JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 9 NO. 1 (2026)

ON CONDITIONS FOR WHICH A FRECHET SPACE NOT CONTAINING C_0 ADMITS THE GELFAND-PHILLIPS PROPERTY

Amos Otieno Wanjara[1*]

[1]*Kaimosi Friends University, Kenya.*
School of Science, Department of Mathematics and Statistics
Email: [1]awanjara78@gmail.com, [1] awanjara@kafu.ac.ke
[*] Corresponding author

Abstract. If X is a Gelfand-Phillips space, then every sequence $(x_n:n\in\mathbb{N})\subset X$ which is equivalent to $(x_n:n\in\mathbb{N})$ contains an infinite subsequence $(x_n:n\in\mathbb{N})$ such that Span $(x_n:n\in\mathbb{N})$ is complemented in X. C_0 is the space of all sequences $a=\{a_n\}_{n\in\mathbb{N}}$ such that $a_n\to 0$ as $n\to\infty$. The norm is $\|a\|_\infty = \sup_n |a_n|\ \forall\ a_n\in C_0$. It is known that if every limited set of X is relatively weakly compact, then every copy of C_0 in X contains a subspace still isomorphic to C_0 which is complemented in X. A Banach space is said to have the Gelfand-Phillips property if every limited subset is relatively norm compact. The question whether if a Fre'chet space does not contain C_0 implies it can admit or have the Gelfand-Phillips property is still open. The purpose of this study was to establish the conditions for which a Fre'chet Space can admit the Gelfand-Phillips property without C_0. We investigated the conditions on which a Fre'chet space not containing C_0 admits the Gelfand-Phillips property. The methodology involved the use of literature study related to Fre'chet spaces and Gelfand-Phillips property. The conditions which imply that a Fre'chet space Ψ has the Gelfand-Phillips property in spite of it not containing C_0 were given. It was established that if a Fre'chet space Ψ has a compact subspace $\left(\upsilon(\Psi'),\ \eta(\Psi',\Psi)\right)$, then it admits the Gelfand-Phillips property even if it doesn't contain C_0 . The results in this study have a momentous contribution in the field of Mathematical Analysis.

Keywords: Gelfand-Phillips Property, Fre'chet Space, Limited Set, relatively norm compact

I. INTRODUCTION

A space X has the Gelfand-Phillips (GP) property if every limited subset of X is relatively compact. The following spaces have the Gelfand-Phillips property: Schur spaces; spaces with $w^* - sequential\ compact\ dual\ unit\ balls$ (for example subspaces of weakly compactly generated spaces, separable spaces, spaces whose duals have the Radon- Nikodym property, reflexive spaces and spaces whose duals do not contain ℓ_1); dual spaces X^* with X not containing ℓ_1; Banach spaces with the separable complementation property, i.e. every separable subspace is contained in a complemented separable subspace(for example $L_1(\mu)$ spaces), where μ is a positive measure (see [1], [2], [3], [4]). In another direction, a Banach space is said to have the Gelfand-Phillips property if every limited subset is relatively norm compact (see [1], [2], [5]). For instance, every Banach space having $w^* - sequentially$ compact dual ball (e.g. WCG or Asplund spaces see [6]) or having the separable complementation property enjoys the Gelfand-Phillips property. In addition, a Banach space X *has* the separable complementation property (SCP) if every separable subspace of X is

contained in a separable complemented subspace of X. Typical examples of Banach spaces with the SCP are WCG spaces and Banach lattice not containing C_0. The following open problem is directly connected with a question on Pettis integration raised by Talagrand [7]. Let X be a Banach space not containing C_0. Does X have the Gelfand-Phillips property? In this study, we give conditions which imply that a Fre'chet space X has the Gelfand-Phillips property in spite of not containing C_0. Where C_0 is the space of all sequences $a = \{a_n\}_{n\in\mathbb{N}}$ such that $a_n \to 0$ as $n \to \infty$ with the norm is $\| a \|_\infty = \sup_n |a_n| \ \forall\ a_n \in C_0$.

The purpose of this study was to establish the conditions for which a Fre'chet Space can admit the Gelfand-Phillips property without C_0. We investigated the conditions on which a Fre'chet space not containing C_0 admits the Gelfand-Phillips property.

II. PRELIMINARIES

In this section, we give the fundamental concepts underlying this research that are very important in understanding the Gelfand-Phillips property and the general Fre'chet space.

Definition 1 [1] *A Banach space E is Gelfand-Phillips if for every $B \in BNP(E)$, there is a $weak^*$ null sequence $(x_n)_{n\in w}$ in E' such that $\| x_n \|_B \nrightarrow 0$.*

Definition 2 [8] *A subset A of a Banach space E is limited in E if every $\boldsymbol{weak^*}$ null sequence in $\boldsymbol{E'}$ converges uniformly to zero(0) on A.*

Definition 3 [8] *A subset of a Banach space X is called limited (resp. Dunford- Pettis(DP)), if every $\boldsymbol{weak^*}$ null(resp. weak null) sequence $\boldsymbol{weak^*}$ null $(\boldsymbol{x_n^*})$ in $\boldsymbol{X^*}$ converges uniformly on A, that is*

$$\boldsymbol{\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^* \rangle| = 0.}$$

Definition 4[6] *If $\boldsymbol{A \subseteq X^*}$ and every weak null sequence $\boldsymbol{(x_n)}$ in X converges uniformly on A, we say that A is an L-set.*

Remark 1 Every relatively compact subset of X is limited and clearly every limited set is DP and every DP subset of a dual Banach space is an L-set, but the converse of these assertions, in general are false. If every limited subset of a Banach space X is relatively compact, then X has the Gelfand-Phillips(GP) property and X has the relatively compact Dunford- Pettis property (DPrcP), if every DP set in X is relatively Compact.

Definition 5 [7] *A Banach space E has the Gelfand-Phillips property $\boldsymbol{E \in (GP)}$ if every limited set in E is relatively compact.*

Proposition 1[2] *If the dual unit ball $\boldsymbol{B_1(X')}$ of a Banach space X contains a $\boldsymbol{weak^*}$sequential pre-compact subset. D which norms the elements of X up to a constant $\boldsymbol{c > 0}$ (i.e if $\boldsymbol{\| x \| \leq c\sup_{x'\in D} |\langle x', x\rangle| \forall\ x \in X}$), then X has the Gelfand-Phillips property.*

The following propositions connect limited sets and the Gelfand-Phillips property.

Proposition 2[3] *Let A be a set. For* $\boldsymbol{A \subset X}$*, the following conditions are equivalent:*

i) A is limited in X

ii) For every sequence $(\boldsymbol{x_n}: \boldsymbol{n} \in \mathbb{N}) \subset \boldsymbol{A}$ *and for every* $\boldsymbol{\sigma(X', X)}$ *−zero sequence,* $\boldsymbol{\lim_{n\to\infty} \langle x_n', x_n \rangle = 0}$

iii) T(A) is relatively compact for every $\boldsymbol{T \in L(X, C_0)}$*. In particular, we conclude that limitedness is countably determined.*

Proposition 3[4] *For* $\boldsymbol{A \subset X}$*, the following conditions are equivalent:*

a) A is limited in X

b)i) A is conditionally $\boldsymbol{\sigma(X, X')}$ *−compact*

ii) each $\boldsymbol{\sigma(X, X')}$ *− zero sequence in a* $\boldsymbol{C_0(A)}$ *is limited in X. In particular, we conclude that X is Gelfand-Phillips if and only if every normed, weakly to zero converging sequence is not limited in X.*

Proposition 4 [5] *Let A and B be subsets of X*

a) If A and B are limited in, then the sets $\boldsymbol{A \cup B, A + B, \overline{A}}$ *and* $\boldsymbol{C_0(A)}$ *have the same property.*

b) Let $\boldsymbol{A \subset B}$*. If B is limited in X, so is A.*

c) Let $\boldsymbol{T \in L(X, Y)}$*; if A is X-limited, then T(A) is Y-limited.*

Lemma 1 [6] *Let A be a bounded subset of a Banach space E and consider a semi norm* $\boldsymbol{\varphi: E^* \to \mathbb{R}_+}$ *given by the formula*

$$\boldsymbol{\varphi^* = \sup_{a \in A} |x^*(a)|}$$

Then: i) $\boldsymbol{\varphi}$ *is* $\boldsymbol{weak^*}$ *sequentially continuous if and only if A is limited.*

ii) $\boldsymbol{\varphi}$ *is* $\boldsymbol{weak^*}$ *continuous on* $\boldsymbol{B_{E^*}}$ *if and only if A is relatively norm compact.*

Definition 6 [9, Definition 1.5] *A Fre'chet space (F for short) is a locally convex topological vector space.*

Trivial example of a Fre'chet space is a Banach space. Every Banach space, in particular,ℓ^P $\forall$ $1 \leq p \leq \infty$ is a Fre'chet space.

Definition 7[10, Definition 3.0 (i)] *The space* C_0 *is the space of all sequences* $a = \{a_n\}_{n\in\mathbb{N}}$ *such that* $a_n \to 0$ *as* $n \to \infty$*. The norm is* $\| a \|_\infty = \sup_n |a_n|$ $\forall$ $a_n \in C_0$.

Definition 8[10, Definition 5.0] *The dual space of* C_0*,* C_0^* *is isometrically isomorphic to* ℓ^1*. The isomorphism is given as follows. For* $a = \{a_n\} \in \ell^1$*, we define an element* $\alpha \in (C_0^*)$ *by* $\alpha(b) = \sum_n a_n b_n$.

Definition 9[10, Definition 3.0(iii)] *The space* $\ell^P \; \forall \; 1 \leq p < \infty$ *is a space of all sequences* $a = \{a_n\}_{n\in\mathbb{N}}$ *such that* $\sum_{n=1}^{\infty} |\, a_n|^p$ *is convergent. The norm is* $\| \, a \, \|_p = (\sum_{n=1}^{\infty} |\, a_n|^p)^{\frac{1}{p}}$.

Definition 10[11, definition 4] *For a Fre'chet space* $(F, \|.\|)$, $B_1(F)$ *denotes its closed ball,* $S(F)$ *denotes the surface of* $B_1(F)$.

The following notations are used in the results and Discussions section:
i) $(\boldsymbol{\Psi}, \|.\|)$ or $\boldsymbol{\Psi}$ - denotes a Fre'chet space
ii) $C(\boldsymbol{\Psi})$ - denotes the closed unit ball of $(\boldsymbol{\Psi}, \|.\|)$
iii) $S(\boldsymbol{\Psi})$ - denotes the surface of $C(\boldsymbol{\Psi})$
iv) $\big(\mho(\boldsymbol{\Psi}'), \eta(\boldsymbol{\Psi}', \boldsymbol{\Psi})\big)$-denotes a compact subspace in $(\boldsymbol{\Psi}, \|.\|)$

III. RESULT AND DISCUSSION

In this section we presented the results and discussions on the research. This section presents theorems showing the conditions for which a Fre'chet space not containing C_0 admits the Gelfand-Phillips property. **Theorem 1** establishes that if a Fre'chet space $\boldsymbol{\Psi}$ has a compact subspace $\big(\mho(\boldsymbol{\Psi}'), \eta(\boldsymbol{\Psi}', \boldsymbol{\Psi})\big)$ then it has the Gelfand-Phillips property even if it doesn't admit C_0. Furthermore, in **Theorem 2,** it is shown that provided a Fre'chet space is separable as a complemented sum of two Gelfand-Phillips spaces whose compact operators are limited, then it admits Gelfand-Phillips property.

Theorem 1. *Let* $(\boldsymbol{\Psi}, \|.\|)$ *be a Fre'chet space,* $C(\boldsymbol{\Psi})$ *a closed ball in* Ψ *and* $\big(\mho(\boldsymbol{\Psi}'), \eta(\boldsymbol{\Psi}', \boldsymbol{\Psi})\big)$ *a compact subspace in* $(\boldsymbol{\Psi}, \|.\|)$ *then* $\boldsymbol{\Psi}$ *has the Gelfand-Phillips property even if it doesn't contain* C_0 *provided the following conditions hold:*

1. $C(\boldsymbol{\Psi}')$ *is* $\eta(\boldsymbol{\Psi}', \boldsymbol{\Psi})$ *sequentially compact.*
2. ***By Lemma 1****,* $C(\boldsymbol{\Psi}')$ *contains a* $\eta(\boldsymbol{\Psi}', \boldsymbol{\Psi})$ *sequentially pre-compact subset of* $\mho$ *which norms* $\boldsymbol{\Psi}$

   i.e. $\| \boldsymbol{\psi} \| = \underset{\psi' \in \mho}{Sup} \big|\langle \psi', \boldsymbol{\psi} \rangle\big| \; \forall \; \boldsymbol{\psi} \in \boldsymbol{\Psi}$**.**
3. $\exists$ *a compact set* $\mathfrak{H}_{|\boldsymbol{\Psi}}$ *containing a sequentially pre- compact and dense subset such that* $\boldsymbol{\Psi}$ *can be isometrically embedded in* $\mho(\mathfrak{H}_{|\boldsymbol{\Psi}})$
4. $\exists$ *an equivalent norm* $\|.\|$ *on* $\boldsymbol{\Psi}$ *such that* $(\boldsymbol{\Psi}, \|.\|)$ *satisfies condition (2)*
5. $\exists$ *an equivalent norm* $\|.\|$ *on* $\boldsymbol{\Psi}$ *such that* $(\boldsymbol{\Psi}, \|.\|)$ *satisfies condition (3)*
6. $\exists$ *a* $\beta > 0$ *and sequentially* $\eta(\boldsymbol{\Psi}', \boldsymbol{\Psi})$ *pre- compact subset* $\mho$ *of* $C(\boldsymbol{\Psi}')$ ***by Proposition 1****such that*

   $\| \boldsymbol{\psi} \| \leq \boldsymbol{\beta} \underset{\psi' \in \mho}{Sup} \big|\langle \psi', \boldsymbol{\psi} \rangle\big| \; \forall \; \boldsymbol{\psi} \in \boldsymbol{\Psi}$**.**
7. $\boldsymbol{\Psi}$ *has the Gelfand-Phillips property.*

Proof. (1⟹ 2) is trivial from the hypothesis.
(2⟹ 3): Let $\mho \subset C(\boldsymbol{\Psi}')$be as in (2). Suppose we set $\mathfrak{H}_{|\boldsymbol{\Psi}} \coloneqq \mho^{-\eta}(\boldsymbol{\Psi}', \boldsymbol{\Psi})$ then $\mathfrak{H}_{|\boldsymbol{\Psi}}{}'$ furnished with $\eta(\boldsymbol{\Psi}', \boldsymbol{\Psi}) \cap \mathfrak{H}_{|\boldsymbol{\Psi}}$ has the desired properties and the operator $\vartheta: \boldsymbol{\Psi} \to \mho(\mathfrak{H}_{|\boldsymbol{\Psi}}), \boldsymbol{\psi} \mapsto (\mathfrak{H}_{|\boldsymbol{\Psi}} \ni \boldsymbol{\psi}' \mapsto \langle \boldsymbol{\psi}', \boldsymbol{\psi} \rangle)$**,** is an isometric embedding.
(3⟹ 2): Let $\mathfrak{H}_{|\boldsymbol{\Psi}}$ be a compact space which contains a dense sequentially pre-compact subset $\widetilde{\mathfrak{H}_{|\boldsymbol{\Psi}}}$ and admits an isometric embedding $\vartheta: \boldsymbol{\Psi} \to \mho(\mathfrak{H}_{|\boldsymbol{\Psi}})$, then the set $\mho \coloneqq \vartheta'(\{\mu_\lambda | \lambda \in \widetilde{\mathfrak{H}_{|\boldsymbol{\Psi}}}\})$ satisfies the conditions of (2).
(2⟹ 4) is trivial from the hypothesis.
(4⟹ 5) follows as in (2⟺ 3)
(4⟹ 6) is a gain trivial from the hypothesis
(6⟹ 7): **By proposition 3**, it suffices to show that a given normed $\eta(\boldsymbol{\Psi}', \boldsymbol{\Psi})$ zero sequence $(\psi_0 : n \in \mathbb{N})$ is not limited in $\boldsymbol{\Psi}$**.** Suppose that $\beta > 0$ and $\mho \subset C(\boldsymbol{\Psi}')$ are as in condition (6). Then there exists $\forall\, n \in \mathbb{N}$ and $\boldsymbol{\psi}'_n \mho$ with $|\langle \boldsymbol{\psi}'_n \boldsymbol{\psi}_n \rangle| > \frac{\beta}{2}$. By condition (6), we may assume that $\boldsymbol{\psi}'_n : n \in \mathbb{N}$ converges in $\eta(\boldsymbol{\Psi}', \boldsymbol{\Psi})$ to an $\boldsymbol{\psi}'_0 \in \boldsymbol{\Psi}'$ and since $(\psi_n : n \in \mathbb{N})$ converges in $\eta(\boldsymbol{\Psi}', \boldsymbol{\Psi})$ to 0, we claim that $|\langle \boldsymbol{\psi}'_0, \boldsymbol{\psi}_n \rangle| > \frac{\beta}{4}\ \forall\, n \in \mathbb{N}$; thus we have $\lim_{n\to\infty} |\langle x'_n - x'_0, x_n \rangle| \geq \frac{\beta}{4}$ **by proposition 2.** ■

Theorem 2. *Suppose* $(\boldsymbol{\Psi}, \|.\|)$ *is a separable Fre'chet space with complemented sum of two Gelfand-Phillips spaces and* $\mho'$ *is a compact subspace of* $\boldsymbol{\Psi}$ *contained in another separable subspace* $\mho'' \in \boldsymbol{\Psi}$ *such that the limited operators from* $\mho''$ *to* $\boldsymbol{\Psi}$ *are compact then in general,* $\boldsymbol{\Psi}$ *admits Gelfand-Phillips property if the following conditions hold:*

1. $\boldsymbol{\Psi}$ *is Gelfand-Phillips*
2. *Every compact subspace* $\mho$ *of* $\boldsymbol{\Psi}$ *is Gelfand- Phillips*
3. *Every separable compact subspace* $\mho'$ *of* $\boldsymbol{\Psi}$ *is contained in a subspace* $\mho'' \in \boldsymbol{\Psi}$ *which has the Gelfand-Phillips property and is complemented in* $\boldsymbol{\Psi}$**.**
4. $\boldsymbol{\Psi}$ *is the complemented sum of two Gelfand-Phillips spaces.*
5. *For every Fre'chet space* $\mho''$*, the limited operators from* $\mho''$ *to* $\boldsymbol{\Psi}$ *are compact.*
6. *Every limited operator from* ℓ^1 *to* $\boldsymbol{\Psi}$ *is compact.*

Proof. (1⟹ 2) follows from proposition 4(c)
(2⟹ 3) is trivial from the hypothesis
(3⟹ 1): here it suffices to show that a non-compact sequence $(\psi_n : n \in \mathbb{N})$ in $\boldsymbol{\Psi}$ is not limited whenever $\boldsymbol{\Psi}$ satisfies (3). By hypothesis from (3) there exists a complemented subspaces $\mho''$ of $\boldsymbol{\Psi}$ satisfying the Gelfand-Phillips property and containing $(\psi_n : n \in \mathbb{N})$. Given that $(\psi_n : n \in \mathbb{N})$ is not limited in $\mho''$ and from Proposition 4(c) then it can't be limited in $\boldsymbol{\Psi}$ since it is the preimage of the projection from $\boldsymbol{\Psi}$ onto $\mho''$ of a non-limited set.
(1⟹ 4) follows from the fact that $\boldsymbol{\Psi} = \boldsymbol{\Psi} \oplus \{\mathbf{0}\}$
(4⟹ 1**)** follows from proposition 4(c)
(1⟹ 5) is trivial from the hypothesis
(5⟹ 6) is again trivial from the hypothesis

(6⟹ 1): In this case, by Proposition 2(iii), let $(\psi_n : n \in \mathbb{N})$ be limited in $\mathbf{\Psi}$ and claim that $\mathbf{\Psi}$ satisfies condition (6); it suffices to show that $(\psi_n : n \in \mathbb{N})$ is relatively compact. The operator $\vartheta : \ell^1 \to \mathbf{\Psi}$,
$\mu = \mu_n \longmapsto \sum_{n\in\mathbb{N}}^{n} \mu_n \psi_n$ is bounded and linear and $\vartheta(\mho, \ell^1) \not\subset a\ C_0\overline{(\psi_n : n \in \mathbb{N})}$ is limited in $\mathbf{\Psi}$. Given that every limited operator $\vartheta : \ell^1 \to \mathbf{\Psi}$ is compact and since $(\psi_n : n \in \mathbb{N}) \subset \vartheta(\mho, \ell^1)$, $(\psi_n : n \in \mathbb{N})$ is relatively compact. ■

IV. CONCLUSIONS

This study establishes some conditions for which a Fre'chet space not containing C_0 admits the Gelfand-Phillips property. The results are obtained using the definitions, propositions and lemmas given in the methodology section. A different way to construct a Fre'chet space and the conditions of Gelfand-Phillips property is given. This makes mathematical analysis more complete.

REFERENCES

[1] L. Drewnowski, On Banach spaces with the Gelfand-Phillips property, Math. Z. 193 (1986), no.3,405-411. MR862887(88d:46024)
[2] J. Bourgain and J. Diestel, Limited operators and Strict Cosinngularity. Math. Nachr. 119. (1984), 55-58. MR 774176(86d:47024)
[3] L. Drewnowski, G. Emmanuele, On Banach spaces with the Gelfand-Phillips property, II, Rend. Circ. Mat. Palermo (2) 38(3) (1989) 377-391
[4] G. Emmanuele, A dual Characterization of Banach Spaces not containing ℓ^1, Bull. Polish Acad.Sci. Math. 34 (3-4) (1986),155-160
[5] Th. Schlumprecht, Limited sets in C(K) –spaces and examples concerning the Gelfand Phillips property, Math. Nachr.157(1992), 51-64. MR 1233046 (94h:46039)
[6] J. Diestel, *Sequences and Series in Banach spaces*, Texts in Mathematics, Vol. 92, Springer-Verlag, New York, 1984. MR 737004 (85i:46020)
[7] M. Talagrand, Pettis integral and Measure Theory, Mem. Amer. Math. Soc. 51 (1984), no. 307, ix+224. MR 756174 (86j: 46042)
[8] Manijeh S. and S. Mohammad, The Gelfand- Phillips property in closed subspaces of some operator spaces. Banach J. Math. Anal. 5(2011) no.2, 84-92
[9] Andreas Kriegl. Fre'chet spaces. http://www.mat.univie.ac.at/kriegl/Skripten/2016SS,(2016)
[10] Kronheimer P.B. Banach spaces, definitions and examples, October,27(2010)
[11] Heydari F. et al. Smoothness and Rotundity in Banach spaces, Vol 14 No. 3 Autum (2014)