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NORMAL ELEMENT ON IDENTIFY PROPERTIES

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Abstract. One type of element in the ring with involution is normal element. Their main properties is commutative with their image by involution in ring. Group invers of element in ring is always commutative with element which is commutative with itself.
In this paper, properties of normal element in ring with involution which also have generalized Moore Penrose invers are constructed by using commutative property of group invers in ring.
Keywords: Normal, Moore Penrose, group, involution

I. INTRODUCTION

Let ring with identity is sybolized by R and $a \in R$ An element $a \in R$ is said regular if there exists an element $b \in R$ such that $aba = a$, [1]. For this reason, then b is called an inner inverse of a. Inner inverse $b \in R$ of $a \in R$ is called group inverse of a if satisfies [2]:

$$bab = b, \qquad ab = ba.$$

For futher $b = a^{\#}$. Elements that have group inverse are collected, and that collect is symbolized with $R^{\#}$. Given by Harte and Mbektha [3], an involution "$*$" in a ring R is a function "$*$" : $a \in R \mapsto a^* \in R$ and for each $a, b \in R$ hold the following :

$$(a^*)^* = a \qquad (a+b)^* = a^* + b^* \qquad (ab)^* = b^*a^*$$

Suppose R is a ring with involution "$*$". Element $a \in R$ which is commute with a^* is called normal element [4] . Symbol R^{nor} The set of normal elements in R is denoted by R^{nor}.

For ring R with involution "$*$", if $a \in R$ is regular element with inner inverse b and satisfy

$$aba = a, \qquad (ab)^* = ab, \qquad (ba)^* = ba. \qquad (1)$$

then element b is called a generalized Moore–Penrose inverse of a [6]. We get a_g^+ and R_g^+ respectively to denote the generalized Moore Penrose inverse of a and the set of elements in R which have generalized Moore Penrose inverse. If a have generalized Moore Penrose invers, then they are can be more than one. That are $a_g^+ = a_g^+ a a_g^+$ or $a_g^+ \neq a_g^+ a a_g^+$. So we can construct $(a_g^+)^-$ for the set of a_g^+ .

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From Equation (1) we see that if $a \in R_g^+$, then a^*a and aa^* are regular element with their inner inverse are respectively $a_g^+ aa_g^+ (a_g^+ aa_g^+)^*$ and $(a_g^+ aa_g^+)^* a_g^+ aa_g^+$ for each $a_g^+ \in (a_g^+)^-$. It is obtained that inner inverse is also group inverse of multiplication between a and a *. So,

$$(a^*a)^\# = a_g^+ aa_g^+ (a_g^+ aa_g^+)^* \tag{2}$$

and

$$(aa^*)^\# = (a_g^+ aa_g^+)^* a_g^+ aa_g^+ \tag{3}$$

for each $a_g^+ \in (a_g^+)^-$, delivered by Titi et al [5, Theorem 2.10].

II. PROPERTIES OF NORMAL ELEMENT

It is known that if $a \in R_g^+$, then $a^*a \in R^\#$. So that, the multiplication of $(a^*a)^\#$ and a^*a is commutative. On the other hand by using definition of normal element, we have a is commutative with a^*a. From here we get the result that element a which has generalized Moore Penrose invers and elements a which are normal elements have the same properties, that is commutative element a^*a. For the next caused consequence is if $a \in R_g^+ \cap R^{nor}$, then by using property of group inverse, we derive

$$\begin{aligned}(a^*a)^\# a &= ((a^*a)^\#)^2 a^*aa = ((a^*a)^\#)^2 aa^*a = ((a^*a)^\#)^2 a(a^*a)^2(a^*a)^\# \\ &= ((a^*a)^\#)^2 (a^*a)^2 a\, (a^*a)^\# = (a^*a)^\# a^*aa\, (a^*a)^\# \\ &= (a^*a)^\# (a^*a)^2 a\big((a^*a)^\#\big)^2 = a^*aa\big((a^*a)^\#\big)^2 \\ &= a(a^*a)^\#. \end{aligned} \tag{4}$$

On the same way if $a \in R_g^+ \cap R^{nor}$, then

$$(aa^*)^\# a = a\,(aa^*)^\# \tag{5}$$

Futhermore by substituting Equation (2) into Equation (4), we get

$$a_g^+ aa_g^+ (a_g^+ aa_g^+)^* a = aa_g^+ aa_g^+ (a_g^+ aa_g^+)^* \tag{6}$$

for each $a_g^+ \in (a_g^+)^-$. Similarly by substituting Equation (3) into Equation (5), we get

$$(a_g^+ aa_g^+)^* a_g^+ aa_g^+ a = a(a_g^+ aa_g^+)^* a_g^+ aa_g^+ \tag{7}$$

for each $a_g^+ \in (a_g^+)^-$.

Inspired by Mosic and Djordjevic ([4], Theorem 2.2.) that is discusses the necessary and sufficient conditions of normal element on the Moore Penrose inverse. Trough Theorem 1 in this paper, we construct the sufficient conditions of normal element on the generalized Moore Penrose inverse.

Theorem 1. *If* $a \in R_g^+ \cap R^{nor}$, *then* $aa_g^+ = a_g^+ a$ *for each* $a_g^+ \in \left(a_g^+\right)^-$.

Proof :

Since $a \in R_g^+ \cap R^{nor}$ and by using Equation (6), then for each $a_g^+ \in \left(a_g^+\right)^-$ we have

$$\begin{aligned} aa_g^+ &= aa_g^+ aa_g^+ = aa_g^+ (aa_g^+)^* = aa_g^+ aa_g^+ (aa_g^+ aa_g^+)^* \\ &= aa_g^+ aa_g^+ (a_g^+ aa_g^+)^* a^* = a_g^+ aa_g^+ (a_g^+ aa_g^+)^* a\, a^* \\ &= a_g^+ aa_g^+ (a_g^+ aa_g^+)^* a^* a = a_g^+ aa_g^+ (aa_g^+ aa_g^+)^* a \\ &= a_g^+ a \end{aligned}$$

Equation (8) in the following describe that if $a \in R_g^+$, then a_g^+ is inner inverse of $a^* \in R_g^+$ for each $a_g^+ \in \left(a_g^+\right)^-$. In other side Equation (9) explain that if $a^* \in R_g^+$, then $(a^*)_g^+$ is inner inverse of $a \in R_g^+$ for each $(a^*)_g^+ \in \left((a^*)_g^+\right)^-$.

Next, suppose $a \in R_g^+$. Then for each $a_g^+ \in \left(a_g^+\right)^-$ we get

$$a^* = (aa_g^+ a)^* = (aa_g^+ aaa_g^+)^* = a^*(a_g^+ aa_g^+)^* a^* \tag{8}$$

In the same way if $a^* \in R_g^+$, then for each $(a^*)_g^+ \in \left((a^*)_g^+\right)^-$ we have

$$a = (a^*)^* = (a^*(a^*)_g^+ a^*)^* = a((a^*)_g^+)^* a \tag{9}$$

Theorem 2 in the following inspired by Equation (8) and Equation (9)

Theorem 2. *Element* $a \in R_g^+$ *if and only if* $a^* \in R_g^+$.

Proof :

Equation (8) explain that if $a \in R_g^+$, then $a^* = a^*(a_g^+ aa_g^+)^* a^*$ for each $a_g^+ \in \left(a_g^+\right)^-$. Futhermore $((a_g^+ aa_g^+)^* a^*)^* = (aa_g^+)^* = (aa_g^+ aa_g^+)^* = (a_g^+ aa_g^+)^* a^*$ and $(a^*(a_g^+ aa_g^+)^*)^* = (a_g^+ a)^* = (a_g^+ aa_g^+ a)^* = a^*(a_g^+ aa_g^+)^*$ for each $a_g^+ \in \left(a_g^+\right)^-$. Conversly if $a^* \in R_g^+$, then we already have $a = a((a^*)_g^+)^* a$ for each $(a^*)_g^+ \in \left((a^*)_g^+\right)^-$ from Equation (9). Next, $(a((a^*)_g^+)^*)^* = ((a^*)^* ((a^*)_g^+)^*)^* = ((a^*)_g^+ a^*)^* = a((a^*)_g^+)^*$ and $(((a^*)_g^+)^* a)^* = (((a^*)_g^+)^*(a^*)^*)^* = (a^*(a^*)_g^+)^* = ((a^*)_g^+)^* a$ for each $(a^*)_g^+ \in \left((a^*)_g^+\right)^-$.

Inspired by Theorem 2 and if $a \in R_g^+ \cap R^{nor}$, then using property of group inverse, we get

$$(a^*a)^\# a^* = ((a^*a)^\#)^2 a^*aa^* = ((a^*a)^\#)^2 a^*a^*a$$
$$= ((a^*a)^\#)^2 a^*(a^*a)^2(a^*a)^\# = ((a^*a)^\#)^2 (a^*a)^2 a^* (a^*a)^\#$$
$$= (a^*a)^\# a^*aa^* (a^*a)^\# = (a^*a)^\#(a^*a)^2 a^*\left((a^*a)^\#\right)^2$$
$$= a^*aa^*((a^*a)^\#)^2 = a^*a^*a((a^*a)^\#)^2$$
$$= a^*(a^*a)^\#. \quad (10)$$

On the same way if $a \in R_g^+ \cap R^{nor}$, then

$$(aa^*)^\# a^* = a^* (aa^*)^\# \quad (11)$$

For the next, by substituting Equation (2) into Equation (10), we get

$$a_g^+ aa_g^+ (a_g^+ aa_g^+)^* a^* = a^* a_g^+ aa_g^+ (a_g^+ aa_g^+)^* \quad (12)$$

for each $a_g^+ \in \left(a_g^+\right)^-$. By substituting Equation (3) into Equation (11) and if $a \in R_g^+ \cap R^{nor}$ then for each $a_g^+ \in \left(a_g^+\right)^-$ we hold

$$(a_g^+ aa_g^+)^* a_g^+ aa_g^+ a^* = a^*(a_g^+ aa_g^+)^* a_g^+ aa_g^+ \quad (13)$$

Motivated by Equation (12) and Equation (13), then we obtain Theorem 3 in the following. That results already done by Mosic and Djordjevic ([4], Theorem 2.2.) for Moore Penrose inverse.

Theorem 3. *Suppose $a \in R_g^+$. Element $a \in R^{nor}$ if $a^* a_g^+ aa_g^+ = a_g^+ aa_g^+ a^*$ for each $a_g^+ \in \left(a_g^+\right)^-$.*

Proof :

Since $a \in R_g^+ \cap R^{nor}$, so by using Equation (13) we obtain

$$a^* a_g^+ aa_g^+ = a^* a_g^+ aa_g^+ aa_g^+ == a^*\left(a_g^+ aa_g^+ a\right)^* a_g^+ aa_g^+ = a^*a^*\left(a_g^+ aa_g^+\right)^* a_g^+ aa_g^+$$
$$= a^*\left(a_g^+ aa_g^+\right)^* a_g^+ aa_g^+ a^* = \left(a_g^+ aa_g^+ a\right)^* a_g^+ aa_g^+ a^*$$
$$= \left(a_g^+ a\right)^* a_g^+ aa_g^+ a^* = a_g^+ aa_g^+ aa_g^+ a^*$$
$$= a_g^+ aa_g^+ a^*$$

for each $a_g^+ \in \left(a_g^+\right)^-$.

III. CONCLUTION

The commutative property of the group inverse can be used for establish the sufficient condition of normal element to have generalized Moore Penrose invers. The problem that arises for the next is whether that the sufficient requirement is also the necessary condition.

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