JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 3 NO. 2 (NOV 2020)
Available online at www.jfma.math.fsm.undip.ac.id

OPTIMAL CONTROL MODELLING OF COVID-19 OUTBREAK

Dhimas Mahardika[1*], R. Heru Tjahjana[2], Sunarsih[3]

[1,2,3]*Departemnet of Mathematics, Diponegoro University, Semarang, Indonesia*
Email: [1]dhimasmahardika@student.undip.ac.id, [2]heru.math.undip@gmail.com, [3]narsih_pdil@yahoo.com
*Corresponding author

Abstract. Corona virus infection is lethal and life threatening to human life, for prevention it is necessary to carry out quarantined for a portion of susceptible, exposed, and infected population, this kind of quarantine is intended to reduce the spread of the corona virus. The optimal control that will be carried out in this research is conducting quarantine for a portion of susceptible, exposed, and infected individuals. This control function will be applied to the dynamic modelling of Covid-19 spread using Pontryagin Minimum Principle. We will describe the formulation of dynamic system of Covid-19 spread with optimal control, then we use Pontryagin Minimum Principle to find optimal solution of the control. The optimal control will aim to minimize the number of infected population and control measures. Numerical experiments will be performed to illustrate and compare the graph of Covid-19 spread model with and without control.
Keywords: Covid-19, Dynamical System, Pontryagin Minimum Principle, Fixed time and free end point optimal control.

I. INTRODUCTION

The use of dynamic system applications in biology has shown significant changes in mathematics and biosciences in recent years [1]. This dynamic system will use the optimal control laws to look for optimization of functional objectives for a certain period of time, to find the optimum control of the dynamic system, Pontryagin Minimum Principle will be used [2], [10].

In 2019, in the city of Wuhan there was a corona virus outbreak that infected and killed thousands of people. and then Covid-19 is the name given by the world researcher and SARS Cov 2 as the name of the virus [3]. This outbreak began in the Hunan area in the seafood market. where many small mammals are traded especially bats, based on information from WHO [4]. Initially palm civets and raccoon dogs were thought by researchers to be the source of the infection, and the researchers suggested that the palm civet might be the secondary hosts [5]. In Hong Kong there is a sample from a person that shows of antibodies against SARS-coronavirus, it means corona virus has already circulating among human in 2003 [6]. The researchers later suggested that The Rhinolophus bats were the source of the virus [7], [17].

The optimal control that will be carried out in this research is to conduct quarantine for a portion of the susceptible, exposed and infected population, the duration of the quarantine is 132 days (January 27th 2020 until June 6th 2020) [12] and the type of the optimal control is fixed time with free end point which means that the time duration of the state model is determined (132 days) and the end points of the state model are free [16] . The purpose of this quarantine is to reduce the spread of Corona virus. Later on we will compare the graph of dynamic model of the spread of Covid-19 without optimal control and with optimal control. MATLAB software will be used to perform numerical calculation (iteration) and generate graphs [8] [14].

II. COVID-19 SPREAD DYNAMIC MODEL

In this section we will divide the total population into seven compartment or class, which is population of susceptible(S), Exposed(E), Infectious(I), Recovered(R), Quarantined susceptible(Q_S), Quarantined exposed(Q_E), Quarantined infected(Q_I) and then we make model of the spread of Covid-19 with optimal control, making formulation of optimal control problem, and the optimality condition and after that we conducting numerical calculation or iteration using MATLAB software with parameter and initial data provided in table 1. The result of the iteration will be in the form of graph of dynamic model of Covid-19 with and without optimal control where later these two graphs will be compared.

Then the mathematical model for covid-19 without control:

$$\begin{aligned}
\dot{S} &= -\beta_1(I+\beta_2 E)S \\
\dot{E} &= \beta_1(I+\beta_2 E)S - \nu_1 E \\
\dot{I} &= \nu_1 E - \gamma_1 I - \delta_1 I \\
\dot{R} &= \gamma_1 I
\end{aligned} \tag{1}$$

And mathematical model for covid-19 with control:

$$\begin{aligned}
\dot{S} &= -\beta_1(I+\beta_2 E)S + \alpha Q_S - u_1 S \\
\dot{E} &= \beta_1(I+\beta_2 E)S - \nu_1 E - u_2 E \\
\dot{I} &= \nu_1 E - u_3 I - \gamma_1 I - \delta_1 I \\
\dot{R} &= \gamma_1 I + \gamma_2 Q_I \\
\dot{Q}_S &= u_1 S - \alpha Q_S - \sigma Qs \\
\dot{Q}_E &= u_2 E + \sigma Qs - \nu_2 Q_E \\
\dot{Q}_I &= u_3 I + \nu_2 Q_E - \gamma_2 Q_I - \delta_2 Q_I
\end{aligned} \tag{2}$$

with $S(0)\geq 0$, $E(0)\geq 0$, $I(0)\geq 0$, $R(0)\geq 0$, $Q_S(0)\geq 0$, $Q_E(0)\geq 0$, $Q_I(0)\geq 0$ as initial condition $\dot{S}$, $\dot{E}$, $\dot{I}$, $\dot{R}$, equal the rate of population change of the susceptible, exposed, infectious, and recovered respectively, $\dot{Q}_S$, $\dot{Q}_E$, $\dot{Q}_I$ equal the rate of population change of the quarantined susceptible, quarantine exposed, and quarantine infected respectively, $\beta_1(I+\beta_2 E)S$ is rate of transmission from susceptible to exposed, $\nu_1 E$ is rate of transmission from exposed to infected, $\nu_2 Q_E$ is rate of transmission from quarantine exposed to quarantine infected, $\delta_1 I$ and $\delta_2 Q_I$ is death rate of infected population and quarantine infected due to Covid-19 respectively, $\gamma_1 I$ is recovery rate of infected population, $\gamma_2 Q_I$ is recovery rate of quarantine infected population, αQ_S is transfer rate from quarantine susceptible to susceptible with, σQ_S is transfer rate from quarantine susceptible to quarantine exposed with β_1, β_2, ν_1, ν_2, δ_1, δ_2 γ_1, γ_2, α, σ are the parameters.

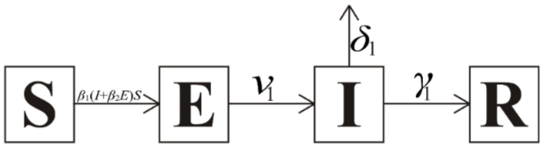

Figure 1. Flow diagram of dynamic system (1).

In Fig. 1 we can see the flow diagram of dynamic system (1) that describe the movement of people between compartment (arrow direction).

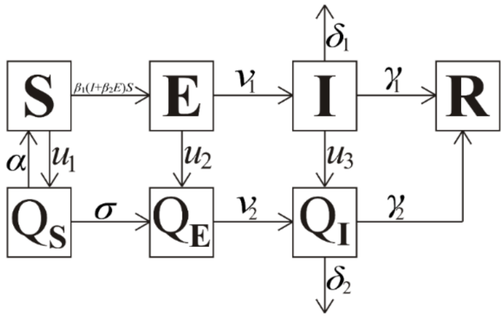

Figure 2. Flow diagram of dynamic system (2).

In Fig. 2 we can see the flow diagram of dynamic system (2) that describe the movement of people between compartment (arrow direction).

Dynamic system(1) is taken from [11], the model in [11] only describe prediction for susceptible, exposed, infected and recovered population for certain time *T*, then we modified in system (2) by providing quarantine controls for susceptible population (u_1S), exposed population (u_2E) , and Infected population (u_3I), then Q_S , Q_E , Q_I in the system (2) means that quarantine for susceptible, exposed and infected population are separated and given new compartment as we can see in fig 2. Dynamic System (2) taken from [12], and the weakness of model in [12] is that the control in there play role as parameter, and we develop the system model (2) by adding the optimum control Pontryagin rather than parameter.

III. OPTIMAL CONTROL ANALYSIS

In this section, we propose dynamic model of Covid-19 spread with control:

$$\begin{aligned}
\dot{S} &= -\beta_1(I+\beta_2E)S + \alpha Q_S - u_1S \\
\dot{E} &= \beta_1(I+\beta_2E)S - \nu_1E - u_2E \\
\dot{I} &= \nu_1E - u_3I - \gamma_1I - \delta_1I \\
\dot{R} &= \gamma_1I + \gamma_2Q_I \\
\dot{Q}_S &= u_1S - \alpha Q_S - \sigma Qs \\
\dot{Q}_E &= u_2E + \sigma Qs - \nu_2Q_E \\
\dot{Q}_I &= u_3I + \nu_2Q_E - \gamma_2Q_I - \delta_2Q_I
\end{aligned} \tag{3}$$

with $S(0)\geq 0$, $E(0)\geq 0$, $I(0)\geq 0$, $R(0)\geq 0$, $Q_S(0)\geq 0$, $Q_E(0)\geq 0$, $Q_I(0)\geq 0$ as initial condition
Then we have the following control variables:

1. $u_1(t)$ control variable of quarantined effectiveness of susceptible people.
2. $u_2(t)$ control variable of quarantined effectiveness of exposed people.
3. $u_3(t)$ control variable of quarantined effectiveness of infectious people.

The purpose of the optimal control is to reduce the number of infected population, and control measures. Then the objective functional correspond with control variables and dynamic model (2):

$$J(u_1,u_2,u_3) = \text{minimize} \int_0^T K_1 I + \frac{K_2}{2}u_1^2(t) + \frac{K_3}{2}u_2^2(t) + \frac{K_4}{2}u_3^2(t)dt \tag{4}$$

subject to (2), With $S(0)\geq 0$, $E(0)\geq 0$, $I(0)\geq 0$, $R(0)\geq 0$, $Q_S(0)\geq 0$, $Q_E(0)\geq 0$, $Q_I(0)\geq 0$ as initial condition. The weight of the infected and the weight of u_1,u_2,u_3 represent as K_1, K_2, K_3, K_4 respectively. The goal of objective functional is to minimize the number of infected individuals and the cost of the control measures. Then define control functions (u_1^*,u_2^*,u_3^*) such that:

$$J\left(u_1^*,u_2^*,u_3^*\right) = \text{minimize}\left\{(u_1,u_2,u_3); u_1,u_2,u_3 \in U\right\} \text{ subject to (2)} \tag{5}$$

and control set by:

$$U = \left\{(u_1,u_2,u_3)\middle| u_i \text{ measurable of Lebesque on} [0,T], 0\leq u_i \leq 1, i=1,2,3\right\}. \tag{6}$$

To determine the controls, let's prove the existence. First we prove its existence. Let $C(\varphi) = G = A\varphi + B(\varphi)$,

$$G = \begin{bmatrix} \dot{S} \\ \dot{E} \\ \dot{I} \\ \dot{R} \\ \dot{Q}_S \\ \dot{Q}_E \\ \dot{Q}_I \end{bmatrix}, \quad \varphi = \begin{bmatrix} S \\ E \\ I \\ R \\ Q_S \\ Q_E \\ Q_I \end{bmatrix}, \quad B(\varphi) = \begin{bmatrix} -\beta SI \\ \beta SI \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

$$A = \begin{bmatrix} -u_1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & -(v_1+u_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & v_1 & -(u_3+\delta_1+\gamma_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_1 & 0 & 0 & 0 & \gamma_2 \\ u_1 & 0 & 0 & 0 & -(\alpha+\sigma) & 0 & 0 \\ 0 & u_2 & 0 & 0 & \sigma & -v_2 & 0 \\ 0 & 0 & u_3 & 0 & 0 & v_2 & \gamma_2+\delta_2 \end{bmatrix} \tag{8}$$

then

$$|A(\varphi_1)-A(\varphi_2)|=\begin{bmatrix} u_1S_2+\alpha QS_1-u_1S_1-\alpha QS_2 \\ \nu_1E_2+u_2E_2-\nu_1E_1-u_2E_1 \\ \nu_1E_1+\gamma_1I_2+\delta_1I_2+u_3I_2-\nu_1E_2-\gamma_1I_1-\delta_1I_1-u_3I_1 \\ \gamma_1I_1+\gamma_2I_1-\gamma_1I_2-\gamma_2I_2 \\ u_1S_1+\alpha QS_2+\sigma QS_2-u_1S_2-\alpha QS_1-\sigma QS_1 \\ u_2E_1+\nu_2QE_2+\sigma QS_1-u_2E_2-\nu_2QE_1-\sigma QS_2 \\ u_3I_1+\nu_2QE_1+\gamma_2QI_1+\delta_2QI_1-u_3I_2-\nu_2QE_2-\gamma_2QI_2-\delta_2QI_1 \end{bmatrix} \quad (9)$$

$$\begin{aligned} &\le V_1|S_1-S_2|+V_2|E_1-E_2|+V_3|I_1-I_2|+V_4|QS_1-QS_2|+V_5|QE_1-QE_2|+V_6|QI_1-QI_2| \\ &\le V(|S_1-S_2|+|E_1-E_2|+|I_1-I_2|+|QS_1-QS_2|+|QE_1-QE_2|+|QI_1-QI_2|) \quad (10) \\ &\le V|\varphi_1-\varphi_2| \end{aligned}$$

with $V_1=2u_1$, $V_2=2(\nu_1+u_2)$, $V_3=2(\gamma_1+u_3)+\delta_1$, $V_4=2(\alpha+\sigma)+\delta_1$, $V_5=2\nu_2$, $V_6=2(\gamma_2+\delta_2)$,

$$V=V_1+V_2+V_3+V_4+V_5+V_6. \quad (11)$$

Define

$$|B(\varphi_1)-B(\varphi_2)|=\begin{bmatrix} \beta_1(I_2+\beta_2E_2)S_2-\beta_1(I_1+\beta_2E_1)S_1 \\ \beta_1(I_1+\beta_2E_1)S_1-\beta_1(I_2+\beta_2E_2)S_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (12)$$

$$|B(\varphi_1)-B(\varphi_2)|=|\beta_1(I_2+\beta_2E_2)S_2-\beta_1(I_1+\beta_2E_1)S_1|+|\beta_1(I_1+\beta_2E_1)S_1-\beta_1(I_2+\beta_2E_2)S_2| \quad (13)$$

$$\begin{aligned} &\le 2\beta_1(|S_1||I_1-I_2|+|S_1-S_2||I_2+\beta_2E_1|+|\beta_2S_2||E_1-E_2|) \\ &\le M|\varphi_1-\varphi_2| \end{aligned} \quad (14)$$

with $M=2\beta_1$.
Define:

$$C(\varphi)=A(\varphi)+B(\varphi) \quad (15)$$

then

$$|C(\varphi_1)-C(\varphi_2)|\le V|\varphi_1-\varphi_2|+M|\varphi_1-\varphi_2|\le W|\varphi_1-\varphi_2| \quad (16)$$

so that function $C(\varphi)$ uniformly Lipschitz continuous, it implies that a unique solution of system (2) exists.

Theorem 1. *There exists optimal control* $u^*=(u_1^*,u_2^*,u_3^*)\in U$ *minimizing* $J(u_1^*,u_2^*,u_3^*)$.

Proof: To prove the existence, we follow ref. [9] . Since
i. The variables of state and variables of control are positive.
ii. Control set (6) is closed and convex.

JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 3 NO. 2 (NOV 2020)
Available online at www.jfma.math.fsm.undip.ac.id

iii. The right hand side of the state system (2) is continuous, is bounded above by a linear combination of the control and state, and can be written as a linear function of *u* with coefficients depending by the time and the state.
iv. The integrand of the objective functional (4) is convex on *u*
v. There exist constant $C_1, C_2 > 0$ and $\phi > 1$ such that the integrand of the objective functional satisfies $L(x,u) = -C_2 + C_1 |u|^{\phi}$. □

IV. CONDITION OF OPTIMAL

To find the optimal solution, we began by defining the Lagrangian and the Hamiltonian associated with our optimal control problem. Let $x = S, E, I, R, Q_S, Q_E, Q_I$ be the vector of state variables and $u = u_1, u_2, u_3$ the control vector.

The Lagrangian is given by:

$$L(x,u) = K_1 I + \frac{K_2}{2} u_1^2(t) + \frac{K_3}{2} u_2^2(t) + \frac{K_4}{2} u_3^2(t) \tag{17}$$

and Hamiltonian is given by:

$$\begin{aligned} &H(x,u,\lambda) = \\ &K_1 I + \frac{K_2}{2} u_1^2(t) + \frac{K_3}{2} u_2^2(t) + \frac{K_4}{2} u_3^2(t) + \\ &\lambda_1(\dot{S}) + \lambda_2(\dot{E}) + \lambda_3(\dot{I}) + \lambda_4(\dot{R}) + \lambda_5(\dot{Q}_S) + \lambda_6(\dot{Q}_E) + \lambda_7(\dot{Q}_I) \end{aligned} \tag{18}$$

were λ_1, λ_2, λ_3, λ_4, λ_5, λ_6, λ_7 are adjoin variables that satisfies :

$$\begin{aligned} \frac{d\lambda_1}{\mathrm{dt}} &= -\frac{\partial H}{\partial S} = \lambda_1\left(\beta_1 I + u_1 + \beta_1\beta_2 E\right) - \lambda_2\left(\beta_1 I + \beta_1\beta_2 E\right) - \lambda_5 u_1 \\ \frac{d\lambda_2}{\mathrm{dt}} &= -\frac{\partial H}{\partial E} = \lambda_2\left(v_1 + u_2 - \beta_1\beta_2 S\right) + \lambda_1\beta_1\beta_2 S - \lambda_3 v_1 - \lambda_6 u_2 \\ \frac{d\lambda_3}{\mathrm{dt}} &= -\frac{\partial H}{\partial I} = -K_1 + \lambda_3\left(\gamma_1 + \delta + u_3\right) + \beta_1 S\left(\lambda_1 - \lambda_2\right) - \lambda_4\gamma_1 - \lambda_7 u_3 \\ \frac{d\lambda_4}{\mathrm{dt}} &= -\frac{\partial H}{\partial R} = 0 \\ \frac{d\lambda_5}{\mathrm{dt}} &= -\frac{\partial H}{\partial Q_S} = \left(\alpha + \sigma\right)\lambda_5 - \alpha\lambda_1 - \sigma\lambda_6 \\ \frac{d\lambda_6}{\mathrm{dt}} &= -\frac{\partial H}{\partial Q_E} = v_2\left(\lambda_6 - \lambda_7\right) \\ \frac{d\lambda_7}{\mathrm{dt}} &= -\frac{\partial H}{\partial Q_I} = \lambda_7\left(\gamma_2 + \delta_2\right) - \lambda_4\gamma_2. \end{aligned} \tag{19}$$

With terminal (transversality) condition [15] we have

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = \lambda_5(T) = \lambda_6(T) = \lambda_7(T) = 0.$$

These adjoin variables will minimize the state variable with respect to the state function. Then the optimal variables of control

$$\frac{\partial H}{\partial u_1} = -S\lambda_1 + S\lambda_5 + K_2 u_1 = 0 \Rightarrow u_1^* = \frac{S(\lambda_1 - \lambda_5)}{K_2} \tag{20}$$

with $u_1 = u_1^*$

$$\frac{\partial H}{\partial u_2} = -E\lambda_2 + E\lambda_6 + K_3 u_2 = 0 \Rightarrow u_2^* = \frac{E(\lambda_2 - \lambda_6)}{K_3} \tag{21}$$

with $u_2 = u_2^*$

$$\frac{\partial H}{\partial u_3} = -I\lambda_3 + I\lambda_7 + K_4 u_3 = 0 \Rightarrow u_3^* = \frac{I(\lambda_3 - \lambda_7)}{K_4}. \tag{22}$$

Since $0 \le u_1 \le 1$, $0 \le u_2 \le 1$ and $0 \le u_3 \le 1$, then we can rewrite for u_1^*:

$$u_1^* = \max\left\{0, \min\left(\frac{S(\lambda_1 - \lambda_5)}{K_2}, 1\right)\right\} \tag{23}$$

u_2^*:

$$u_2^* = \max\left\{0, \min\left(\frac{E(\lambda_2 - \lambda_6)}{K_3}, 1\right)\right\} \tag{24}$$

and u_3^*:

$$u_3^* = \max\left\{0, \min\left(\frac{I(\lambda_3 - \lambda_7)}{K_4}, 1\right)\right\}. \tag{25}$$

V. NUMERICAL ANALYSIS AND DISCUSSION

The numerical result of the optimal control problem in equation system (1), (2) will be in the form of graph and calculated using MATLAB programming [8]. We will compare the graph of dynamic system (1) and dynamic system (2) with fixed time (T = 132 days(January 27th 2020 until June 6th 2020)) and free end point ($x(0)$ determined, but $x(T)$ are free), where x is the variable state $(x = S, E, I, R, Q_S, Q_E, Q_I)$. The parameters' values used in this numerical analysis are shown in Table 1 whereas the results are shown in Fig. 3 to Fig. 9.

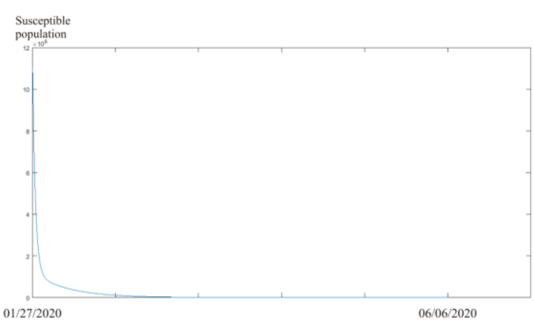

Figure 3a. Dynamics of susceptible population with control

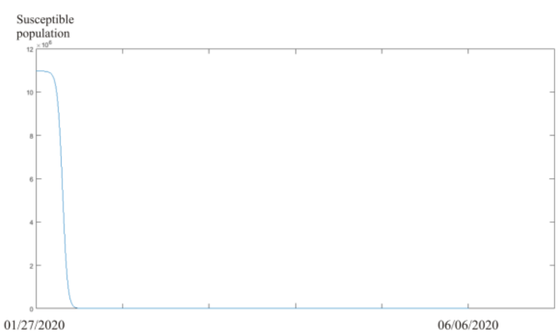

Figure 3b. Dynamics of susceptible population without control

Table 1. Parameter data of the dynamic system of equation (1) and (2)

Parameter	Description	Value	Reference
$S(0)$	Initial point/value of susceptible	10967505	[12]
$E(0)$	Initial point/value of exposed	220	[12]
$I(0)$	Initial point/value of infected	130	[12]
$R(0)$	Initial point/value of recovered	10	[12]
$Q_S(0)$	Initial point/value of quarantine for susceptible	45	[12]
$Q_E(0)$	Initial point/value of quarantine for exposed	110810	[12]
$Q_I(0)$	Initial point/value of quarantine for infected	1460	[12]
β_1	Infectivity contact rate	2.06×10^{-7}	[12]
β_2	Infectivity contact rate	0.63	[12]
v_1	Transfer rate from exposed to infectious	0.13	[12]
v_2	Transfer rate from quarantine exposed to quarantine infectious	0.13	[12]
γ_1	Transfer rate from infectious to recovery	0.154	[11]
γ_2	Transfer rate from quarantine infectious to recovery	0.16	[12]
δ_1	Death rate due to COVID-19	0.002789	[12]
δ_2	Death rate due to COVID-19	0.002789	[12]
α	Transfer rate from quarantine susceptible to susceptible	0.2	Fitted
σ	Transfer rate from quarantine susceptible to quarantine exposed	0.3	Fitted
K_1	Relative weight of the infected	1	Fitted
K_2	Relative weight of control u_1	1	Fitted
K_3	Relative weight of control u_2	1	Fitted
K_4	Relative weight of control u_3	1	Fitted

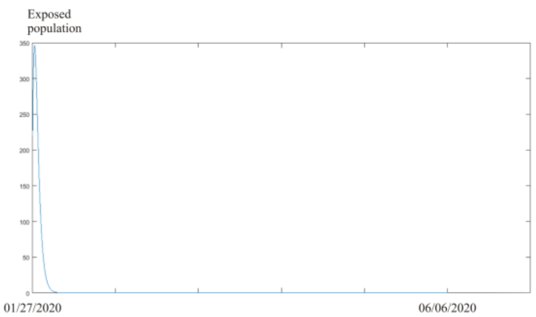

Figure 4a. Dynamics of exposed population with control

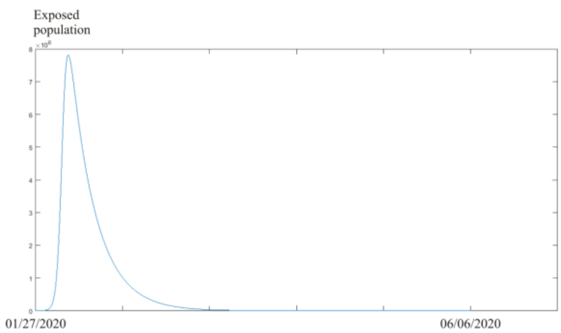

Figure 4b. Dynamics of exposed population without control

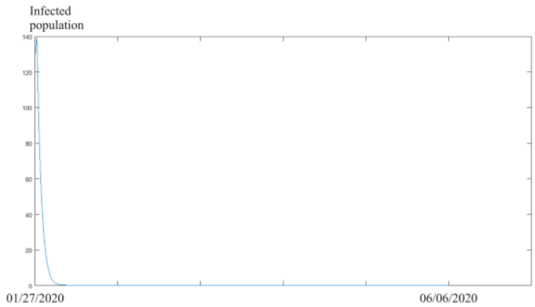

Figure 5a. Dynamics of infected population with control

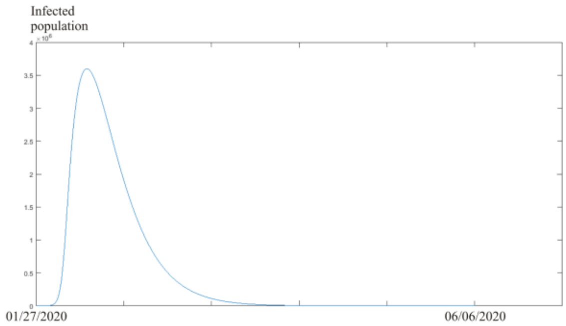

Figure 5b. Dynamics of infected population without control

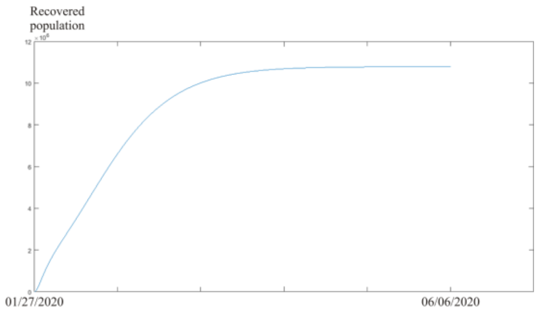

Figure 6a. Dynamics of recovered population with control

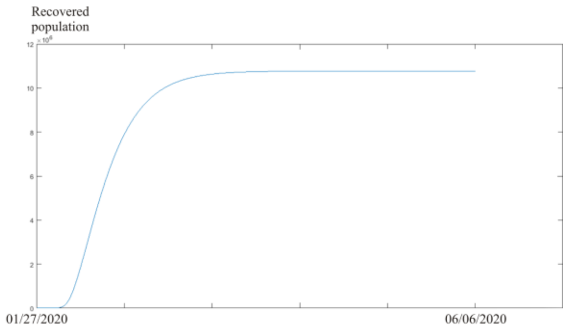

Figure 6b. Dynamics of recovered population without control

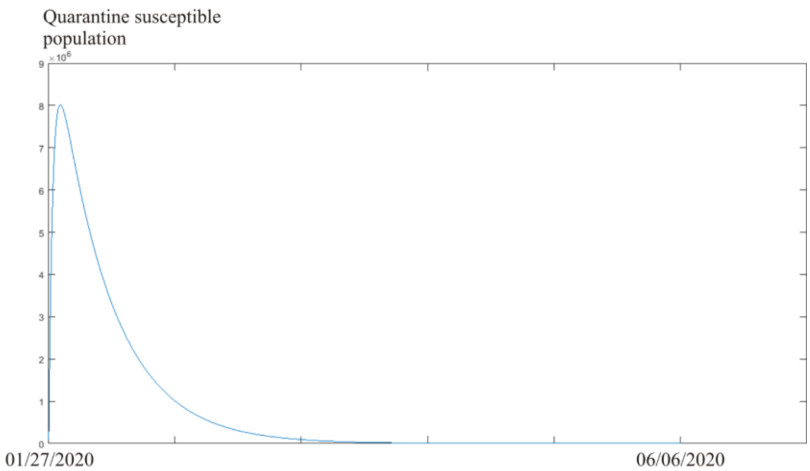

Figure 7. Dynamics of quarantine susceptible population

The terms $-u_1(t)S(t)$, $-u_2(t)E(t)$, $-u_3(t)I(t)$ in the dynamic system (2) means that there is a reduction in the rate of change of the susceptible, exposed, infectious population by $u_1(t)S(t)$, $u_2(t)E(t)$, $u_3(t)I(t)$ respectively, to be detailed description will be given the following example: By numerical calculation (iteration) using MATLAB software it is known that $u_1(7)S(7) = 645953.34$, $u_2(7)E(7) = 1745.8$, $u_3(7)I(7) = 0.717$ with t =7 days, means on the seventh day we must quarantined 645954 susceptible individuals (rounding up from 645953.34) at home, 1746 exposed individuals (rounding up from 1745.8) and one infectious individuals (rounding up from 0.717) simultaneously. This control measure is used as a benchmark for how many susceptible, exposed, infectious people should be quarantined on the seventh day, u_1, u_2 and u_3 here play role as effectiveness of the isolation [13] since $0 \le u_1 \le 1$, $0 \le u_2 \le 1$ and $0 \le u_3 \le 1$.

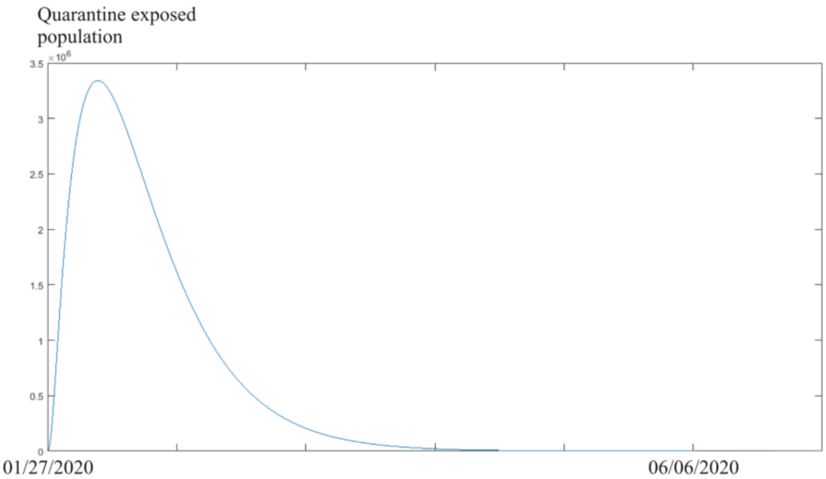

Figure 8. Dynamics of quarantine exposed population

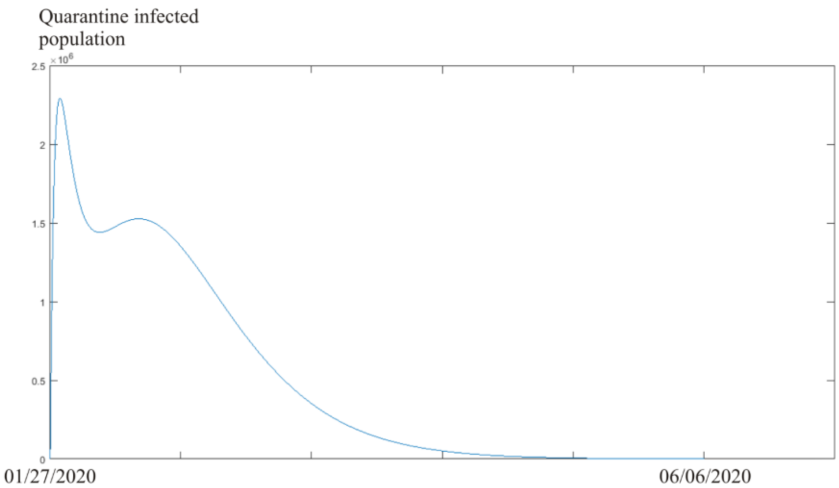

Figure 9. Dynamics of quarantine infected population

VI. CONCLUSION

The objectives of this paper is to find the value of optimal control (u_1, u_2, u_3) using Pontryagin Minimum Principle method which is applied to the dynamic modeling of Covid-19 spread, the value of optimal control here is how many susceptible individuals should be quarantined $(u_1(t)S(t))$, how many exposed individuals should be quarantined $(u_2(t)E(t))$ and how many infectious individuals should be quarantined $(u_3(t)I(t))$ simultaneously correspond with time t. This quarantine is intended to reduce the spread of the corona virus. In the end, use of control function in the dynamic model is to provide advice to the authorities to handle and control Covid-19 outbreak.

REFERENCES

[1] F. Brauer and C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*. New York, NY: Springer New York, 2012.

[2] L. S. Pontryagin, *Mathematical Theory of Optimal Processes*. Routledge, 2018.

[3] M. A. Shereen, S. Khan, A. Kazmi, N. Bashir, and R. Siddique, “COVID-19 infection: Origin, transmission, and characteristics of human coronaviruses,” *Journal of Advanced Research*, vol. 24, pp. 91–98, 2020. https://doi.org/10.1016/j.jare.2020.03.005.

[4] C. Wang, P. W. Horby, F. G. Hayden, and G. F. Gao, “A novel coronavirus outbreak of global health concern,” *The Lancet*, vol. 395, no. 10223, pp. 470–473, 2020. https://doi.org/10.1016/s0140-6736(20)30185-9.

[5] B. Kan, M. Wang, H. Jing, H. Xu, X. Jiang, M. Yan, W. Liang, H. Zheng, K. Wan, Q. Liu, B. Cui, Y. Xu, E. Zhang, H. Wang, J. Ye, G. Li, M. Li, Z. Cui, X. Qi, K. Chen, L. Du, K. Gao, Y.-T. Zhao, X.-Z. Zou, Y.-J. Feng, Y.-F. Gao, R. Hai, D. Yu, Y. Guan, and J. Xu, “Molecular Evolution Analysis and Geographic Investigation of Severe Acute Respiratory Syndrome Coronavirus-Like Virus in Palm Civets at an Animal Market and on Farms,” *Journal of Virology*, vol. 79, no. 18, pp. 11892–11900, 2005. https://doi.org/10.1016/s0140-6736(20)30185-9

[6] B. J. Zheng, Y. Guan, K. H. Wong, J. Zhou, K. L. Wong, B. W. Y. Young, L. W. Lu, and S. S. Lee, “SARS-related Virus Predating SARS Outbreak, Hong Kong,” *Emerging Infectious Diseases*, vol. 10, no. 2, pp. 176–178, 2004. https://doi.org/10.3201/eid1002.030533

[7] Z. Shi and Z. Hu, “A review of studies on animal reservoirs of the SARS coronavirus,” *Virus Research*, vol. 133, no. 1, pp. 74–87, 2008. https://doi.org/10.1016/j.virusres.2007.03.012

[8] Anita S, Arnautu V, & Capasso V, *An introduction to optimal control problems in life sciences and economics: From mathematical models to numerical simulation with* MATLAB®, Birkhäuser 2010.

[9] Kamien MI , Schwartz NL Dynamic Optimization, *Advanced Textbooks in Economics* 31. second ed. Amsterdam: North-Holland Publishing Co 1991.

[10] Naidu, D. S., *Pontryagin minimum principle in Optimal Control Systems*, Boca Raton: CRC Press, pp. 249-291, 2003. https://doi.org/10.1201/9781315214429-6 2018.

[11] Yang Z, Zeng Z, Wang K, Wong S-S, Liang W, Zanin M, et al., "Modified SEIR and AI prediction of the epidemics trend of COVID-19 in China under public health interventions," *Journal of Thoracic Disease*, vol. 12, no. 3, pp. 165-174 2020. https://doi.org/10.21037/jtd.2020.02.64.

[12] Cui, Q., Hu, Z., Li, Y., Han, J., Teng, Z., & Qian, J., " Dynamic variations of the COVID-19 disease at different quarantine strategies in Wuhan and Mainland China," *Journal of Infection and Public Health*, vol. 13, no. 6, pp. 849-855, 2020 https://doi.org/10.1016/j.jiph.2020.05.014.

[13] Driss K, Abderrahim L, Omar B, Mostafa R, El H L., "Spread of COVID-19 in Morocco discrete mathematical modeling: Optimal control strategies and cost-effectiveness analysis," *Journal of Mathematical and Computational Science*, vol. 5, pp. 2070-2093, 2020. https://doi.org/10.28919/jmcs/4817.

[14] Pérez López, C., *Numerical Calculus with MATLAB. Applications to differential equations*, Berkeley, CA: Apress, pp. 73-100, 2014. https://doi.org/10.1007/978-1-4842-0310-1_6.

[15] Cannarsa, P., & Frankowska, H., "Infinite horizon optimal control: Transversality

conditions and sensitivity relations," *American Control Conference (ACC) 2017*. https://doi.org/10.23919/acc.2017.7963349

[16] Frank L. Lewis Draguna L. Vrabie Vassilis L., *Syrmos Final-time-free and constrained input control in Optimal Control*, (eds F.L. Lewis, D.L. Vrabie and V.L. Syrmos), pp. 213-259, 2012. https://doi.org/10.1002/9781118122631.ch5

[17] Wang, M and Hu, Z ., "Bats as animal reservoirs for the SARS coronavirus: Hypothesis proved after 10 years of virus hunting," *Virologica Sinica*, vol. 28, no. 6, pp. 315-317, 2013. https://doi.org/10.1007/s12250-013-3402-x