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Coprime Graph of Integer Modulo n Group and its Subgroups

Rina Juliana[1], Masriani[2], I Gede Adhitya Wisnu Wardhana[3*], Ni Wayan Switrayni[4], Irwansyah[5]

[1,2,3,4,5] *Department of Mathematics, Universitas Mataram*
Email: [1]rina.juliana@unram.ac.id, [2]masriani@unram.ac.id, [3]adhitya.wardhana@unram.ac.id, [4]niwayan.switrayni@unram.ac.id, [5]irw@unram.ac.id
*Corresponding author

Abstract. Coprime Graph is a geometric representation of a group in the form of undirected graph. The coprime graph of a group G, denoted by Γ_G is a graph whose vertices are all elements of group G; and two distinct vertices a and b are adjacent if and only if $(|a|;|b|) = 1$. In this paper, we study coprime graph of integers modulo n group and its subgroups. One of the results is if n is a prime number, then coprime graph of integers modulo n group is a bipartite graph.
Keywords: bipartite graph, coprime graph, integer modulo, multipartite graph.

I. INTRODUCTION

Mathematicians define specific graphs on algebraic structures, and use graph properties as a geometric representations of an algebraic structure. In 2014, Ma *et al* [1] define a coprime graph of a group as follows: take G as the vertices of Γ_G and two distinct vertices x and y are adjacent if and only if $(|x|,|y|) = 1$. In this paper, we will study the coprime graph of cyclic group, $\mathbb{Z}_n$. In 2016 Dorbidi [2] classify all the groups which Γ_G is a complete r-partite graph or a planar graph, he also studied the automorphism group of Γ_G.

II. Result

2.1. Coprime Graph of $\mathbb{Z}_n$

Some terminology of group and graph that used in this paper are given as follows.

Definition 1 ([3]) *Two vertices on the non-directed graph G are said to be neighbors if they are connected directly by an edge. In other words, u is adjacent to v if (u,v) is an edge on graph G.*

Definition 2 *If G is a group with identity e and $x \in G$, the order of x is the least natural number k such that $x^k = e$ and we write $|x| = k$.*

Definition 3 ([1]) *The coprime graph of a group G, denoted by Γ_G is a graph whose vertices are elements of G and two distinct vertices u and v are adjacent if and only if $(|a|,|b|) = 1$.*

Definition 4 ([3]) *Graph G, whose set of vertices can be partitioned into two subsets V_1 and V_2, such that each edge in G connecting a vertice in V_1 to a vertice in V_2, is called a bipartite graph* *and is expressed as $G(V_1, V_2)$. In other words, each pair of vertices in V_1 (as well as vertices in V_2) are not neighbors. If each node in V_1 is adjacent to all vertices at V_2, then $G(V_1, V_2)$ is called a complete bipartite graph, denoted by $K_(m, n)$, where $m = |V_1|$ and $n = |V_2|$.*

Definition 5 ([1]) *A k-partite graph is a graph whose vertices can be partitioned into k disjoint sets so that no two vertices within the same set are adjacent.*

As we know, $\mathbb{Z}_n$ is a cyclic group. The elements of $\mathbb{Z}_n$ can be written as $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$. Some examples of coprime graphs that obtained from the group $\mathbb{Z}_n$ are as follow.

Example 1 Let $\mathbb{Z}_3 = \{0, 1, 2\}$. We can see that the order of its elements are $|0| = 1, |1| = 3, |2| = 3$. Therefore, we have the coprime graph of $\mathbb{Z}_3$ as shown in Figure 1..

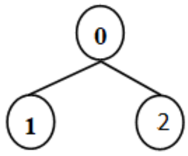

Figure 1. Coprime graph of $\mathbb{Z}_3$

Example 2 Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. We can check that the order of its elements are $|0| = 1, |1| = 4, |2| = 2, |3| = 4$. Therefore, we have the coprime graph of $\mathbb{Z}_4$ as shown in Figure 2..

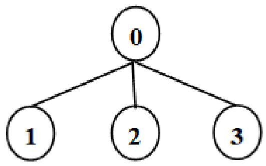

Figure 2. Coprime graph of $\mathbb{Z}_4$

By following the above examples, we can obtain some properties of the coprime graph of Group $\mathbb{Z}_n$ as follow. The first results we obtained is the coprime graph of $\mathbb{Z}_n$ is a complete bipartite graph whenever n is a prime.

Theorem 1 *If n is a prime number, then the coprime graph of $\mathbb{Z}_n$ is a complete bipartite graph.*

Proof. Clearly $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ with $|0| = 1$. Since n is a prime number, then $|1| = |2| = ... = |n-1| = n$. So, the set $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ can be partitioned into $V_1 = \{0\}$ and $V_2 = \{1, 2, \cdots, n-1\}$. For all $a, b \in V_2$, we have $(|a|, |b|) = n > 1$. This implies a and b are not neighbors. Because $|0| = 1$, then for each $a \in V_2$, we have $(|0|, |a|) = 1$. So 0 is adjacent to a. Thus coprime graph of the group $\mathbb{Z}_n$ is a complete bipartite graph. □

The second results we obtained is the coprime graph of $\mathbb{Z}_n$ is a complete bipartite graph whenever n is a prime power.

Theorem 2 *If $n = p^k$, for some prime p and $k \in \mathbb{N}$, then the coprime graph of $\mathbb{Z}_n$ is a complete bipartite graph.*

Proof. Clearly $\mathbb{Z}_n = \{0, 1, 2, \cdots, p^{k-1}\}$ with $|0| = 1$. Since p is a prime number, every $a \in \mathbb{Z}_n$ with $(p^k, a) \neq 1$, can be written as $a = p^l q$, for some l with $l < k$. This implies $|a| = p^{k-l}$. Also, for every $b \in \mathbb{Z}_n$ with $(p^k, b) = 1$, we have $|b| = p^k$. So, for every $a, b \in \mathbb{Z}_n$ with $a, b \neq 0$, we have $(|a|, |b|) \neq 1$. Thus, $\mathbb{Z}_n = \{0, 1, 2, \cdots, p^{k-1}\}$ can be partitioned into $V_1 = \{0\}$ and $V_2 = \{1, 2, \cdots, p^{k-1}\}$. Because $|0| = 1$, then for each $a \in V_2$, we have $(|a|, |0|) = 1$. Then, for all $a \in V_2$, a is adjacent to 0, thus coprime graph which is formed from $\mathbb{Z}_n$ is a complete bipartite graph. □

The second results we obtained is the coprime graph of $\mathbb{Z}_n$ is a t-partite graph whenever n is not a prime power.

Theorem 3 *If $n = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$, where $p_1, p_2, \cdots, p_j$ are distinct prime numbers and $k_1, k_2, \cdots, k_j$ are natural numbers, then coprime graph of $\mathbb{Z}_n$ is a $(j+1)-$partite graph.*

Proof. Let $\mathbb{Z}_n$ be the group of integers modulo n,with $n = p_1^{k_1} p_2^{k_2} \cdots p_j k_j$, where $p_1, p_2, \cdots, p_j$ are distinct prime numbers and $k_1, k_2, \cdots, k_j \in \mathbb{N}$. Clearly $\mathbb{Z}_n = \{0, 1, 2, \cdots, (p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}) - 1\}$. Every $a \in \mathbb{Z}_n$ with $(a, n) \neq 1$, can be written as $a = p_1^{l_1} p_2^{k_2} \cdots p_j^{k_j}$ with $l_i \leq k_i$. This implies, $|a| = (p_1^{k_1 - l_1} p_2^{k_2 - l_2} \cdots p_j^{k_j - l_j})$. Any $b \in \mathbb{Z}_n$ with $(b, n) = 1$, we have $|b| = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$. So, $\mathbb{Z}_n = \{0, 1, 2, \cdots, (p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}) - 1\}$ can be partitioned into the following sets.

$$V_1 = \{0\}$$

$$V_2 = \{a_1, a_2, \cdots, a_j\} \text{ with } |a_i| = \prod_{w=1}^{j} p_w^{\alpha_w}, 0 \leq \alpha_w \leq k_w, \alpha_1 \neq 0$$

$$V_3 = \{b_1, b_2, \cdots, b_j\} \text{ with} |\ |b_i| = \prod_{w=2}^{j} p_w^{\alpha_w}, 0 \leq \alpha_w \leq k_w, \alpha_2 \neq 0$$

$$\vdots$$

$$V_{j+1} = \{q_1, q_2, \cdots, q_j\} \text{ with } |q_i| = p_j^{\alpha_j}, 0 \leq \alpha_j \leq k_j$$

So, 0 is adjacent to all $x \in V_i, i = 2, 3, \cdots, j+1$. Also, some $u \in V_i$ is adjacent to $v \in V_l, i \neq l$. Thus, coprime graph that formed from $\mathbb{Z}_n$ is a graph $(j+1)-$partite. □

2.2. Coprime Graph of Subgroups of $\mathbb{Z}_n$

In this part, we will describe coprime graphs of subgroups of $\mathbb{Z}_n$. The first result is the coprime graphs of nontrivial subgroups of $\mathbb{Z}_n$ are bipartite whenever n is a prime power.

Theorem 4 *If $n = p^k$, for some prime number p and $k \in \mathbb{N}$, then coprime graphs of nontrivial subgroups of $\mathbb{Z}_n$ are bipartite.*

Proof. Any non-trivial subgroup of $\mathbb{Z}_{p^k}$ is isomorphic to $\mathbb{Z}_{p^l}$, for some $0 < l < k$. Therefore, by Theorem 2, coprime graph of any nontrivial subgroup of $\mathbb{Z}_{p^k}$ is bipartite. $\square$

The second result is whenever n is a product of two prime power, the the coprime graphs of nontrivial subgroups of $\mathbb{Z}_n$ are bipartite or tripartite.

Theorem 5 *If $n = p_1^{k_1} p_2^{k_2}$, with p_1, p_2 are distinct prime numbers, and k_1, k_2 are natural numbers, then coprime graphs of nontrivial subgroups of $\mathbb{Z}_n$ are bipartite or multipartite (3-partite).*

Proof. Any non-trivial subgroup of $\mathbb{Z}_{p_1^{k_1} p_2^{k_2}}$ is isomorphic to $\mathbb{Z}_{p_1^{l_1} p_2^{l_2}}$, for some $l_1 < k_1$ and $l_2 < k_2$. When $l_1 = 0$ or $l_2 = 0$, then by Theorem 2, the coprime graph of the corresponding subgroup is bipartite. Otherwise, by Theorem 3, the coprime graph of the corresponding subgroup is 3-partite. $\square$

The third result is whenever n is not a prime power, the coprime graphs of nontrivial subgroups of $\mathbb{Z}_n$ are multipartite.

Theorem 6 *If $n = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$, where $p_1, p_2, \cdots, p_j$ are distinct prime numbers and $k_1, k_2, \cdots, k_j \in \mathbb{N}$, then the coprime graph of non-trivial subgroups of $\mathbb{Z}_n$ is multipartite.*

Proof. Any non-trivial subgroup of $\mathbb{Z}_{p_1^{k_1} \cdots p_j^{k_j}}$ is isomorphic to $\mathbb{Z}_{p_1^{l_1} \cdots p_j^{l_j}}$, for some $l_i < k_i$, for all $i = 1, 2, \ldots, j$. If $l_{i_1}, l_{i_2}, \ldots, l_{i_t}$ are the only non-zero powers, then by Theorem 3, the coprime graph of the corresponding subgroup is $(t + 1)$-partite. $\square$

III. CONCLUSIONS

We described coprime graphs of Z_n and its subgroups for all n. In general, the resulting coprime graphs are bipartite whenever n is a prime power and multipartite whenever n is not a prime power. But when we consider its subgroups, the coprime graph subgroup of Z_n may a bipartite even if n is not a prime power.

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