimizing the performance of several events occur in the given lapse of time, and a stochastic approach that considers the occurrence of events under certain statistical conditions. A conventional analysis of stability that employs appropriate Lyapunov functions can be used for logical DES that was applied on a manufacturing system [1]. In real life, many phenomena such as manufacturing systems, telecommunication networks, and transportation systems can be described as discrete event systems. The general nature of these examples is that the start of an activity depends on the cessation of several other activities, and the system describes naturally exhibiting periodic behavior. The problems of discrete event systems, if modeled on a conventional discrete system, will result in a nonlinear system. Therefore, the system cannot conveniently be described to analyze its properties. One framework exists for studying DES, which is with Max-Plus Algebra ([2], [3], and [4]). It was shown that linear models could express DES, which only contains synchronization in Max-Plus Algebra. These models are called Max-Plus Linear (MPL) systems. With the MPL system, the system that was initially nonlinear when modeled with a conventional discrete system becomes a linear system in Max-Plus Algebra. Application of MPL systems arises in the context of manufacturing plants and traffic management ([5] & [6]).

In general, an appropriate model to use in modeling is a nonlinear system. Such is the case in conventional algebra, max-plus-linear systems are included in discrete event systems; for example, manufacturing companies, where manufacturing companies have a significant role in life. Because of the existence of a manufacturing company, we can utilize raw materials that previously could not be used into finished goods that could be used. The function of production in manufacturing companies is to make an item or service that is needed and following consumer needs. The most important to be considered by manufacturing companies before carrying out the production process is to have a production plan to achieve the production function appropriately and adequately ([7], [8]). Several strategies can be done, including predicting consumer demand, controlling inventory, preparing employees, managing time stages, and identifying problems that occur in the company. In the manufacturing system, time accuracy is crucial because of the increasing production demands to meet market needs. Therefore, punctuality is needed in the manufacturing system. Model predictive control (MPC) is one of the popular methods used in the industry because MPC is exactly in practical control processes and is widely accepted in industrial processes. To produce the performance as desired, the MPL system and systems in general, given a control [9]. One of them can be used; predictive model control techniques is a form of control that uses a system process model to predict future system behavior over a specified period. The advantage of predictive model control is able to provide certain constraints on the input and output control signals [10].
The MPC used here is for discrete event systems [11]. The predictive control model uses the principle of receding horizon, namely the calculation of predictive output and the optimal control sequence at each time step along the horizon involving calculations that have been obtained in the previous time step. Only the first part of the control sequence that has been achieved in the last time step is applied to the system. The use of Model predictive control to predict the output of a process that will come within a specific timeframe is called the optimization of the prediction control horizon. Boom and Schutter decide equilibrium of the control system for this max-plus-linear system with guaranteed stability [12]. Much work has been done in modeling, analysis, control, and optimization of max-plus-linear systems, including reachability analysis MPL systems ([13], [14]). Further discussion of the MPL system presented the necessary and sufficient conditions for the existence and uniqueness of globally optimal solutions MPL systems ([15], [16]).

In this paper, the MPC control is discussed in the control of a finite horizon predictive model with a reference trajectory for the Max-Plus Linear system of a manufacturing company. The objective function for the finite horizon MPC control problem is designed to minimize errors, which is the difference between MPL system output and predetermined reference trajectories. A manufacturing company case study on a production system modeling that has a job shop production process design is modeled as a max-plus-linear system. Where the production process, in this case, is a type of production process that has high flexibility, or the sequence of operations can occur erratically. The application of MPC to the production system aims to stabilize the MPL system. Furthermore, the model is implemented via Matlab to get the predicted optimal time in the production system.

II. MATHEMATICAL MODELLING

2.1 Model Predictive Control

This part is a short introduction to MPC, its roots in optimal control, to use a dynamic model to optimize the forecast and forecast system behavior to produce the best decision at the moment. The initial state of the system is determined to use measurement records. Therefore, the initial state influences the optimal control of the dynamic system. So, from that condition, a model forecast is used to result in the optimal control action, and the estimation problem, in which the record of measurements to provide an optimal state estimate, implicate dynamic models and optimization. The most general linear state-space model is the time-varying continuous model with initial condition $x(0)$,

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + B(t)u, \quad x(0) = x_0 \\ y &= C(t)x + D(t)u \end{aligned} \tag{1}$$

where $y \in \mathbb{R}^p$ is the output, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $A(t) \in \mathbb{R}^{n\times n}$ is the state transition matrix, $B(t) \in \mathbb{R}^{n\times m}$ is the input matrix, $C(t) \in \mathbb{R}^{p\times n}$ is the output matrix, and $D(t) \in \mathbb{R}^{p\times m}$ is a relation coupling between u and y. In many applications $D = 0$. If A, B, C, and D are time-invariant, the linear model reduces to

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx + Du \end{aligned} \tag{2}$$

Discrete-time models are often suitable if the sampling rate is chosen appropriately. The model can describe the behavior exclusively at the sample times. The finite-dimensional, time-invariant, discrete-time, linear model [17] is

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0 \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{3}$$

where $k \in Z^+$ is a nonnegative integer denoting the sample number, is connected to time by $t = k\Delta$. The linear discrete-time model is a linear differential equation. So, analytical solutions are readily derived. The solution (1) is

$$x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} Bu(j). \tag{4}$$

So, consider the following linear model:

$$x(j+1) = Ax(j) + Bu(j) + B_d d_s, \tag{5}$$

where in the state is $x \in \mathbb{R}^n$, the input system is $u \in \mathbb{R}^m$, and the trouble of the system is $d_s \in \mathbb{R}^d$. Assumed from the system (A, B) in the condition is stabilizable. The following are constraints states and inputs:

$$x(j) \in X,$$
$$u(j) \in U. \tag{6}$$

Next, the MPC problem gives an objective function, $\ell_T(x,u;z_t)$ is the tracking stage cost for suggestion input u from state x which contract the cost function by adding a terminal penalty (penalize deviations) from a chosen steady-state $z_t = (x_t, u_t)$. The optimal steady-state problem for the tracking cost as follows [17]:

$$\min_{x,u} \ell_T(x(k),u(k);z_t) = (x(k)-x_t)^T Q(x(k)-x_t) + (u(k)-u_t)^T R(u(k)-u_t)$$
s.t.
$$x = Ax + Bu + B_d d_s, \tag{7}$$
$$x(k) \in \mathrm{X},$$
$$u(k) \in U,$$

in which matrices Q and R are positive semi-definite matrices which guide and maintain states and inputs to their respective steady-state. In general, when the objective function of the MPC problem is stated in the following quadratic form:

$$J(k) = \sum_{i=N_w}^{N_p} (y(k+i|k) - r(k+i|k))^T Q(i)(y(k+i|k) - r(k+i|k)) + \sum_{i=0}^{N_u-1} (\Delta u^T(k+i|k) R(i) \Delta u(k+i|k) \tag{8}$$

with $Q(i)$ are positive semi-definite matrices, $R(i)$ are positive definite matrices, and $N_w \geq 1$. The first term of the right-hand side of the equation (8) expresses an error, which is the difference between the predicted output and the reference trajectory term. Thus, the optimization problem for MPC can be seen as a matter of determining optimal control changes that minimize errors. The objective function (8) can also be expressed as

$$J(k) = (\Upsilon(k) - \tau(k))^T Q(\Upsilon(k) - \tau(k)) + \Delta\upsilon^T(k) R \Delta\upsilon(k) \tag{9}$$

with

$$\tau(k) = \begin{bmatrix} r(k+N_w|k) \\ r(k+N_w+1|k) \\ \vdots \\ r(k+N_p|k) \end{bmatrix}, Q = \begin{bmatrix} Q(N_w) & 0 & \cdots & 0 \\ 0 & Q(N_w+1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q(N_p) \end{bmatrix},$$

$$\text{and } R = \begin{bmatrix} R(0) & 0 & \cdots & 0 \\ 0 & R(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R(N_u-1) \end{bmatrix}.$$

Next, define a tracking error $\varepsilon(k) = \tau(k) - \Psi x(k) - \Upsilon u(k-1)$. So,

$$\tau(k) = \Psi x(k) + \Upsilon u(k-1) + \varepsilon(k) \tag{10}$$

belongs to the output equation and from (10) we have

$$\begin{aligned} \Upsilon(k) - \tau(k) &= \Psi x(k) + \Upsilon u(k-1) + \Theta\Delta\upsilon(k) - (\Psi x(k) + \Upsilon u(k-1) + \varepsilon(k)) \\ &= \Theta\Delta\upsilon(k) - \varepsilon(k) \end{aligned} \tag{11}$$

Substituting (11) into (9) derives

$$\begin{aligned} J(k) &= (\Theta\Delta\upsilon(k)-\varepsilon(k))^T Q(\Theta\Delta\upsilon(k)-\varepsilon(k))+\Delta\upsilon(k)^T R\Delta\upsilon(k) \\ &= \varepsilon(k)^T Q\varepsilon(k) - 2\Delta\upsilon(k)^T\Theta^T Q\varepsilon(k)+\Delta\upsilon(k)^T[\Theta^T Q\Theta + R]\Delta\upsilon(k) \\ &= \varepsilon(k)^T Q\varepsilon(k) - \Delta\upsilon(k)^T G + \Delta\upsilon(k)^T H\Delta\upsilon(k) \end{aligned}$$

with $G = 2\Theta^T Q\varepsilon(k)$ and $H = \Theta^T Q\Theta + R$. To determine $\Delta\upsilon(k)$ optimally, then the gradient of J is $\nabla_{\Delta\upsilon(k)} J = 0$.

2.2 Max-Plus Linear Systems

We will present about Max-Plus Algebra (MPA) before we proposed the Max-Plus Linear system. Standart operations of the max-plus algebra are addition ($\oplus$) and multiplication ($\otimes$), in which the structure ($\mathbb{R}_\varepsilon$,$\oplus$,$\otimes$) is called the max-plus algebra defined by [2]:

$$x \oplus y = \max(x, y), x \otimes y = x + y \text{ for } x, y \in \mathbb{R}_\varepsilon$$

Define $\varepsilon = -\infty$, for $x, y \in \mathbb{R}_\varepsilon \stackrel{\text{def}}{=} \mathbb{R} \cup \{-\infty\}, \mathbb{R}_+ = \{x \in \mathbb{R}: x \geq 0\}$. The operation of max-plus-algebraic addition is symbolized with $\oplus$, and the operation of max-plus-algebraic multiplication is symbolized with $\otimes$. Through many attributes and concepts from linear algebra.

The $m \times n$ matrix set for $n \in \mathbb{N}$ in $\mathbb{R}_\varepsilon$, max-plus-algebraic zero matrix, is denoted by $(\varepsilon_{m\times n})_{ij} = \varepsilon$ for all i, j; and E_n denotes the $n \times n$ max-plus-algebraic identity matrix: $(E_n)_{ii} = 0$ for all i and $(E_n)_{ii} = \varepsilon$ for all i, j with $i \neq j$. If $A, B \in \mathbb{R}_\varepsilon{}^{m\times n}, C \in \mathbb{R}_\varepsilon{}^{n\times p}$ accordingly, we have

$$(A \oplus B)_{ij} = \left(a_{ij} \oplus b_{ij}\right) = \max\left(a_{ij}, b_{ij}\right),$$
$$(A \otimes C)_{ij} = \oplus_{k=1}^{n}\left(a_{ik} \otimes c_{kj}\right) = \max_k(a_{ik} + c_{kj})$$

for all i, j. $A \in \mathbb{R}_\varepsilon{}^{n\times n}$ is the max-plus-algebraic matrix and defined as follows: $A^{\otimes^0} = E_n$ and $A^{\otimes^k} = A \otimes A^{\otimes^{k-1}}$ For $k = 1, 2, 3, ...$ Trace a matrix in MPA is also defined as the sum of the main diagonal entries of the matrix i.e.

$$trace(A) = \oplus_{i=1}^{n}[A]_{ii}.$$

Matrix B is called the inverse of the matrix A and is defined as $B = A^{\otimes^{-1}}$. Then the matrix A is an invertible matrix here.

Max-plus linear systems are a particular class of max-plus algebraic theories that modeled in linear behavior, which is synchronized with discrete linear systems, and there is no resistances in the system. Therefore, max-plus-linear DES is efficient in assessing and analyzing the system's characteristics and it can be modeled by the following form [2, 18]:

$$\begin{aligned} x(k+1) &= A_k \otimes x(k) \oplus B_k \otimes u(k) \\ y(k) &= C_k \otimes x(k) \end{aligned} \tag{12}$$

where the index k is the event counter which indicates the number of event occurrences from the initial state. The state variable is $x(k) \in \mathbb{R}^n$, control input is $u(k) \in \mathbb{R}^m$, and $y(k) \in \mathbb{R}^l$ is system output. Moreover, $A_k \in \mathbb{R}_\varepsilon{}^{n\times n}, B_k \in \mathbb{R}_\varepsilon{}^{n\times m}$ dan $C_k \in \mathbb{R}_\varepsilon{}^{l\times n}$, are system matrices, where m is the number of inputs and l is the number of outputs. (12) is enhanced at each k cycle so it can be modeled as the discrete event system and the corresponding result will be called max-plus-linear (MPL) system or max-plus-linear time-invariant discrete event system

[19]. Note that for MPL systems, the sequences are non-decreasing functions. This because the MPL system input is time, so it applies:

$$u(k) \leq u(k+1) \tag{13}$$

for each $k \geq 0$. In constructing MPC for MPL systems without constraint, (13) will be used. The following definition describes the stabilizing control for MPL discrete event system.

Definition 1 [20] *Given a state feedback controller* $\mu : \mathbb{R}_\varepsilon^n \to \mathbb{R}_\varepsilon^m$, *then the closed-loop system* $x(k) = A \otimes x(k-1) \oplus B \otimes \mu(x(k-1))$ *is stable if the state remains bounded, i.e., for every,* $\delta > 0$ *there exists a real-valued function* $\theta(\delta) > 0$ *such that* $\| x(0) - x_{e1} \|_\infty \leq \delta$ *implies* $\| x(k) - x_{e1} \|_\infty \leq \theta(\delta)$ *for all* $k \geq 0$.

Next, changing the coordinate for MPL systems (12)-(13) is processed. Since the largest eigenvalue of matrices A_k is λ_{max} that is finite, then there is an invertible matrix $P \in \mathbb{R}_\varepsilon{}^{n \times n}$ so that matrices $\tilde{A} = P^{\otimes^{-1}} \otimes A_k \otimes P$ satisfy $[\tilde{A}]_{ij} \leq \lambda_{max}$ for each $i, j = 1,2, \ldots, n$. Based on the existence of P matrices, the coordinate changing for the MPL system (12)-(13) becomes:

$$\begin{aligned}
A &= P^{\otimes^{-1}} \otimes A_k \otimes P \Rightarrow A_k = P \otimes A \otimes P^{\otimes^{-1}} \\
x(k) &= P^{\otimes^{-1}} \otimes x(k) \Rightarrow x(k) = P \otimes x(k) \\
B &= P^{\otimes^{-1}} \otimes B_k \Rightarrow B_k = P \otimes B \\
C &= C_k \otimes P \Rightarrow C_k = C \otimes P^{\otimes^{-1}} \\
y(k) &= y(k) \\
u(k) &= u(k)
\end{aligned} \tag{14}$$

and, substituting equation (14) into (12) derives:

$$\begin{aligned}
P \otimes x(k+1) &= P \otimes A \otimes P^{\otimes^{-1}} \otimes P \otimes x(k) \oplus P \otimes B \otimes u(k) \\
&= P \otimes A \otimes x(k) \oplus P \otimes B \otimes u(k) \\
&= P \otimes (A \otimes x(k) \oplus B \otimes u(k))
\end{aligned} \tag{15}$$

and

$$y(k) = C \otimes P^{\otimes^{-1}} \otimes P \otimes x(k) = C \otimes x(k)\,. \tag{16}$$

By multiplying (15) with $P^{\otimes^{-1}}$ we have

$$\begin{aligned}
P^{\otimes^{-1}} \otimes P \otimes x(k+1) &= P^{\otimes^{-1}} \otimes P \otimes (A \otimes x(k) \oplus B \otimes u(k)) \\
x(k+1) &= A \otimes x(k) \oplus B \otimes u(k).
\end{aligned} \tag{17}$$

Thus, we have the system

$$\begin{aligned}
x(k+1) &= A \otimes x(k) \oplus B \otimes u(k) \\
y(k) &= C \otimes x(k)
\end{aligned} \tag{18}$$

The next step is to normalize system (18) by subtracting the state vector $\tilde{x}$, input $\tilde{x}$, dan output $\tilde{y}$ with $\rho k, \rho$ vector with positive entries, and subtracting each entry of $\tilde{A}$ with ρ. Then we have

$$x(k) = x(k) - \rho k,$$
$$u(k) = u(k) - \rho k,$$
$$y(k) = y(k) - \rho k,$$
$$[A]_{ij} = [A]_{ij} - \rho,$$
$$B = B$$
$$C = C$$

and the normalized system is

$$\begin{aligned} x(k+1) &= A_k \otimes x(k) \oplus B_k \otimes u(k) \\ y(k) &= C_k \otimes x(k). \end{aligned} \tag{19}$$

The MPL system (19) is controllable if and only if each component of the state can be made arbitrarily large by applying an appropriate controller to the system initially at the rest [20]. In other words, the system is controllable if all states are connected to some input.

2.3 Model Predictive Control for MPL Systems

In this subsection, predictive control system tracking in Max-Plus Algebra is used to determine solutions to the nonlinear system. Consider the deterministic, the case for max-plus-linear systems, plant with m inputs and l outputs that can be modeled by a state-space of the form

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k). \end{aligned} \tag{20}$$

The vector x represents the state, u the input, and y the output. A system that can be modeled by (20)-**Error! Reference source not found.** will be called a plus-time-linear system. In MPC, a performance cost criterion J is formulated as a function that reflects the reference tracking error (J_{out}) and the control effort (J_{in}) as follows

$$J = J_{out} + \lambda J_{in} = \sum_{j=1}^{N_p} \| y(k+j \mid k) - r(k+j) \|^2 + \lambda \sum_{j=1}^{N_p} \| u(k+j-1) \|^2, \tag{21}$$

where $\hat{y}(k+j|k)$ is the estimate of the output at time step $k+j$ based on the information available at time step k, r is the reference signal, λ is a nonnegative scalar, and N_p is the prediction horizon. In MPC, the input is taken to be constant from a certain point on $u(k+j) = u(k+N_c-1)$ for $j = N_c, \dots, N_p - 1$ where N_c is the control horizon. The use of a control horizon leads to a reduction in the number of optimization variables. This results in the decrease of the computational burden, a smoother controller signal (because of the emphasis on the average behavior rather than on aggressive noise reduction), and a stabilizing (since the output signal is forced to its steady-state value). MPC uses a receding horizon principle. At time step k the future control sequence $u(k), \dots, u(k+N_c-1)$ is determined such that the cost criterion is minimized subject to the constraints. At time step k, the first element of the optimal sequence $\left(u(k)\right)$ is applied to the process. At the next time instant, the horizon is shifted, the model is updated with new information of the measurements, and a new optimization at time step $k+1$ is performed. By successive substitution of (20) into **Error! Reference source not found.**, estimates of the future values of the output can be computed [21]. In matrix notation we have

$$y(k) = Hu(k) + g(k)$$

with

$$\tilde{y}(k) = \begin{bmatrix} y(k+1|k) \\ y(k+2|k) \\ \vdots \\ y(k+N_p|k) \end{bmatrix}, \tilde{r}(k) = \begin{bmatrix} r(k+1) \\ r(k+2) \\ \vdots \\ r(k+N_p) \end{bmatrix}, u(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N_p-1) \end{bmatrix},$$
$$H = \begin{bmatrix} CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N_p-1}B & CA^{N_p-2}B & \cdots & CB \end{bmatrix}, g(k) = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N_p} \end{bmatrix} x(k) \quad (22)$$

The MPC problem for plus-time linear systems at time step k is defined as follows: find the input sequence $u(k), \dots, u(k+N_c-1)$ that minimizes the performance index J subject to the linear constraint

$$E(k)u(k) + F(k)y \leq h(k) \quad (23)$$

with $E(k) \in \mathbb{R}^{p\times mN_p}, F(k) \in \mathbb{R}^{p\times 1N_p}, h(k) \in \mathbb{R}^p$ for some integer p, subject to the control horizon constraint

$$u(k+1) = u(k+N_c-1) \text{ for } j = N_c, N_c+1, \dots. \quad (24)$$

Minimizing J subject to (23) and (24) is a convex quadratic programming problem which can be solved efficiently. The parameters $N_c, N_p,$ and λ are the three basic MPC tuning parameters [19]. The length of the step response of the process the prediction horizon is N_p, and the time interval $(1, N_p)$. The control horizon $N_c \leq N_p$ is usually taken be equal to the system order. The parameter $\lambda \geq 0$ makes a trade-off between the tracking error and the control effort and is usually chosen as small as possible.

Based on [22], it is shown that more conventional analysis of stability which employs appropriate Lyapunov functions can be used for logical DES and explained further about a general characterization of the stability properties of automata-theoretic DES models, finite-state systems, and the Lyapunov stability analysis approach is illustrated on a manufacturing system. The application of MPC design methods to various types of control system models is very appropriate. Due to the linear property that satisfies the operation of a vector and multiplication between a matrix on the max-plus algebra, then the prediction equation can be obtained directly. In this section, explain the MPC systems [2]. The following is given a direct calculation of the state variable in the upcoming event counter $k+1, \dots k+N$:

$$x(k+1) = A_k \otimes x(k) \oplus B_k \otimes u(k+1)$$
$$x(k+2) = A_{k+1} \otimes A_k \otimes x(k) \oplus A_{k+1} \otimes B_k \otimes u(k+1) \oplus B_{k+1} \otimes u(k+2)$$
$$\vdots$$
$$x(k+N) = A_{k+N+1} \otimes \cdots \otimes A_k \otimes x(k) \oplus A_{k+N-1} \otimes \cdots \otimes A_{k+1} \otimes B_k \otimes u(k+1) \oplus A_{k+N-1}$$
$$\otimes \cdots \otimes A_{k+2} \otimes B_{k+1} \otimes u(k+2) \oplus \cdots \oplus B_{k+N-1} \otimes u(k+N)$$

Also in [2], $y(k)$, the output prediction of future values (19), can be done by successive substitution that leads to the expression

$$y(k) = C \otimes x(k) \oplus D \otimes u(k)$$

where $\widetilde{C}$ and $\widetilde{D}$ are given by

$$D = \begin{bmatrix} C \otimes B & \varepsilon & \cdots & \varepsilon \\ C \otimes A \otimes B & C \otimes B & \cdots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ C \otimes A^{\otimes^{N_p - 1}} \otimes B & C \otimes A^{\otimes^{N_p - 2}} \otimes B & \cdots & C \otimes B \end{bmatrix}, C = \begin{bmatrix} C \otimes A \\ C \otimes A^{\otimes^2} \\ \vdots \\ C \otimes A^{\otimes^{N_p}} \end{bmatrix}$$

and $\tilde{u}(k), \widetilde{y}\,(k)$ are defined as

$$y(k) = \begin{bmatrix} y(k+1) \\ y(k+2) \\ \vdots \\ y(k+N_p) \end{bmatrix}, u(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N_p-1) \end{bmatrix}.$$

The MPC formulation for the normalized MPL systems is explained as follows. With $\hat{y}(k+j)$ denotes the prediction of $y(k+j)$ based on the state that is known in steps k and N_p is the prediction horizon, the Predictive Control Model for Max-Plus Linear system problems, called the MPL-MPC, is formulated as follows [2]:

$$\min_{u(k),y(k)} J(u(k), y(k)) = \min_{u(k),y(k)} J_{out}(y(k)) + \lambda J_{in}(u(k)) \tag{22}$$

subject to,

$$y(k) = C \otimes x(k) \oplus D \otimes u(k), \tag{25}$$

$$E(k)u(k) + F(k)y(k) \leq h(k), \tag{26}$$

$$\Delta u(k+j) \geq 0 \ \text{ for } \ j = 0,1,\ldots, N_p - 1 \tag{27}$$

$$\Delta^2 u(k+j) \geq 0 \ \text{ for } \ j = N_c, \ldots, N_p - 1 \tag{28}$$

where $\Delta u(k) = u(k) - u(k-1)$ and $\Delta^2 u(k) = \Delta u(k) - \Delta u(k-1) = u(k) - 2u(k-1) + u(k-2)$. Equation (26) reflects the constraints of the maximum line for the output stage at separated time of the input and output stage, (27) guarantees the nondecreasing input signal, and (28) is the constraint appeared from the control horizon N_c.

Theorem 2 [19]. *Let the mapping $\widetilde{y} \longrightarrow F(k)\widetilde{y}$ be a monotonically non-decreasing function of $\widetilde{y}$. Let $(\widetilde{u}^*, \widetilde{y}^*)$ be an optimal solution of the Predictive Control Model for Max-Plus Linear system problems. If we define $\widetilde{y}^{\#} = \widetilde{C} \otimes x(k) \oplus \widetilde{D} \otimes \tilde{u}^*$ then $(\widetilde{u}^*, \widetilde{y}^{\#})$ is an optimal solution of the original Predictive Control Model for Max-Plus Linear system problems.*

The Predictive Control Model for MPL system problems can be written as a convex optimization problem. In general conditions, if the system is closed-loop system, controller will be piecewise affine in the state $x(k)$ and the reference $r(k)$. In this situation, all signals in the system must be maintained so that the system is stable.

Definition 3 [23]. *A discrete event system is called stable if all its buffer levels remain bounded.*

Based on [12], The Predictive Control Model for Max-Plus Linear system problems is stable. The existence of a solution of the Predictive Control Model for Max-Plus Linear system problems at step k problem can be verified by solving the system, which describes the feasible set of the problem. The theorem that explains the stabilizing nature of MPC control is explained as follows.

Theorem 4 [20] *Given a prediction horizon* N *such that* $\beta < \frac{1}{mN}$, *then*

a. *The following inequalities hold*

$$\begin{cases} u^c(k) \le u^{RHC,N}(k) \le u^f(k) \\ x^c(k) \le x^{RHC,N}(k) \le x^f(k) \end{cases} \tag{29}$$

Then it can be said the receding horizon controller stabilizes the system (29).

b. *If* $N = 1$ *then* $u^{RHC,1}(k) = u^f(k)$. *For two prediction horizon,* $N_1 < N_2$ *we have*

$$\begin{cases} u^{RHC,N_1}(k) \ge u^{RHC,N_2}(k) \\ x^{RHC,N_1}(k) \ge x^{RHC,N_2}(k) \end{cases}$$

In the MPC application for production systems, there are several stages of completion, which are as follows:

1. Knowing the arrangement of the scheme of the production machine that will be modeled from the input, processing, and output
2. Modeling the production system according to the stages of the production machine and its processing time
3. Obtain the MPL system modeled from the production system
4. Obtain the predicted optimal time of the production system by applying MPC to the system.

As well as in design MPC for MPL systems, in the design of production system, it is assumed that $x(k)$ is the state at step k, which is obtained based on measurements or estimates using previous measurements. Then, using equation (19) to suppose the process of changing the output proses for the input sequence $u(k), \dots, u(k + N_p - 1)$, the steady-state problem can be written as

$$J = J_{out} + \lambda J_{in} = \sum_{j=1}^{N_p} \| y(k+j\,|\,k) - r(k+j) \|^2 + \lambda \sum_{j=1}^{N_p} \| u(k+j-1) \|^2,$$

with constraints

$$\begin{aligned} &x(0) = x_0, \\ &x(k+1) = A_k \otimes x(k) \oplus B_k \otimes u(k), \\ &x(k) \in \mathbb{X}, \\ &u(k) \in \mathbb{U}, \\ &x(N) = x_s, \\ &k \in \{0, 1, 2, \dots, N-1\}. \end{aligned}$$

The prediction equation is obtained in the form of a vector, which is expressed as

$$y(k) = C \otimes x(k) \oplus D \otimes u(k)$$

where

$$y(k)=\begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+N-1)\end{bmatrix}, u(k)=\begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N-1)\end{bmatrix},$$

and where C and D are given by

$$C=\begin{bmatrix} C_{k+1}A_k \\ C_{k+2}A_{k+1}A_k \\ \vdots \\ C_{k+N}A_{k+N-1}\dots A_k\end{bmatrix}=\begin{bmatrix} C\otimes A \\ C\otimes A^{\otimes^2} \\ \vdots \\ C\otimes A^{\otimes^N}\end{bmatrix}$$

$$D=\begin{bmatrix} C\otimes B & \varepsilon & \cdots & \varepsilon \\ C\otimes A\otimes B & C\otimes B & \cdots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ C\otimes A^{\otimes^{N-1}}\otimes B & C\otimes A^{\otimes^{N-2}}\otimes B & \cdots & C\otimes B\end{bmatrix}.$$

The next step is to use the output prediction equation that can be derived from the optimal input value. Consider the corresponding reference signal is as follows

$$R(k+1)=\begin{bmatrix} r(k+1) \\ r(k+2) \\ \vdots \\ r(k+N)\end{bmatrix}.$$

The solution of this equation obtained by considering the desired control input $U(k+1)$ for the given reference signal is

$$R(k+1)=Cx(k)\oplus Du(k+1).$$

From the residual theory in the max-plus algebra, the solution of this equation by solving the transformed linear equation is given as

$$Du(k+1)=R(k+1)\oplus Cx(k)$$

that implicitly shows that the input shaped by $R(k+1)=Cx(k)\oplus Du(k+1)$ can produce a solution. The settlement of equation $Du(k+1)=R(k+1)\oplus Cx(k)$ is showed by utilizing the most excellent sub-solution method as follows:

$$u(k+1)=\mathrm{D}^T\odot\{R(k+1)\oplus C\mathrm{x(k)}\}$$

The system's input used to predict within a specific time frame is determined by the receding horizon method. The first input of $u(k+1)$ from $u(k+1)=D^T\odot\{R(k+1)\oplus \mathrm{Cx(k)}\}$ is used to the controlled system so that

$$u(k+1)=[e_p,\epsilon_{pp},\epsilon_{pp},\dots,\epsilon_{pp}]u(k).$$

The input after time step $(k+1)^{th}$ is determined by equation $u(k+1)=[e_p,\epsilon_{pp},\epsilon_{pp},\dots,\epsilon_{pp}]u(k+1)$. From this result, feedback control on the change in the internal conditions occurred in the system can be realized.

III. NUMERICAL EXPERIMENT RESULTS

In this paper, we take a case study about the manufacturing system to consider the production system described in Fig 1. This was developed from [24] with a simple manufacturing system consisting of three processing units. The scheme in the production system proposed in this example is explained as follows.

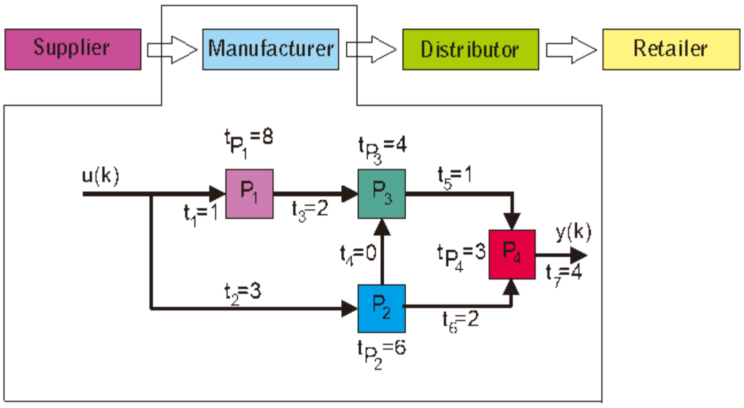

Fig 1. Scheme for Manufacturing System.

We consider the production system of Fig. 1 that consists of four processing units: P_1, P_2, P_3 and P_4. At P_1 and P_2 raw materials are processed and sent to P_3 where assembly takes place. It is continued to process P_4 as the last process in the production system. The processing time for P_1, P_2, P_3 and P_4 are $t_{P_1} = 8, t_{P_2} = 6, t_{P_3} = 4$, and $t_{P_4} = 3$ time unit respectively. It takes $t_1 = 1$ time unit for the raw material to get from the input source to P_1, while $t_2 = 3$ time units for the raw material to get from the input source to P_2. Then for P_1 toward P_3 needs $t_3 = 2$ time units, P_2 toward P_3 do not need time, and $t_7 = 4$ time units for a finished product to be continued in next step. The processing unit can begin only if it finished the processing of the foregoing product, and it is assumed that for the arrangement time or transportation time in the processing unit is ignored. After all parts are available, then each processing unit starts to work immediately. The system is explained by the following space models [24]:

$$x(k+1) = \begin{bmatrix} 8 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon & \varepsilon \\ 18 & 12 & 4 & \varepsilon \\ 23 & 17 & 9 & 3 \end{bmatrix} \otimes x(k) \oplus \begin{bmatrix} 1 \\ 3 \\ 11 \\ 16 \end{bmatrix} \otimes u(k+1),$$

$$y(k) = [\varepsilon \quad \varepsilon \quad \varepsilon \quad 7] \otimes x(k)$$

with $u(k)$ is the time at which a batch of raw material is fed to the system for the $(k+1)^{th}$ time, $x_i(k)$ is the time at which P_i starts working for the k^{th} time, and $y(k)$ is the time at which the k^{th} the finished product can leave the system.

The evolution of the manufacturing system can be described by the state-space model above and the solution can be found by solving the MPL system via receding horizon method. The results of the closed-loop simulations are displayed in Fig. 2. Noted that MPC input could reach the steady-state behavior in a finite number of steps and is known that the condition is nondecreasing.

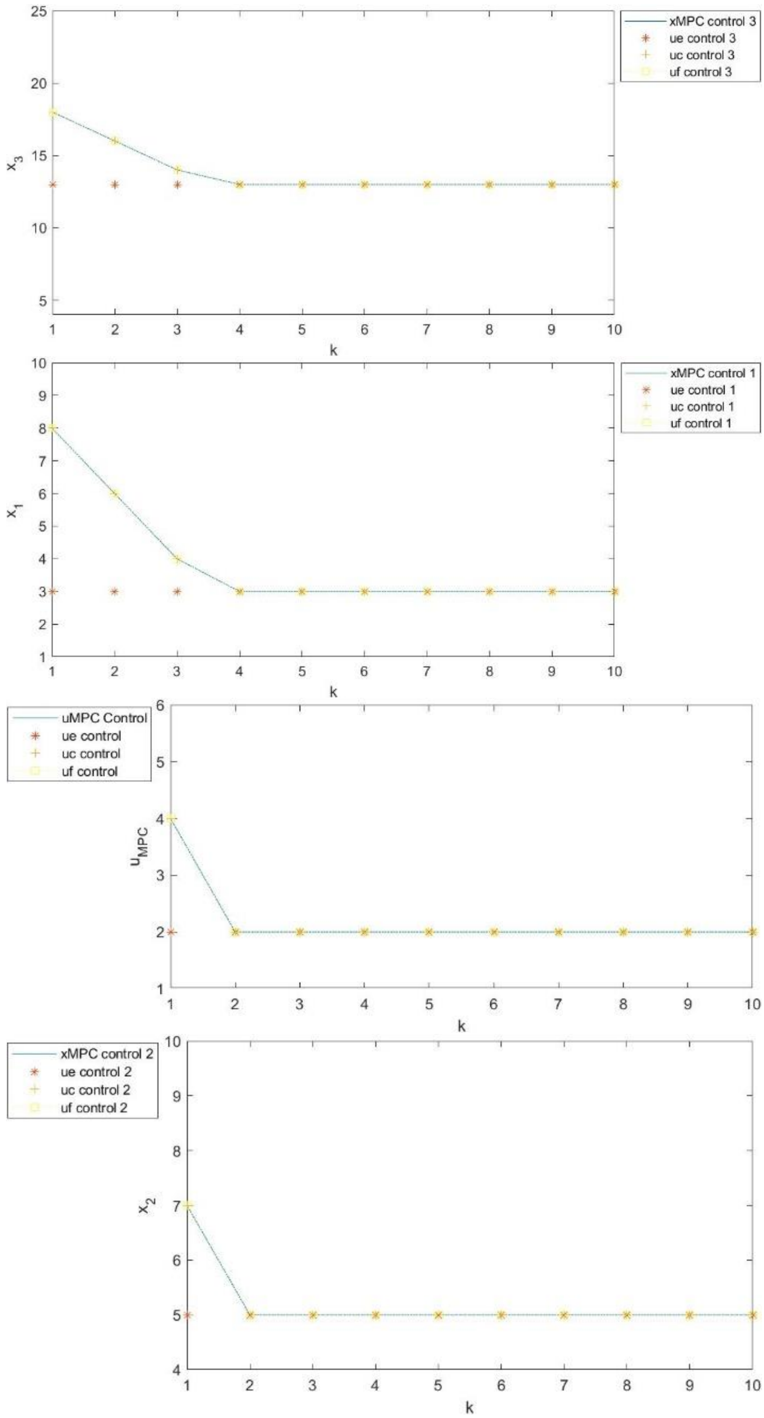

Figure 2. Numerical experiment results

Based on Fig. 2, two types of control are defined which are the feedback control and the Ultimately Constant Slope (UCS) control. Both of which are stabilizing [20]. The MPC control is employed between the feedback control and the UCS control for $k = 1,2,3,\ldots,10$. It can be seen that at $k = 4$, the state system, after controlled by MPC, is equal to the system equilibrium state. It can be also seen that the MPL system is Lyapunov stable in a finite step.

IV. CONCLUSIONS

In this paper, Max-Plus Linear system model combined with model predictive control was successfully implemented to a production system. The MPL system model was developed from a production system that has job shop production process scheme. The predicted optimal time of the production system was obtained by inputting the MPL system parameter matrix along with the term and constraints that meet the MPC into the Matlab toolbox program. Based on the simulation results, the MPC can stabilize the MPL system for the production system model of a manufacturing company.

V. ACKNOWLEDGMENT

The authors would like to thank Diponegoro University for funding this research through RPI Research Grant for the year 2019-2020.

VI. REFERENCES

[1] K. M. Passino, A. N. Michel and P. J. Antsaklis, "Stability Analysis of Discrete Event Systems," in *Proceedings of The Allerton Conference on Communication, Control, and Computing*, Urbana, 1990.

[2] F. Baccelli, G. Cohen, G. . J. Olsder and J. P. Quadrat, Synchronization and Linearity, New York: John Wiley & Sons, 1992.

[3] J. Komenda, S. Lahaye and J. L. Boimond, "Max-Plus Algebra and Discrete Event Systems," in *IFAC PapersOnline*, 2017.

[4] J. Komenda, S. Lahaye , J. -L. Boimond and T. v. d. Boom, "Max-Plus Algebra in The History of Discrete Event Systems," *Annual Reviews in Control*, vol. 45, pp. 240-249, 2018.

[5] B. Cottenceau, L. Hardouin and J. Trunk, "Weight-Balanced Timed Event Graphs to Model Periodic Phenomena in Manufacturing Systems," *IEEE Transactions on Automation Science and Engineering* , vol. 14, pp. 1731-1742, 2017.

[6] B. Kersbergen , J. Rudan , T. v. d. Boom and B. D. Schutter, "Towards railway traffic management using switching Max-Plus Linear Systems," *Discrete Event Dyn Syst*, vol. 26, pp. 183-223, 2016.

[7] T. v. d. Boom and B. D. Schutter, "Modelling and Control of Discrete Event System Using Switching Max-Plus Linear Systems," *Control Engineering Practice*, vol. 14, pp. 1199-1211, 2006.

[8] M. E. Leusin , E. M. Frazzom, M. U. Maldonanda, M. Kück and M. Freitag, "Solving the Job-Shop Scheduling Problem in the," *Technologies*, vol. 6, p. 107, 2018.

[9] H. Zhang, Y. Tao and Z. Zhang, "Strong Solvability of Interval Max-Plus Systems and Applications to Optimal Control," *Systems & Control Letters*, vol. 96, pp. 88-94, 2016.

[10] J. Maciejowski, Predictive Control with Constraints, Prentice, Hall, Harlow: England, 2002.

[11] W. Chen, H. Liu and E. Qi, "Discrete Event-Driven Model Predictive Control for Real-Time Work-in-Process Optimization in Serial Production Systems," *Journal Of Manufacturing Systems*, vol. 55, pp. 132-142, 2020.

[12] T. van den Boom and B. D. Schutter, "MPC for Max-Plus -Linear Systems with Guaranteed Stability," *IFAC*, 2005.

[13] D. Adzkiya, B. D. Schutter and A. Abate, "Computational Techniques for Reachability Analysis of Max-Plus Linear Systems," *Automatica*, vol. 53, pp. 293-302, 2015.

[14] D. Adzkiya, B. D. Schutter and A. Abate, "Backward Reachability of Autonomous Max-Plus Linear Systems," *IEEE Workshop on Discrete Event Systems*, pp. 117-122, 2014.

[15] Y. Tao and C. Wang, "Global Optimization for Max-Plus Linear Systems and Applications in Distributed Systems," *Automatica*, vol. 119, p. 109104, 2020.

[16] N. Shinzawa, "Ultra Discrete Permanent and The Consistency of Max-Plus Linear Equations," *Linear Algebra and Its Applications*, vol. 506, pp. 445-477, 2016.

[17] J. B. Rawlings, D. Q. Mayne and M. M. Diehl, Model Predictive Control: Theory, Computation, and Design, Santa Barbara, California: Nob Hill Publishing, LLC, 2014.

[18] R. Cuninghame-Green, Minimax Algebra, Berlin, Germany: Springer-Verlag, 1979.

[19] . S. B. De and T. van den Boom, "Model predictive control for max-plus-linear discrete event systems," *Automatica*, vol. vo.37, no. no.7 pp, p. 1049–1056, 2001.

[20] N. Ion, T. J. v. d. Boom, B. D. Schutter and H. Hellendoorn, "Stabilization of max-plus-linear systems using model predictive control: The unconstrained case," *Automatica*, vol. 44, pp. 971-981, 2008.

[21] E. Camacho and C. Bordons, Model Predictive Control in the Process Industry, Berlin, Germany: Springer-Verlag, 1995.

[22] K. M. Passino, A. N. Michael and P. J. Antsaklis, "Lyapunov Stability of Class of Discrete Event Systems," *IEEE Transactions on Automatic Control*, vol. vol.2, no. no.2, pp. 269-279, 1994.

[23] T. van den Boom and B. D. Schutter, "Properties of MPC for max-plus-linear systems," *European Journal of Control*, vol. vol.8, no. n0.5,pp, pp. 453-462, 2002.

[24] B. De Schutter and T. van den Boom, "Model predictive control for max-plus-linear discrete event systems: Extended Report & addendum," Technical Report bds:99-10a, Control Lab, ITS, TU Delft, Delft, The Netherlands, 2000.