

M-POLYNOMIAL OF REGULAR DIVISOR GRAPH OF FINITE COMMUTATIVE RING

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Abstract. Let R be a finite commutative ring with identity, for all non-zero elements a and b in the ring R are adjacent if and only if $a = a.b.a$ or $b = b.a.b$ and $a \neq b$, this graph is called by regular divisor graph and denoted by $\mathfrak{R}_\partial(R)$, with vertex set $V(\mathfrak{R}_\partial(R))$ consists of elements of $r^*(R) = \text{Reg}(R) - \{0\}$ and edge set $E(\mathfrak{R}_\partial(R)) = \{(a, b) : a = a.b.a \text{ or } b = b.a.b, a \neq b \neq 0, a, b \in \text{Reg}(R) - \{0\}\}$. We classify the finite commutative ring to get a special graph and we are going to study some properties of this graph, up to the R which is local, Boolean and field. And we study the M-polynomial of the regular divisor graph $\mathfrak{R}_\partial(R)$, and the general form of regular divisor graph of $R = \mathbb{Z}_p \times \mathbb{Z}_q$ for any prime number p and q . In this paper we got some new results with appear with the new graph.

Keywords: Regular divisor graph, M- polynomial, direct product of fields $\mathbb{Z}_p \times \mathbb{Z}_q$.

I. INTRODUCTION

Graph theory is a branch of mathematics that studies graphs—mathematical structures used to model pairwise relations between objects. The origin of graph theory dates back to 1736, when Leonhard Euler solved the famous *Königsberg Bridge Problem*. Euler's formulation of this problem laid the foundation of what is now called graph theory [1].

Graph theory has grown from a purely mathematical curiosity to a core tool in various scientific and technological fields. Specially in mathematics the relationship between graph theory and abstract algebra has grown increasingly significant in contemporary mathematics, this branch explicitly applies algebraic methods (e.g., linear algebra, group theory) to graph problems. The relation between the theory of graphs and the theory of rings was set up by Beck [2] in 1988. He defined a zero-divisor graph denoted by $\Gamma_0(R)$, with all the elements of a ring as graph vertices and two distinct vertices x and y adjacent provided $xy = 0$. After updating the Beck definition [2], Anderson and Livingston [3, 4] were the first to implement the zero-divisor graph definition. Anderson and Badawi [5], introduced the total graph of R , as the simple graph with all elements of R as vertices, and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. Authman, Shuker, and Mohammad [6] started implementing the Idempotent Divisor Graphs of Commutative Rings.

In this paper we combined these two objects (Regular ring in algebra with new property of graph) to gate a new and distinctive type with special specifications of graph called regular

divisor graph. To explore the relationship between regular rings in algebra and regular divisor graphs of certain rings, we first need to understand the fundamental concepts involved:

In Graph theory: A graph G is finite non-empty set consist of two sets, the set of vertices $V(G)$ and the set of edges $E(G)$. $V(G)$ is a non-empty set of elements named vertices, while $E(G)$ is the set (which is possible empty) of unordered pairs of vertices of $V(G)$ called edges.[7]. If $e = \{v_i, v_j\}$ be any edge in $E(G)$ with $v_i, v_j \in V(G)$ then e joins a vertex v_i with the vertex v_j we say v_i and v_j are adjacent in G , each of v_i and v_j called the end vertices of the edge e . If v_i is adjacent with v_j in G , we denote $e = \{v_i, v_j\}$. The order of the graph is the number of vertices which is denoted by $n(G)$, that is $n(G) = |V(G)|$, and the number of edges of G is called the Size of G and is denoted by $m(G)$, that is $m(G) = |E(G)|$. The degree of a vertex v of a graph G , is the number of all edge's incident to v in G . We denote the degree of the vertex v of G by $\deg_G(v)$. If $S \subseteq V(G)$, then $\langle S \rangle$ is a graph with $V(\langle S \rangle) = S$ and $E(\langle S \rangle) = \{ (u,v): u,v \in S \text{ and } (u,v) \in E(G) \}$. If any two distinct vertices of G are adjacent then G is called complete graph. The complete graph of order $n \in \mathbb{N}$ is denoted by K_n . If H be a graph, $H \cup G$ is a graph with $V(H \cup G) = V(H) \cup V(G)$ and $E(H \cup G) = E(H) \cup E(G)$. $nG = \bigcup_{i=1}^n G$ for some $n \in \mathbb{N}$.

Introduced by Deutsch and Klavzar in 2015, [8], the M -polynomial of a graph $G=(V,E)$ is defined as $M(G; x, y) = \sum_{i \leq j} m_{ij}(G) x^i y^j$, where $m_{ij}(G)$ is the number of edges connecting vertices of degree i and degree j . This polynomial acts as a master algebraic producer for degree-based topological indices. Instead of evaluating multiple topological descriptors independently, one computes the M -polynomial once. From M -polynomial, we can obtain first and second Zagreb Indices, general Randic Index, Harmonic Index, Inverse Sum Index, and other indices. See [8-11].

In a ring theory, a regular ring is a type of ring with certain nice properties related to the existence of "pseudo-inverses." is the Von-Neumann regular ring.[12] Let R be a finite commutative ring with identity. A ring R is called Von-Neumann regular if for every $a \in R$, there exists $x \in R$ such that $a = axa$, The set of all regular elements of the ring R denoted by $\text{Reg}(R)$. The ideal M of a ring R , is called maximal ideal if $M \subseteq I$, for any ideal I , implies $I = R$. R is a local ring if it has exactly one maximal ideal. The element $e \in R$ is called idempotent-element if $e = e^2$. $I(R)$ is the set of all idempotent elements of R . R is Boolean ring if and only if $R = I(R)$. For the element $a \in R$, if there is $n \in \mathbb{N}$ such that $a^n = 0$, then a is called to be nilpotent element. R is a reduced ring if and only if $R - \{0\}$ without nilpotent elements. If $R = R_1 \times R_2 \times \dots \times R_k$, for some $k \in \mathbb{N}$, then R is a direct product of the rings R_i for $2 \leq k$. Any finite non-local ring is a direct product of finite local rings. A field F is a commutative ring, with identity, and any non-zero element in it has an inverse. Any field is a local ring, but the converse is not true in general. The finite field has (a power of prime number) elements. R is a reduced ring if and only if it is a direct product of finite fields. $U(R)$ and $Z(R)$ are the unite and zero-divisor elements of R , respectively. If R is a finite, then $R = U(R) \cup Z(R)$. The ring of integers modulo n , denoted by Z_n . Z_n is a filed if and only if n is a prime number. See [13-15].

Divisor Graphs: In the context of ring theory, a divisor graph is a graph associated with the ring's elements or its ideals, where vertices represent certain elements (like non-zero elements, non-units, etc.), and edges represent divisibility relations. This graph illustrates the structure of rings and the relationships among their elements. Each type of divisor graph captures these relationships in different way. For example, the zero-divisor graph highlights

the connection between elements whose product is zero. In fact, many definitions of divisor graph of rings have been introduced in the literature, each emphasizing a particular aspect of ring structure. The regular divisor-graph of a commutative ring R , in particular, describes the relationship between the regular element $a \in \text{Reg}(R) - \{0\} = r^*(R)$ satisfying $a = a b a$ and the another element b within the ring structure. The prominent role of the divisor graph has led to the introduction of several different definitions. See, [2,3], [13], and [16-23].

The new idea in this paper, is to construct a new Graph from the definition of regularity of the ring in algebra, considering that if two elements u and v in the ring are regular with each other, there will be an edge between them in the new graph as two non-zero vertices in the new graph. The new graph that we obtained from the regular ring is called the regular divisor graph. Then we define the regular divisor graph of the ring R , $\mathfrak{R}_\partial(R)$, as a graph that contains those vertices that all non-zero elements of the ring R , $r^*(R) = R - \{0\}$, and every two vertices $u, v \in r^*(R)$ are adjacent if and only if they regular together in the ring R , that is $u = u.v.u$ or $v = v.u.v$. The ring we study in this work will be a commutative ring [20].

In this paper we worked in two directions, the first is classifying the graph according to whether n (the order of ring) is a prime number, odd or even, because whenever we change the value of n in the ring, the regular elements changed and light of that the graph changes, so we classified it to study each type separately so that we can study and analyze the characteristics and properties of each graph to obtain valuable results. On the other hand, we studied the characteristics of some rings (Boolean, local and reduced) and the regularity to obtain the graph from the elements of the ring.

We obtain robust and novel results in the polynomials, especially M -polynomial of the divisor regular ring of the commutative ring. The interplay between abstract graph theory and ring theory reveals that under certain conditions, the regularity of a ring can lead to the regularity of its associated divisor graph.

Recall that the Regular divisor graph of the commutative ring R , which is symbolized by $\mathfrak{R}_\partial(R)$, where two non-zero distinct elements u and v in $r^*(R)$, are adjacent as a vertex in the graph if and only if $u = u.v.u$ or $v = v.u.v$ for the non-zero elements $u, v \in r^*(R)$.

Then the vertex set of the graph is $V(\mathfrak{R}_\partial(R))$, the set of all non-zero regular elements of the ring R , and edge set $E(\mathfrak{R}_\partial(R)) = \{(u, v) : u = u.v.u \text{ or } v = v.u.v, u \neq v \neq 0, u, v \in r^*(R)\}$.

The commutative ring $Z_n, n \geq 1$ for n equal to (prime, composite, odd, or even), is regular ring or not regular ring but in each case, it has some regular elements depending on n .

To study the regular divisor graph of the commutative ring Z_n , which is (undirected) graph and symbolized by $\mathfrak{R}_\partial(Z_n)$, where two non-zero distinct elements in Z_n , a and b are adjacent as a vertex if and only if $a = a.b.a$ or $b = b.a.b$ for the regular elements $a, b \in Z_n$. the regular divisor graph of commutative ring Z_n is simple, undirected loop less graph $\mathfrak{R}_\partial(Z_n)$ with vertex set $V(Z_n)$ and edge set $E(Z_n) = \{(a, b) : a = a.b.a \text{ or } b = b.a.b, a \neq b \neq 0 \in Z_n\}$.

The regular divisor graph of commutative ring $Z_n, n \geq 1$ has no well-known form (certain form) it is changed with respect on n (prime, composite, odd or even) to find the certain form of the graph we must classify the ring Z_n with respect to the order of ring n , as the following:

II SOME PROPERTIES OF REGULAR DIVISOR GRAPH OF FINITE COMMUTATIVE RINGS

In this section, we find some fundamental properties of the regular divisor graph of R , where R is a field, local, non-local, Boolean, or isomorphism with Z_n for some positive integer number n .

The ring Z_p is a regular ring for all prime p , since Z_p is a division ring and every division ring is a regular ring. The regular divisor graph of the ring Z_p is special graph, for all non-zero element $a \in Z_p$ there exist $a^{-1} \in Z_p$ such that a and a^{-1} are adjacent.

Theorem 2.1 *The regular divisor graph of the ring Z_p is a bipartite graph.*

Proof: Since the ring Z_p is division ring then Z_p is regular ring it means for all $a \in Z_p$, a is a regular element. Then the vertex set $V(Z_p) = Z_p - \{0\}$, two elements 1 and $p-1$ in Z_p are self-regular and self-inverse $1.1.1 = 1$, $2.2.2 = 2, \dots$, $(p-1)(p-1)(p-1) = (p-1)$ and they make a loop in the regular divisor graph so we exclude 1 and $p-1$ in the vertex set, then $|V(Z_p)| = p-3$, and each $a \in Z_p$, a is adjacent with a^{-1} , $(a_i \cdot a_i^{-1} \cdot a_i = a_i)$ for all $a_i \in Z_p$ by the regularity so we have two types of vertices such that $V_1(Z_p) = \{a_1, a_2, \dots, a_i\}$ and $V_2(Z_p) = \{a_1^{-1}, a_2^{-1}, \dots, a_i^{-1}\}$, $i = \frac{p-3}{2}$, and each element in V_1 is adjacent with only one element in V_2 , since the inverse element is unique. There is no another adjacent vertex in the graph.

Theorem 2.2 *Let p be a prime number, m be a positive integer. Then*

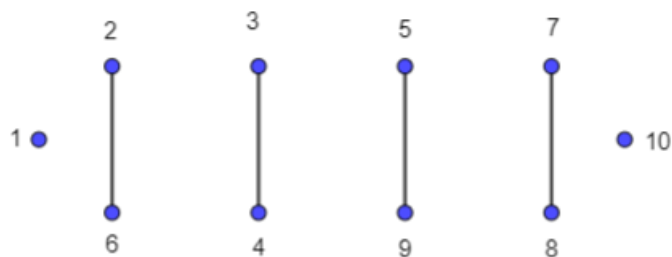
$$\mathfrak{R}_\partial(F_{p^m}) \cong \begin{cases} K_1 \cup \frac{p^m-2}{2} K_2 & , \text{if } p = 2 \\ 2 K_1 \cup \frac{p^m-3}{2} K_2 & , \text{if } p > 2 \end{cases}$$

Proof: Let $a, b \in F_{p^m} - \{0\}$, such that a adjacent with b in the graph $\mathfrak{R}_\partial(F_{p^m})$, then we get $a^2b = a$, thus $ab = 1$, therefore $a = b^{-1}$. Hence, a adjacent with b if and only if $a = b^{-1}$, on the other hand the only solution of $a = a^{-1}$ in F_{p^m} is $a = 1, -1$. Since the order of F_{p^m} is even iff $p=2$, then $1 = -1$ if and only if $p=2$. Thus $1 \neq -1$ in any field F_{p^m} for any prime $p > 2$ and $m \in \mathbb{Z}^+$. Therefore, we have two isolated vertices 1 and -1 , and $\frac{p^m-3}{2}$ pairs of adjacent vertices a and a^{-1} where $p > 2$. So, we have one isolated vertex 1, and $\frac{p^m-2}{2}$ pairs of adjacent vertices a and a^{-1} where $p = 2$. ■

Example 1: Consider the ring Z_{11} , $p = 11$.

The non-zero regular elements of Z_{11} are $r^*(Z_{11}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ the elements 1 and 10 are self-regular then exclude them in the vertex set. And the vertex set is $V(Z_{11}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

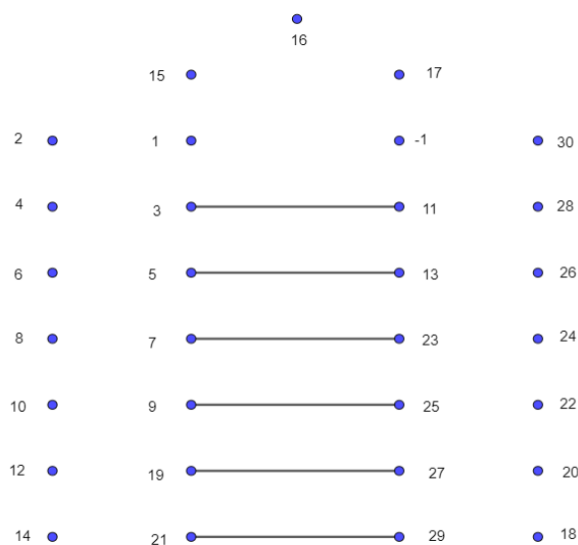
The regular divisor graph of Z_{11} is shown in figure-2.1


 Figure 2.1 Regular divisor graph $\mathfrak{R}_\theta(Z_{11})$

Theorem 2.3 Let R be a ring. Then $\langle U(R) - \{1, -1\} \rangle_{\mathfrak{R}_\theta(R)} \cong \left(\frac{1}{2}|U(R)| - 2\right) K_2$.

Proof: The proof is obvious.

Example 2: Let $R = \mathbb{Z}_{32}$, then $\mathfrak{R}_\theta(\mathbb{Z}_{32})$ is the graph in figure 2.2


 Figure 2.2 $\mathfrak{R}_\theta(\mathbb{Z}_{32})$

We see that, $|U(\mathbb{Z}_{32})| = 14$, and we get $\langle U(\mathbb{Z}_{32}) - \{1, -1\} \rangle_{\mathfrak{R}_\theta(\mathbb{Z}_{32})} \cong \left(\frac{1}{2}(14 - 2)\right) K_2 = 6K_2$.

Proposition 2.4 Let R be a finite local ring, $a, b \in R - \{0\}$, $a \neq b$. Then a adjacent with b in the graph $\mathfrak{R}_\theta(R)$ if and only if $a, b \in U(R)$ with $a = b^{-1}$.

Proof: If R be a field, then by Theorem 2.2 the proof is obvious. If $a = b^{-1}$ then $a^2b = a$, hence a adjacent with b .

Now, let $a, b \in R$, such that a adjacent with b , that is $a^2b = a$, then we have four cases:

Case1: $a, b \in U(R)$, then a and ba are units, since $a^2b = a$ then $ab = 1$, thus $a = b^{-1}$.

Case2: $a, b \in Z(R) - \{0\}$, since $a^2b = a$ then $a(1-ab) = 0$. Clearly, $ab \in Z(R)$ and $(1-ab) \in U(R)$, then we get $a=0$ which is a contradiction.

Case3: $a \in U(R)$ and $b \in Z(R) - \{0\}$. As case2 above, $ab \in Z(R)$ and $(1-ab) \in U(R)$. If $a^2b = a$ then $a(1-ab) = 0$ implies $a=0$ which is a contradiction, else if $b^2a = b$ then $b(1-ab) = 0$ implies $b=0$ which is a contradiction also.

Case4: $a \in Z(R) - \{0\}$ and $b \in U(R)$. Similarly, as case3, this case is impassible. ■

Definition 2.5 Let R be a ring. Define the set $\mathcal{U}(R) = \{u \in U(R) : u \neq u^{-1}\}$.

By Theorem 2.3, Proposition 2.4 and Definition 2.5 we give the general form of the regular divisor graph of any local ring:

Corollary 2.6 Let R be a finite local ring. Then $\mathfrak{R}_\partial(R) \cong (R / -|\mathcal{U}(R)| - 1) K_1 \cup \left(\frac{1}{2}|\mathcal{U}(R)|\right) K_2$.

In Example 2 above, We see that, $\mathcal{U}(R) = \{3, 5, 7, 9, 11, 13, 19, 23, 25, 27\}$, $|\mathcal{U}(R)| = 10$, and $|R| - |\mathcal{U}(R)| - 1 = 32 - 10 - 1 = 21$. Clearly $\mathfrak{R}_\partial(Z_{32}) \cong 21K_1 \cup 5K_2$.

Theorem 2.7 Let R be a non-local ring. Then, the graph $\mathfrak{R}_\partial(R)$ contains the star graph $K_{1, |I(R)|-2}$ as a subgraph.

Proof: Let $a \in I(R) - \{0, 1\}$, then $a^2 \cdot 1 = a$, then a adjacent with 1, that is, 1 adjacent with all vertices of $I(R) - \{0, 1\}$.

Corollary 2.8 Let R be a Boolean ring. Then $\mathfrak{R}_\partial(R)$ is a connected graph.

Example 3: Let $R = Z_2 \times Z_2 \times Z_2$, then $\mathfrak{R}_\partial(R)$ is the graph in figure2.3

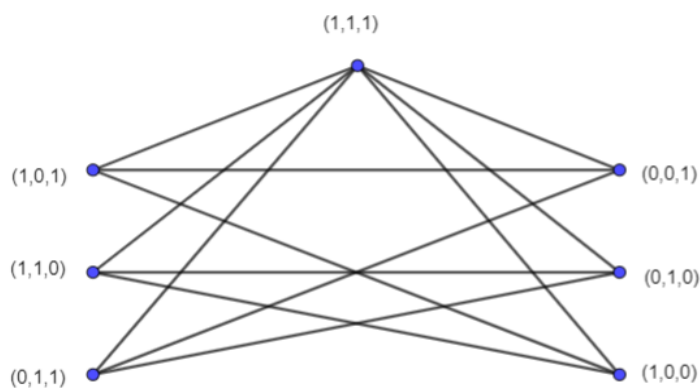


Figure 2.3 $\mathfrak{R}_\partial(Z_2 \times Z_2 \times Z_2)$

We see that R is Boolean ring, and $\mathfrak{R}_\partial(R)$ is a connected.

Corollary 2.8 *Let R be a reduced ring. Then the graph $\mathfrak{R}_\partial(R)$ is a connected graph.*

Proof: Since R is a finite reduced ring, then there are finite fields $F_1, F_2, \dots, F_m$ for some $n \in \mathbb{Z}^+$ such that $R \cong F_1 \times F_2 \times \dots \times F_m$. If R be a Boolean ring, then the proof is clear.

Suppose that R is not Boolean ring, then there is at least one of F_i with $F_i \not\cong \mathbb{Z}_2$. Let $a \in R - I(R)$, then a has unique representation $a = (a_1, a_2, \dots, a_m)$ where $a_i \in F_i$ for $i=1, 2, \dots, m$. We show that there is at least one path between a and the vertices of $I(R)$. There are two cases:

Case1: If $a \in U(R)$, then $a_i \neq 0_i$ for any $i=1, 2, \dots, m$. We see that there is a path $P: a, b, c, d$, where $b = (a_1^{-1}, 0_2, \dots, 0_m) \in Z(R)$, $c = (a_1, 1_2, 0_3, \dots, 0_m)$ and $d = (0_1, 1_2, 0_3, \dots, 0_m) \in I(R)$.

Case2: If $a \in Z(R) - I(R)$, then $a_i = 0_i$ for at least one of $i=1, 2, \dots, m$, without lose the generality suppose that $i=1$, that is $a_1 \neq 0_1$, hence a has an inverse. Also, we see that there is a path $P: a, b, c, d$, where $b = (a_1^{-1}, 0_2, \dots, 0_m)$, $c = (a_1, 1_2, 0_3, \dots, 0_m)$ and $d = (0_1, 1_2, 0_3, \dots, 0_m) \in I(R)$.

Since $\langle I(R) - \{0\} \rangle$ is a connected, therefore $\mathfrak{R}_\partial(R)$ is a connected. ■

III M-POLYNOMIAL OF REGULAR DIVISOR GRAPH OF FINITE COMMUTATIVE RING

In this section, we find the general form of $\mathfrak{R}_\partial(Z_p \times Z_q)$, the degrees of its vertices, and its M-polynomial, for any odd prime numbers p and q .

For any prime p , Z_p is a field. Let $a, b \in Z_p$, then $a b a = a$ hold if and only if either $a=0$ and $b \in Z_p$ or $a \neq 0$ and $b = a^{-1}$. Also, we see that $a = a^{-1}$ if and only if $a=1$ or $a = -1$ (where $-1 = p-1$). We start this section by the following proposition.

Proposition 3.1 *Let p and q are odd prime numbers. Then $\mathfrak{R}_\partial(Z_p \times Z_q)$ is a connected graph. Furthermore, the distinct vertices $(a, b), (c, d) \in Z_p \times Z_q - \{(0, 0)\}$ are adjacent if and only if either $a.c=0$ and $d=b^{-1}$, or $b.d=0$ and $c=a^{-1}$, or $c=a^{-1}$ and $d=b^{-1}$.*

Proof: Clearly, the rings Z_p and Z_q are fields, hence $Z_p \times Z_q$ is a reduced ring, then by Corollary 2.8 $\mathfrak{R}_\partial(Z_p \times Z_q)$ is a connected.

(a, b) adjacent with (c, d) in the graph $\mathfrak{R}_\partial(Z_p \times Z_q)$ if and only if $(a, b) (c, d) (a, b) = (a, b)$, thus $(a c a, b d b) = (a, b)$, implies $a c a = a$ and $b d b = b$.

Since $(a, b), (c, d) \neq (0, 0)$ then either $a=0$ and $b \neq 0$ (i.e b has an inverse in Z_q), or $b=0$ and $a \neq 0$ (i.e. the element a has an inverse in Z_p), or $a, b \neq 0$ (i.e. a has an inverse in Z_p , and b has an inverse in Z_q). Similarly, either $c=0$ and $d \neq 0$ (i.e d has an inverse in Z_q), or $d=0$ and $c \neq 0$ (i.e c has an inverse in Z_p), or $c, d \neq 0$ (i.e c has an inverse in Z_p , and d has an inverse in Z_q). Then, we have the following four cases:

Case1: if $a=0, b \neq 0, c=0$ and $d \neq 0$, or $a=0, b \neq 0, c \neq 0$ and $d \neq 0$, or $a \neq 0, b \neq 0, c=0$ and $d \neq 0$. Then (a, b) adjacent with (c, d) in this case if and only if $a.c=0$ and $d=b^{-1}$.

Case2: if $a \neq 0$, $b=0$, $c \neq 0$ and $d=0$, or $a \neq 0$, $b=0$, $c \neq 0$ and $d \neq 0$, or $a \neq 0$, $b \neq 0$, $c \neq 0$ and $d=0$. Then (a,b) adjacent with (c,d) in this case if and only if $b.d=0$ and $c=a^{-1}$.

Case3: if $a \neq 0$, $b \neq 0$, $d \neq 0$ and $c \neq 0$. Then (a,b) adjacent with (c,d) in this case if and only if $c=a^{-1}$ and $d=b^{-1}$.

Case4: if $a=0$, $b \neq 0$, $c \neq 0$ and $d=0$, or $a \neq 0$, $b=0$, $c=0$ and $d \neq 0$. Then $(a,b).(c,d).(a,b)=(0,0)$, hence (a,b) is not adjacent with (c,d) in this case. ■

By Proposition 3.1, we find $\mathfrak{R}_\partial(Z_2 \times Z_2) \cong K_{1,2}$, $\mathfrak{R}_\partial(Z_2 \times Z_3) \cong P_5$ and $\mathfrak{R}_\partial(Z_3 \times Z_3) \cong C_8$ as shown in figure below:

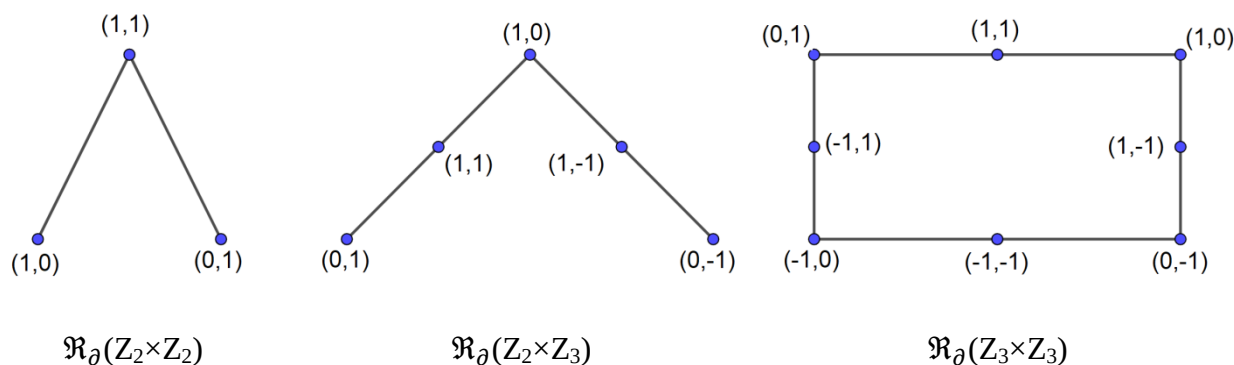


Figure 3.1

Also, $\mathfrak{R}_\partial(Z_3 \times Z_5)$ is the graph in the following figure:

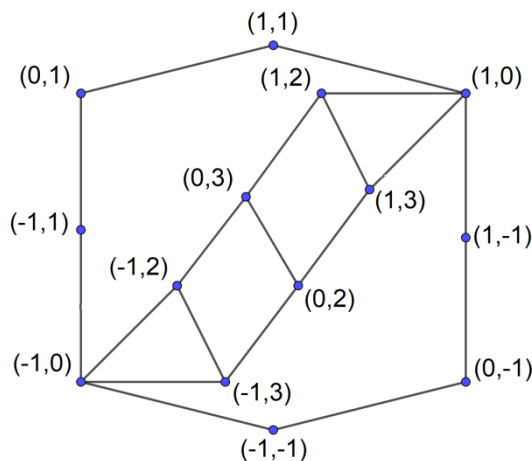


Figure 3.2 $\mathfrak{R}_\partial(Z_3 \times Z_5)$

To find the general form of $\mathfrak{R}_\partial(Z_p \times Z_q)$ for any prime numbers p and q , we give the following definition:

Definition 3.2 Let p and q are prime numbers with $p, q \geq 5$. Define the sets:

$A = \{x \in Z_p - \{0, 1, -1\} : x^{-1} \notin A\}$, $\bar{A} = Z_p - \{x \in A \cup \{0, 1, -1\}\}$, $B = \{x \in Z_q - \{0, 1, -1\} : x^{-1} \notin B\}$ and $\bar{B} = Z_q - \{x \in B \cup \{0, 1, -1\}\}$.

Directly by definition, we see that $A \cup A'' \cup \{0, 1, -1\} = Z_p$ and $B \cup B'' \cup \{0, 1, -1\} = Z_q$. Clearly, if $x \in Z_p$, then $x = x^{-1}$ if and only if $x = 1, -1$, hence $A \cap A'' = \emptyset$, similarly $B \cap B'' = \emptyset$. Also, $|A| = |A''| = \frac{1}{2}(p-3)$ and $|B| = |B''| = \frac{1}{2}(q-3)$.

Example 4: Let $R = Z_5 \times Z_7$, then we have $A = \{2\}$, $A'' = \{3\}$, $B = \{2, 3\}$ and $B'' = \{4, 5\}$. We see $A \cup A'' \cup \{0, 1, -1\} = Z_5$, $B \cup B'' \cup \{0, 1, -1\} = Z_7$, $A \cap A'' = B \cap B'' = \emptyset$, $|A| = |A''| = \frac{1}{2}(5-3) = 1$ and $|B| = |B''| = \frac{1}{2}(7-3) = 2$.

In [20] the general form of $\mathfrak{R}_\partial(Z_2 \times Z_p)$ and $\mathfrak{R}_\partial(Z_3 \times Z_q)$ was given for any prime numbers with $p > 2$ and $q > 3$. By using Definition 3.2 we can give these general form graphs as the following:

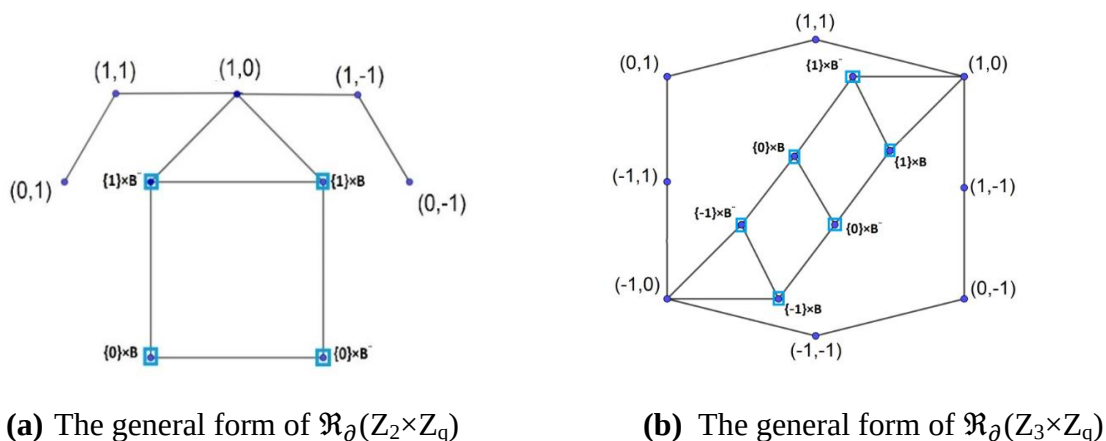


Figure 3.3

By these general forms, directly we can give the following properties:

Proposition 3.3. Let q be a prime number with $q \geq 5$, B and B'' are the sets whose defined in Definition 2.2. If $(x, y) \in Z_2 \times Z_q - \{(0, 0)\}$, then:

$$\deg_{\mathfrak{R}_\partial(Z_2 \times Z_q)}(x, y) = \begin{cases} 1 & : \text{if } x = 0 \text{ and } y \in \{1, -1\} \\ 2 & : \text{if } x \in \{1, -1\} \text{ and } y \in \{1, -1\} \\ & \text{or } x = 0 \text{ and } y \in B \cup B'' \\ 3 & : \text{if } x = 1 \text{ and } y \in B \cup B'' \\ q - 1 & : \text{if } x = 1 \text{ and } y = 0 \end{cases}$$

Proposition 3.4. Let q be a prime number with $q \geq 5$, B and B'' are the sets whose defined in Definition 3.2. If $(x, y) \in Z_3 \times Z_q - \{(0, 0)\}$, then:

$$\deg_{\mathfrak{R}_\partial(Z_3 \times Z_q)}(x, y) = \begin{cases} 2 & : \text{if } x \in \{0, 1, -1\} \text{ and } y \in \{1, -1\} \\ 3 & : \text{if } x \in \{0, 1, -1\} \text{ and } y \in B \cup B'' \\ q - 1 & : \text{if } x \in \{1, -1\} \text{ and } y = 0 \end{cases}$$

Now, by Proposition 3.1 and Definition 3.2 we give the general form of $\mathfrak{R}_\partial(Z_p \times Z_p)$ and $\mathfrak{R}_\partial(Z_p \times Z_q)$ for any prime numbers with $p, q \geq 5$.

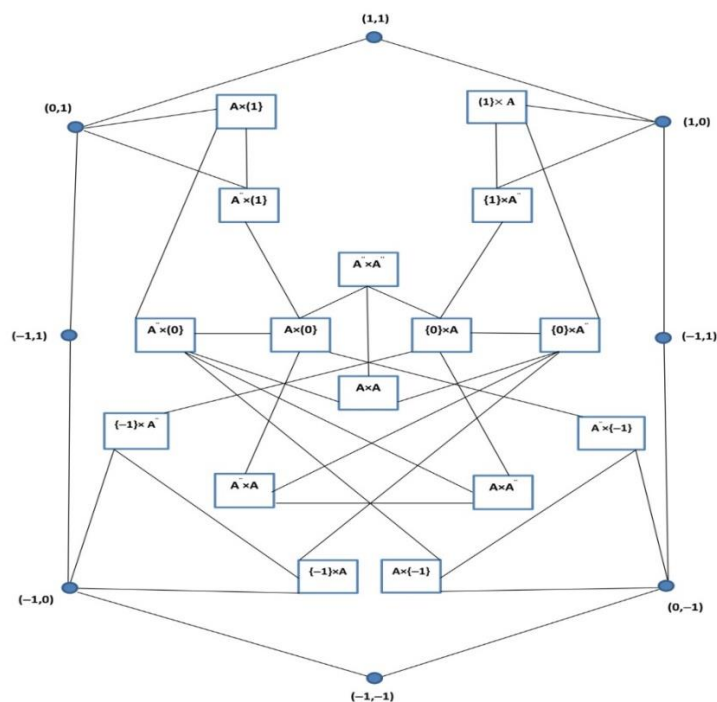


Figure 3.4 The general form of $\mathfrak{R}_\partial(\mathbb{Z}_p \times \mathbb{Z}_p)$ for any prime number $p \geq 5$

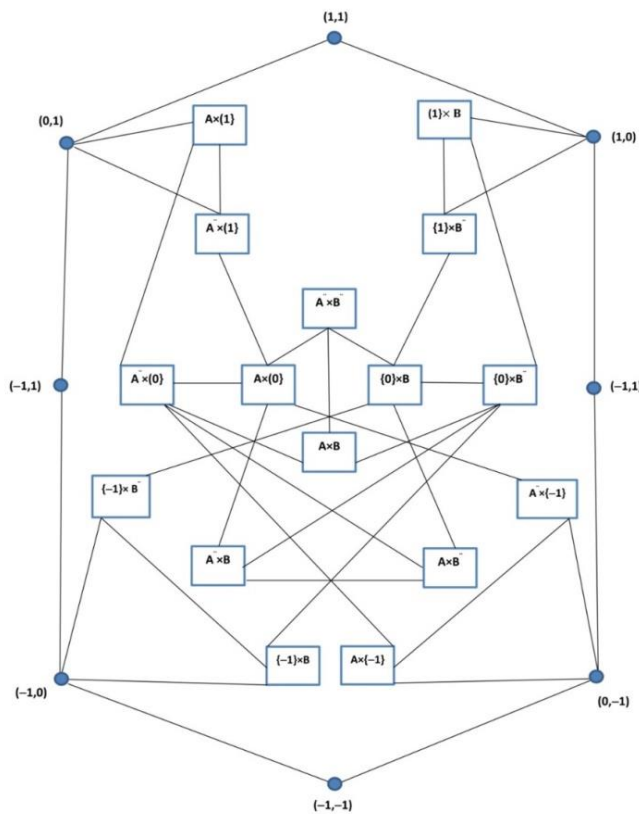


Figure 3.5 The general form of $\mathfrak{R}_\partial(\mathbb{Z}_p \times \mathbb{Z}_q)$ for any prime numbers with $p, q \geq 5$

Theorem 3.5 Let p and q are prime numbers with $p, q \geq 5$, $p < q$, and $A, A^{\sim}, B$ and $B^{\sim}$ are the sets who's defined in Definition 3.2. If $(x, y) \in Z_p \times Z_q - \{(0, 0)\}$, then:

$$\deg_{\mathfrak{R}_\partial(Z_p \times Z_q)}(x, y) = \begin{cases} 2 & : \text{if } x \in \{1, -1\} \text{ and } y \in \{1, -1\} \\ 3 & : \text{if } x \in A \cup A^{\sim} \text{ and } y \in \{1, -1\} \\ & \text{or } x \in \{1, -1\} \text{ and } y \in B \cup B^{\sim} \\ & \text{or } x \in A \cup A^{\sim} \text{ and } y \in B \cup B^{\sim} \\ p-1 & : \text{if } x = 0 \text{ and } y \in \{1, -1\} \\ q-1 & : \text{if } x \in \{1, -1\} \text{ and } y = 0 \\ p & : \text{if } x = 0 \text{ and } y \in B \cup B^{\sim} \\ q & : \text{if } x \in A \cup A^{\sim} \text{ and } y = 0 \end{cases}.$$

Proof: Let $(x, y) \in Z_p \times Z_q - \{(0, 0)\} = (A \cup A^{\sim} \cup \{0, 1, -1\}) \times (B \cup B^{\sim} \cup \{0, 1, -1\}) - \{(0, 0)\}$. By Proposition 3.1 and Figure 3.5 we have the following cases:

Case1: if $(x, y) \in \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$. Since $(1, 1)$ adjacent with only $(1, 0)$ and $(0, 1)$ then $N((1, 1)) = \{(1, 0), (0, 1)\}$, similarly $N((1, -1)) = \{(1, 0), (0, -1)\}$, $N((-1, 1)) = \{(-1, 0), (0, 1)\}$ and $N((-1, -1)) = \{(-1, 0), (0, -1)\}$. Then $\deg_{\mathfrak{R}_\partial(Z_p \times Z_q)}(x, y) = 2$ in this case.

Case2: if $(x, y) \in \{(x, 1) : x \in A\} \cup \{(x, -1) : x \in A\} \cup \{(x^{-1}, 1) : x^{-1} \in A^{\sim}\} \cup \{(x^{-1}, -1) : x^{-1} \in A^{\sim}\}$. Since $(x, 1)$ adjacent only with the vertices $(0, 1)$, $(x^{-1}, 0)$ and $(x^{-1}, 1)$ for any $x \in A$, then $N((x, 1)) = \{(0, 1), (x^{-1}, 0), (x^{-1}, 1)\}$, thus $|N((x, 1))| = 3$ for every $x \in A$. Similarly, $|N((x, -1))| = |N((x^{-1}, 1))| = |N((x^{-1}, -1))| = 3$ for every $x \in A$ and $x^{-1} \in A^{\sim}$. Thus $\deg_{\mathfrak{R}_\partial(Z_p \times Z_q)}(x, y) = 3$ in this case.

Case3: if $(x, y) \in \{(1, y) : y \in B\} \cup \{(-1, y) : y \in B\} \cup \{(1, y^{-1}) : y^{-1} \in B^{\sim}\} \cup \{(-1, y^{-1}) : y^{-1} \in B^{\sim}\}$. Similarly as case2 above we get $|N((1, y))| = |N((-1, y))| = |N((1, y^{-1}))| = |N((-1, y^{-1}))| = 3$ for any $y \in B$ and $y^{-1} \in B^{\sim}$. Thus $\deg_{\mathfrak{R}_\partial(Z_p \times Z_q)}(x, y) = 3$ in this case.

Case4: if $(x, y) \in \{(x, y) : x \in A \cup A^{\sim}, y \in B \cup B^{\sim}\}$. Clearly, $N((x, y)) = \{(x^{-1}, 0), (0, y^{-1}), (x^{-1}, y^{-1})\}$, hence $|N((x, y))| = 3$ for every $x \in A$ and $y \in B$, similarly $|N((x, y^{-1}))| = |N((x^{-1}, y))| = |N((x^{-1}, y^{-1}))| = 3$ for any $x \in A \cup A^{\sim}$ $y \in B \cup B^{\sim}$. Thus $\deg_{\mathfrak{R}_\partial(Z_p \times Z_q)}(x, y) = 3$ in this case.

Case5: if $(x, y) \in \{(0, 1), (0, -1)\}$. Since $(0, 1)$ adjacent with $(a, 1)$ for any $a \in Z_p - \{0\}$, then $N((0, 1)) = \{(a, 1) \in Z_p \times Z_q - \{(0, 0)\} : a \in Z_p - \{0\}\}$, similarly $N((0, -1)) = \{(a, -1) \in Z_p \times Z_q - \{(0, 0)\} : a \in Z_p - \{0\}\}$. Then $\deg_{\mathfrak{R}_\partial(Z_p \times Z_q)}(x, y) = |N((0, 1))| = |N((0, -1))| = p-1$.

Case6: if $(x, y) \in \{(1, 0), (-1, 0)\}$. Similarly, as case5 above, $\deg_{\mathfrak{R}_\partial(Z_p \times Z_q)}(x, y) = |N((1, 0))| = |N((-1, 0))| = q-1$.

Case7: if $(x, y) \in \{(0, y) : y \in B \cup B^{\sim}\}$. Since $(0, b)$ adjacent with (a, b^{-1}) for any $a \in Z_p$ then $N((0, y)) = \{(a, y^{-1}) : a \in Z_p\}$ and $N((0, y^{-1})) = \{(a, y) : a \in Z_p\}$, then $\deg_{\mathfrak{R}_\partial(Z_p \times Z_q)}(x, y) = |N((0, y))| = |N((0, y^{-1}))| = p$.

Case8: if $(x,y) \in \{(x,0) : x \in A \cup A^c\}$. Similarly, as case7 above, $\deg_{\mathcal{R}_\partial(Z_p \times Z_q)}(x,y) = |N((x,0))| = |N(x^{-1},0)| = q$. ■

Proposition 3.6 Let p be prime number with $p \geq 5$, and A, A^c, B and B^c are the sets whose defined in Definition3.2. If $(x,y) \in Z_p \times Z_p - \{(0,0)\}$, then:

$$\deg_{\mathcal{R}_\partial(Z_p \times Z_p)}(x,y) = \begin{cases} 2 & : \text{if } x \in \{1, -1\} \text{ and } y \in \{1, -1\} \\ 3 & : \text{if } x \in A \cup A^c \text{ and } y \in \{1, -1\} \\ & \text{or } x \in \{1, -1\} \text{ and } y \in A \cup A^c \\ & \text{or } x, y \in A \cup A^c \\ p-1 & : \text{if } x = 0 \text{ and } y \in \{1, -1\} \\ & \text{or } x \in \{1, -1\} \text{ and } y = 0 \\ p & : \text{if } x = 0 \text{ and } y \in A \cup A^c \\ & \text{or } x \in A \cup A^c \text{ and } y = 0 \end{cases}$$

Definition 3.7 Let G be a graph, define the subsets V_i of $V(G)$ for $\delta(G) \leq i \leq \Delta(G)$ by $V_i = \{v \in V(G) : \deg_G(v) = i\}$. Directly by definition, we see that $V(G) = \bigcup_{i=\delta(G)}^{\Delta(G)} V_i$.

Now, we give the main results of this paper:

Theorem 3.8 $M(\mathcal{R}_\partial(Z_2 \times Z_q), x, y) = 2xy^2 + \frac{1}{2}(q-3)x^2y^2 + (q-3)x^2y^3 + 2x^2y^{q-1} + \frac{1}{2}(q-3)x^3y^3 + (q-3)x^3y^{q-1}$.

Proof: By Proposition3.3, $V_1 = \{(0,1), (0,-1)\}$, $V_2 = \{(0,y) \in Z_p \times Z_q - \{(0,0)\} : y \in B \cup B^c\} \cup \{(1,1), (1,-1)\}$, $V_3 = \{(1,y) \in Z_p \times Z_q - \{(0,0)\} : y \in B \cup B^c\}$ and $V_{q-1} = \{(1,0)\}$. By Figure3.3(A) we have: $\langle V_1 \rangle \cong 2K_1$ then $|E_{1,1}| = 0$, $\langle V_2 \rangle \cong |B|K_2 \cup 2K_1 = \frac{1}{2}(q-3)K_2 \cup 2K_1$, then $|E_{2,2}| = \frac{1}{2}(q-3)$, $\langle V_3 \rangle \cong \frac{1}{2}(q-3)K_2$ then $|E_{3,3}| = \frac{1}{2}(q-3)$, and $\langle V_{q-1} \rangle \cong K_1$ then $|E_{q-1, q-1}| = 0$. Also, $E_{1,2} = \{(0,1), (1,1)\}, \{(0,-1), (1,-1)\}$ then $|E_{1,2}| = 2$, $|E_{1,3}| = |E_{1,q-1}| = 0$. $E_{2,3} = E(\langle V_2 \cup V_3 \rangle) - E(\langle V_2 \rangle \cup \langle V_3 \rangle)$, by Figure3.3(A) $\langle V_2 \cup V_3 \rangle - E(\langle V_2 \rangle \cup \langle V_3 \rangle) \cong 2|B|K_2 = (q-3)K_2$, then $|E_{2,3}| = q-3$, $E_{2, q-1} = \{(1,0), (1,1)\}, \{(1,0), (1,-1)\}$ then $|E_{2, q-1}| = 2$. Finally, $E_{3, q-1} = E(\langle V_3 \cup V_{q-1} \rangle) - E(\langle V_3 \rangle \cup \langle V_{q-1} \rangle)$, by Figure3.3(A) $\langle V_3 \cup V_{q-1} \rangle - E(\langle V_3 \rangle \cup \langle V_{q-1} \rangle) \cong K_1 + 2|B|K_1 = K_{1,q-3}$, then $|E_{3, q-1}| = q-3$. ■

Theorem 3.9 $M(\mathcal{R}_\partial(Z_3 \times Z_q), x, y) = 4x^2y^2 + 7(\frac{1}{2}(q-3))x^3y^3 + 4x^3y^{q-1}$.

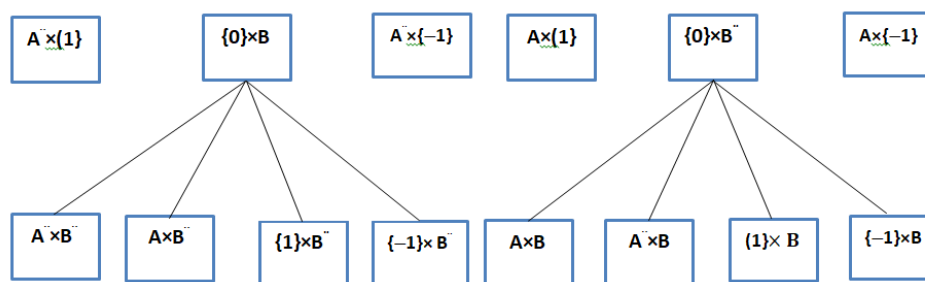
Proof: Similarly, by Figure3.3 (B), and as a proof of a last theorem, we get $|E_{2,2}| = 4, |E_{3,3}| = 7(\frac{1}{2}(q-3)), |E_{q-1, q-1}| = |E_{2,3}| = |E_{2, q-1}| = 0, |E_{3, q-1}| = 4$. ■

Theorem 3.10 $M(\mathcal{R}_\partial(Z_p \times Z_q), x, y) = 4x^2y^{p-1} + 4x^2y^{q-1} + \frac{1}{2}(pq-p-q-3)x^3y^3 + 2(q-1)x^3y^{p-1} + 2(p-1)x^3y^{q-1} + (p-1)(q-1)x^3y^p + (p-1)(q-1)x^3y^q + \frac{1}{2}(q-3)x^py^p + \frac{1}{2}(p-3)x^qy^q$.

Proof: By Theorem3.5, $V_2 = \{(x,y) \in Z_p \times Z_q - \{(0,0)\} : x \in \{1, -1\} \text{ and } y \in \{1, -1\}\}$, $V_3 = \{(x,y) \in Z_p \times Z_q - \{(0,0)\} : x \in A \cup A^c \text{ and } y \in \{1, -1\} \text{ or } x \in \{1, -1\} \text{ and } y \in B \cup B^c \text{ or } x, y \in A \cup A^c \text{ and } y = 0\}$

$x \in A \cup A^{\sim}$ and $y \in B \cup B^{\sim}$, $V_{p-1} = \{(x, y) \in Z_p \times Z_q - \{(0, 0)\} : x = 0 \text{ and } y \in \{1, -1\}\}$,
 $V_{q-1} = \{(x, y) \in Z_p \times Z_q - \{(0, 0)\} : x \in \{1, -1\} \text{ and } y = 0\}$, $V_p = \{(x, y) \in Z_p \times Z_q - \{(0, 0)\} : x = 0 \text{ and } y \in B \cup B^{\sim}\}$, and $V_q = \{(x, y) \in Z_p \times Z_q - \{(0, 0)\} : x \in A \cup A^{\sim} \text{ and } y = 0\}$. Clearly, $\bigcup_{i=2}^q V_i = Z_p \times Z_q - \{(0, 0)\}$. By Theorem 2.3, Theorem 3.5 and Figure 3.5, we have $\langle V_2 \rangle \cong 4K_1$, $\langle V_3 \rangle \cong \frac{1}{2}[2(p-3)+2(q-3)+(p-3)(q-3)] K_2$, $\langle V_{p-1} \rangle \cong \langle V_{q-1} \rangle \cong 4K_1$, $\langle V_p \rangle \cong |B|K_2 = \frac{1}{2}(q-3) K_2$ and $\langle V_q \rangle \cong |A|K_2 = \frac{1}{2}(p-3) K_2$.

Now, $E_{2,2} = E(\langle V_2 \rangle) = E(4K_1)$, thus $|E_{2,2}| = 0$. $E_{2,3} = E(\langle V_2 \cup V_3 \rangle) - E(\langle V_2 \rangle \cup \langle V_3 \rangle)$, since there is no vertex in V_2 adjacent with any vertex in V_3 , then $|E_{2,3}| = 0$. Clearly, $E_{2,p-1} = E(\langle V_2 \cup V_{p-1} \rangle) - E(\langle V_2 \rangle \cup \langle V_{p-1} \rangle)$, by Figure 3.5 we get $\langle V_2 \cup V_{p-1} \rangle = 2P_3 \cup 8K_1$, thus $|E_{2,p-1}| = |E(2P_3)| = 4$. Similarly, $|E_{2,q-1}| = 4$. Clearly, $E_{2,p} = E(\langle V_2 \cup V_p \rangle) - E(\langle V_2 \rangle \cup \langle V_p \rangle)$, and $E_{2,q} = E(\langle V_2 \cup V_q \rangle) - E(\langle V_2 \rangle \cup \langle V_q \rangle)$, since there is no vertex in V_2 adjacent with any vertex in V_i for $i \in \{p, q\}$, then $|E_{2,p}| = |E_{2,q}| = 0$. $E_{3,3} = E(\langle V_3 \rangle)$, thus $|E_{3,3}| = |E(\langle V_3 \rangle)| = \frac{1}{2}[2(p-3)+2(q-3)+(p-3)(q-3)] = \frac{1}{2}(pq-p-q-3)$. Clearly $E_{3,p-1} = E(\langle V_3 \cup V_{p-1} \rangle) - E(\langle V_3 \rangle \cup \langle V_{p-1} \rangle)$, by Theorem 2.3 and Figure 3.5 we see that the vertex $(0, 1)$ adjacent only with the vertices of $\{(x, y) : x \in A \cup A^{\sim} \cup \{1, -1\}, y = 1\}$ in the subgraph $\langle V_3 \cup V_{p-1} \rangle$, so the vertex $(0, -1)$ adjacent only with all vertices of $\{(x, y) : x \in A \cup A^{\sim} \cup \{1, -1\}, y = -1\}$, thus the vertex $(0, 1)$ adjacent only with $(\frac{p-3}{2} + \frac{p-3}{2} + 2)$ vertices, also the vertex $(0, -1)$ adjacent only with $(\frac{q-3}{2} + \frac{q-3}{2} + 2)$ vertices of $\langle V_3 \cup V_{p-1} \rangle$, therefore thus $|E_{3,p-1}| = 2(\frac{q-3}{2} + \frac{q-3}{2} + 2) = 2(q-1)$. Similarly, $|E_{3,q-1}| = 2(p-1)$. To find $E_{3,p}$ we must find the general form of the graph $\langle V_3 \cup V_p \rangle - E(\langle V_3 \rangle \cup \langle V_p \rangle)$ as follow:



Thus $\langle V_3 \cup V_p \rangle - E(\langle V_3 \rangle \cup \langle V_p \rangle) \cong [\frac{1}{2}(q-3)K_1 + (p-1)K_1] \cup [\frac{1}{2}(q-3)K_1 + (p-1)K_1] \cup 2(p-3)K_1$. Therefore $E_{3,p} = E(\frac{1}{2}(q-3)K_1 + (p-1)K_1) \cup [\frac{1}{2}(q-3)K_1 + (p-1)K_1] = E(2K_{\frac{1}{2}(q-1), p-1})$, hence $|E_{3,p}| = 2[\frac{1}{2}(q-1)(p-1)] = (p-1)(q-1)$. Similarly, $|E_{3,q}| = (p-1)(q-1)$.

Clearly, $|E_{p-1,q-1}| = |E_{p-1,p-1}| = |E_{p-1,q}| = |E_{q-1,p}| = |E_{q-1,q-1}| = |E_{q-1,q}| = 0$,
 $|E_{p,p}| = \frac{1}{2}(q-3)$, $|E_{p,q}| = 0$ and $|E_{q,q}| = \frac{1}{2}(p-3)$.

IV Conclusion and future directions

In this paper, we study some properties of the regular divisor graph of the commutative ring R , $\mathfrak{R}_\partial(R)$, where R is a field, local ring, Boolean ring, and reduced ring. And we find the M-polynomial and the general form of $\mathfrak{R}_\partial(R)$, where $R = \mathbb{Z}_p \times \mathbb{Z}_q$, for any prime numbers p and q .

Building upon the results presented in this work, we propose the following directions for future research:

- 1- Computing M-polynomials for other divisor graphs.
- 2- Studying the regular divisor direct graph of the non-commutative ring.

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