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A Novel Approach to Topological Indices of the Power Graph Associated with the Quaternion of a Certain Order

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Abstract. The power graph of a group G, denoted Γ_G, is defined as a graph whose vertices correspond to the elements of G. Two distinct vertices $a, b \in G$ are adjacent if and only if there exists a positive integer m such that $a^m = b$ or $b^m = a$. This study investigates a new approach for computing the topological indices of the power graph associated with the generalized quaternion group Q_{4n}, where $n = p^k$, p is a prime number, and $k \in \mathbb{Z}$. The proposed method relies on structural properties of the power graph, particularly its diameter, allowing the indices to be determined directly from the number of vertices and edges. This enables a more direct, simplified calculation process than conventional methods, which often require more detailed combinatorial analysis or case-based enumeration. The results obtained using this new approach match exactly with those derived through traditional techniques. This confirms that the proposed method is not only valid but also efficient in computing topological indices for the power graph of the generalized quaternion group. Overall, the findings show that structural characteristics of the power graph can be used to streamline the computation of its associated indices.
Keywords: new approach, power graph, generalized quaternion group, topological indices.

I. INTRODUCTION

Graph-based representations of algebraic groups have attracted increasing attention in recent years. To capture different structural properties of a group, several types of graphs have been introduced, such as the coprime graph [1], non-coprime graph [2], relative coprime graph [3], intersection graph [4], and power graph [5]. The notion of the power graph of a finite group G, denoted by $\Gamma(G)$, was originally introduced in the directed setting by Kelarev and Quinn [6]. In their formulation, each element of G corresponds to a vertex, and two elements $a, b \in G$ are adjacent whenever there exists a natural number k such that $a^k = b$, resulting in a directed graph. Subsequently, Chakrabarty et al. proposed an undirected version for semigroups, where two distinct elements a and b are adjacent if and only if $a^k = b$ or $b^k = a$ for some natural number k [7].

In recent years, research on power graphs has expanded in several directions. Ali et al. examined the power graphs of several classes of finite groups [8], while Romdhini et al. investigated spectral properties of power graphs associated with generalized quaternion groups [9]. These studies show that power graphs provide a useful framework for connecting algebraic properties of finite groups with structural and spectral properties of graphs.

Researchers have also extensively studied topological indices in the context of power

graphs. Ismail et al. investigated a wide range of Zagreb indices for power graphs of finite cyclic, non-cyclic, dihedral, and generalized quaternion groups [10]. Ali et al. subsequently studied several degree- and distance-based topological descriptors of power graphs of finite non-abelian groups, including the Wiener, Harary, and other related indices for generalized quaternion groups [11]. These studies show that computing topological indices for power graphs of finite groups has received considerable attention.

Alongside these developments, researchers have also proposed structural approaches to simplify the computation of topological indices. Maulana et al. introduced an approach for obtaining several topological indices of graphs with diameter at most two by expressing the indices in terms of the numbers of vertices and edges [12]. This approach was subsequently applied to other graph families, including the power graph associated with a dihedral group of order $4n$ for $n = p^k$ [13]. These studies indicate that global structural properties of a graph can be exploited to obtain compact formulas for several topological indices and to verify the resulting expressions through direct computation.

Despite the growing literature on topological indices of power graphs, there remains room for a unified structural treatment of several indices for specific families of non-abelian groups. In particular, the generalized quaternion group Q_{4n} with $n = p^k$ has a highly structured power graph whose degree classes and diameter can be used together to compute several topological indices. Existing studies have mainly considered individual topological descriptors through degree-based, distance-based, or combinatorial calculations. A systematic treatment that emphasizes the common structural mechanism underlying several indices of the power graph of Q_{4n} is therefore of interest.

Motivated by this observation, this paper develops a structural approach to compute the Wiener, Hyper-Wiener, Harary, and Gutman indices of the power graph associated with the generalized quaternion group Q_{4n}, where $n = p^k$, p is a prime number, and $k \in \mathbb{Z}$. The approach begins by determining the power graph's degree structure and number of edges. Since the resulting graph has diameter at most two, the Wiener, Hyper-Wiener, and Harary indices can then be expressed in terms of the number of vertices and edges. At the same time, the Gutman index is obtained by combining the edge count with the first and second Zagreb indices. Thus, rather than treating each topological index independently, the proposed approach exploits the power graph's common structural properties to derive several indices within a unified framework.

The novelty of this work lies in the use of this structural framework for the generalized quaternion power graph $\Gamma_{Q_{4n}}$ with $n = p^k$, leading to explicit formulas for several topological indices from the degree structure, vertex count, and edge count of the graph. In addition to the theoretical derivations, we provide an illustrative example in which we explicitly construct the corresponding power graph, calculate the relevant topological indices manually, and compare the results with the formulas obtained in this paper. This direct verification demonstrates the consistency of the proposed formulas and illustrates how the structural approach can simplify the computation of the considered indices.

The remainder of this paper is organized as follows. Section 2 recalls the necessary definitions and known structural results. Section 3 presents the main results on the topological indices. Section 4 provides an illustrative example and verification of the obtained formulas, and Section 5 concludes the paper.

II. Preliminaries

In this section, we briefly recall the basic concepts and known results required in the subsequent analysis. The generalized quaternion group Q_{4n} of order $4n$ is given by

$$Q_{4n} = \left\langle x, y \mid x^{2n} = 1,\ x^n = y^2,\ yxy^{-1} = x^{-1} \right\rangle,$$

where the structure of the group and its relevant properties have been studied in [14]. The power graph of a group G, denoted by $\Gamma(G)$, is the graph whose vertex set is G, in which two distinct vertices $a, b \in G$ are adjacent whenever $a^m = b$ or $b^m = a$ for some natural number m [7].

For a finite group G, the diameter of its power graph is determined by its group structure. In particular, $\mathrm{diam}(\Gamma_G) = 1$ when G is a cyclic group of prime-power order, whereas $\mathrm{diam}(\Gamma_G) = 2$ otherwise [8]. Since the generalized quaternion group considered in this paper is not cyclic, this implies that

$$\mathrm{diam}(\Gamma_{Q_{4n}}) = 2.$$

Furthermore, the degree structure of $\Gamma_{Q_{4n}}$ is known from [9]. The vertices can be partitioned into three classes,

$$V_1 = \{e, a^n\}, \qquad V_2 = \{a^i b : 0 \le i \le 2n-1\},$$

and

$$V_3 = \{a^i : 1 \le i \le 2n\} \setminus V_1,$$

with the corresponding degrees

$$d_v = \begin{cases} 4n-1, & v \in V_1, \\ 3, & v \in V_2, \\ 2n-1, & v \in V_3. \end{cases}$$

Thus, V_1, V_2, and V_3 contain 2, $2n$, and $2n-2$ vertices, respectively. We use this degree classification directly in Section 3 to determine the number of edges and the degree-based quantities required to compute the topological indices.

We next recall the topological indices considered in this paper. For a connected graph G, let $d(u, v)$ denote the distance between two vertices u and v. The Wiener index [15], Hyper-Wiener index [16], and Harary index [17] are respectively given by

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v),$$

$$WW(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} \left[d(u, v) + d(u, v)^2 \right],$$

and

$$H(G) = \sum_{\{u,v\} \subseteq V(G)} \frac{1}{d(u, v)}.$$

Here and throughout the paper, we take sums involving distances over unordered pairs of distinct vertices.

The first and second Zagreb indices [18] are given by

$$M_1(G) = \sum_{u \in V(G)} d_u^2$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_u d_v,$$

respectively, where d_u denotes the degree of the vertex u. The Gutman index [19], which combines vertex degrees and distances, is defined by

$$Gut(G) = \sum_{\{u,v\} \subseteq V(G)} d_u d_v d(u,v).$$

For graphs of diameter at most two, Maulana et al. [12] established useful expressions that allow these indices to be computed from basic structural quantities. In particular, if Γ satisfies $\mathrm{diam}(\Gamma) \leq 2$, then

$$W(\Gamma) = |V(\Gamma)|\big(|V(\Gamma)| - 1\big) - |E(\Gamma)|,$$

$$WW(\Gamma) = \frac{3}{2}|V(\Gamma)|\big(|V(\Gamma)| - 1\big) - 2|E(\Gamma)|,$$

and

$$H(\Gamma) = \frac{1}{4}|V(\Gamma)|\big(|V(\Gamma)| - 1\big) + \frac{1}{2}|E(\Gamma)|.$$

For the Gutman index, the corresponding expression is

$$Gut(\Gamma) = 4|E(\Gamma)|^2 - M_1(\Gamma) - M_2(\Gamma).$$

We apply these formulas directly to $\Gamma_{Q_{4n}}$ after determining its edge count and the required Zagreb indices.

III. Result

In this section, we apply the structural properties recalled in Section 2 to determine the topological indices of the power graph $\Gamma_{Q_{4n}}$, where $n = p^k$, p is a prime number, and $k \in \mathbb{Z}$. We first determine the number of edges from the vertex degree classification. We then use the resulting vertex and edge counts to derive the Wiener, Hyper-Wiener, and Harary indices. Finally, we determine the first and second Zagreb indices from the degree structure and use them to obtain the Gutman index.

Lemma 1 *The Number of Edge*
Consider $\Gamma_{Q_{4n}}$ to be the power graph associated with the generalized quaternion group Q_{4n}. Assuming $n = p^k$, where p is a prime number and $k \in \mathbb{Z}$, the number of edges in $\Gamma_{Q_{4n}}$ can be expressed as

$$|E\left(\Gamma_{Q_{4n}}\right)| = 2n^2 + 4n.$$

Proof. To prove this result, we use the Handshake Lemma together with the degree classifica-

tion recalled in Section 2. Thus,

$$\begin{aligned} 2|E\left(\Gamma_{Q_{4n}}\right)| &= \sum_{u \in Q_{4n}} d_u \\ &= \sum_{u \in V_1} d_u + \sum_{u \in V_2} d_u + \sum_{u \in V_3} d_u \\ &= 2(4n-1) + (2n)3 + (2n-2)(2n-1) \\ &= 4n^2 + 8n. \end{aligned}$$

Therefore,

$$|E\left(\Gamma_{Q_{4n}}\right)| = 2n^2 + 4n.$$

□

Since $\mathrm{diam}(\Gamma_{Q_{4n}}) = 2$, the diameter-two formulas recalled in Section 2 can be applied to compute the distance-based topological indices. In particular, the number of unordered pairs of vertices in $\Gamma_{Q_{4n}}$ is

$$\binom{4n}{2} = 2n(4n-1),$$

while the number of non-adjacent pairs is

$$\binom{4n}{2} - |E(\Gamma_{Q_{4n}})|.$$

Theorem 1 *Wiener Index*
Consider $\Gamma_{Q_{4n}}$ to be the power graph associated with the generalized quaternion group Q_{4n}. Assuming $n = p^k$, where p is a prime number and $k \in \mathbb{Z}$, the Wiener index of $\Gamma_{Q_{4n}}$ can be expressed as

$$W\left(\Gamma_{Q_{4n}}\right) = 14n^2 - 8n.$$

Proof. Since $\mathrm{diam}(\Gamma_{Q_{4n}}) = 2$, the Wiener index formula recalled in Section 2 gives

$$\begin{aligned} W\left(\Gamma_{Q_{4n}}\right) &= |V(\Gamma_{Q_{4n}})|\left(|V(\Gamma_{Q_{4n}})| - 1\right) - |E(\Gamma_{Q_{4n}})| \\ &= 4n(4n-1) - (2n^2 + 4n) \\ &= 14n^2 - 8n. \end{aligned}$$

Alternatively, we can obtain this result directly from the definition of the Wiener index. Since every pair of adjacent vertices has distance one and every non-adjacent pair has distance

two,

$$\begin{aligned} W\left(\Gamma_{Q_{4n}}\right) &= |E(\Gamma_{Q_{4n}})| + 2\left(\binom{4n}{2} - |E(\Gamma_{Q_{4n}})|\right) \\ &= 2\binom{4n}{2} - |E(\Gamma_{Q_{4n}})| \\ &= 2\left(2n(4n-1)\right) - (2n^2+4n) \\ &= 14n^2 - 8n. \end{aligned}$$

□

Theorem 2 *Hyper-Wiener index*
Consider $\Gamma_{Q_{4n}}$ to be the power graph associated with the generalized quaternion group Q_{4n}. Assuming $n = p^k$, where p is a prime number and $k \in \mathbb{Z}$, the Hyper-Wiener index of $\Gamma_{Q_{4n}}$ can be expressed as

$$WW\left(\Gamma_{Q_{4n}}\right) = 20n^2 - 14n.$$

Proof. Since every adjacent pair of vertices has distance one and every non-adjacent pair has distance two, the Hyper-Wiener index can be calculated directly as

$$\begin{aligned} WW(\Gamma_{Q_{4n}}) &= \frac{1}{2}\Bigg[|E(\Gamma_{Q_{4n}})|(1+1^2) \\ &\qquad + \left(\binom{4n}{2} - |E(\Gamma_{Q_{4n}})|\right)(2+2^2)\Bigg] \\ &= (2n^2+4n)(2) \\ &\qquad + \left(2n(4n-1) - (2n^2+4n)\right)(6) \\ &= 20n^2 - 14n. \end{aligned}$$

Equivalently, using the diameter-two Hyper-Wiener formula recalled in Section 2,

$$\begin{aligned} WW(\Gamma_{Q_{4n}}) &= \frac{3}{2}|V(\Gamma_{Q_{4n}})|\left(|V(\Gamma_{Q_{4n}})| - 1\right) - 2|E(\Gamma_{Q_{4n}})| \\ &= \frac{3}{2}(4n)(4n-1) - 2(2n^2+4n) \\ &= 20n^2 - 14n. \end{aligned}$$

□

Theorem 3 *Harary index*
Consider $\Gamma_{Q_{4n}}$ to be the power graph associated with the generalized quaternion group Q_{4n}. Assuming $n = p^k$, where p is a prime number and $k \in \mathbb{Z}$, the Harary index of $\Gamma_{Q_{4n}}$ can be expressed as

$$H(\Gamma_{Q_{4n}}) = 5n^2 + n.$$

Proof. Since every adjacent pair of vertices has distance one and every non-adjacent pair has

distance two, the Harary index is

$$\begin{aligned} H(\Gamma_{Q_{4n}}) &= |E(\Gamma_{Q_{4n}})| + \frac{1}{2}\left(\binom{4n}{2} - |E(\Gamma_{Q_{4n}})|\right) \\ &= 2n^2 + 4n + \frac{1}{2}\left(2n(4n-1) - (2n^2+4n)\right) \\ &= 5n^2 + n. \end{aligned}$$

Equivalently, using the diameter-two Harary formula recalled in Section 2,

$$\begin{aligned} H(\Gamma_{Q_{4n}}) &= \frac{1}{4}|V(\Gamma_{Q_{4n}})|\left(|V(\Gamma_{Q_{4n}})| - 1\right) + \frac{1}{2}|E(\Gamma_{Q_{4n}})| \\ &= \frac{1}{4}(4n)(4n-1) + \frac{1}{2}(2n^2+4n) \\ &= 5n^2 + n. \end{aligned}$$

□

The first and second Zagreb indices are obtained directly from the degree classification recalled in Section 2. These quantities are needed to compute the Gutman index.

Lemma 2 *First and Second Zagreb indices*
Consider $\Gamma_{Q_{4n}}$ to be the power graph associated with the generalized quaternion group Q_{4n}. Assuming $n = p^k$, where p is a prime number and $k \in \mathbb{Z}$, the first and second Zagreb indices of $\Gamma_{Q_{4n}}$ are given by

$$M_1(\Gamma_{Q_{4n}}) = 8n^3 + 16n^2 + 12n$$

and

$$M_2(\Gamma_{Q_{4n}}) = 8n^4 + 4n^3 + 42n^2.$$

Proof. For the first Zagreb index, using the degree classes recalled in Section 2, we obtain

$$\begin{aligned} M_1(\Gamma_{Q_{4n}}) &= \sum_{u \in V(\Gamma_{Q_{4n}})} d_u^2 \\ &= \sum_{u \in V_1} d_u^2 + \sum_{u \in V_2} d_u^2 + \sum_{u \in V_3} d_u^2 \\ &= 2(4n-1)^2 + (2n)(3)^2 + (2n-2)(2n-1)^2 \\ &= 8n^3 + 16n^2 + 12n. \end{aligned}$$

For the second Zagreb index, we classify edges by the degrees of their endpoints. There is one edge joining the two vertices in V_1, there are $2(2n) = 4n$ edges joining V_1 and V_2, there are $2(2n-2) = 4n-4$ edges joining V_1 and V_3, there are n edges whose endpoints both belong to V_2, and there are

$$\binom{2n-2}{2}$$

edges whose endpoints both belong to V_3. Hence,

$$\begin{aligned} M_2(\Gamma_{Q_{4n}}) &= (4n-1)^2 + 2(2n)(4n-1)(3) \\ &\quad + 2(2n-2)(4n-1)(2n-1) \\ &\quad + n(3)^2 + \binom{2n-2}{2}(2n-1)^2 \\ &= 8n^4 + 4n^3 + 42n^2. \end{aligned}$$

□

Theorem 4 *Gutman index*
Consider $\Gamma_{Q_{4n}}$ to be the power graph associated with the generalized quaternion group Q_{4n}. Assuming $n = p^k$, where p is a prime number and $k \in \mathbb{Z}$, the Gutman index of $\Gamma_{Q_{4n}}$ can be expressed as

$$Gut(\Gamma_{Q_{4n}}) = 8n^4 + 52n^3 + 6n^2 - 12n.$$

Proof. Since $\mathrm{diam}(\Gamma_{Q_{4n}}) = 2$, the Gutman index can be expressed using the structural formula recalled in Section 2:

$$Gut(\Gamma) = 4|E(\Gamma)|^2 - M_1(\Gamma) - M_2(\Gamma).$$

Substituting the edge count and the Zagreb indices obtained above gives

$$\begin{aligned} Gut(\Gamma_{Q_{4n}}) &= 4(2n^2+4n)^2 \\ &\quad - (8n^3 + 16n^2 + 12n) \\ &\quad - (8n^4 + 4n^3 + 42n^2) \\ &= 8n^4 + 52n^3 + 6n^2 - 12n. \end{aligned}$$

The same result follows directly from the definition of the Gutman index. Since every edge has distance one and every non-edge has distance two,

$$Gut(\Gamma_{Q_{4n}}) = \sum_{uv \in E(\Gamma_{Q_{4n}})} d_u d_v + 2 \sum_{uv \notin E(\Gamma_{Q_{4n}})} d_u d_v.$$

Moreover,

$$\sum_{\{u,v\} \subseteq V(\Gamma_{Q_{4n}})} d_u d_v = \frac{\left(\sum_{u \in V(\Gamma_{Q_{4n}})} d_u\right)^2 - M_1(\Gamma_{Q_{4n}})}{2}.$$

Since

$$\sum_{u \in V(\Gamma_{Q_{4n}})} d_u = 2|E(\Gamma_{Q_{4n}})| = 4n^2 + 8n,$$

we obtain

$$\sum_{uv \notin E(\Gamma_{Q_{4n}})} d_u d_v = \frac{(4n^2+8n)^2 - M_1(\Gamma_{Q_{4n}})}{2} - M_2(\Gamma_{Q_{4n}}).$$

Therefore,

$$\begin{aligned} Gut(\Gamma_{Q_{4n}}) &= M_2(\Gamma_{Q_{4n}}) + 2\left[\frac{(4n^2+8n)^2 - M_1(\Gamma_{Q_{4n}})}{2} - M_2(\Gamma_{Q_{4n}})\right] \\ &= (4n^2+8n)^2 - M_1(\Gamma_{Q_{4n}}) - M_2(\Gamma_{Q_{4n}}) \\ &= 8n^4 + 52n^3 + 6n^2 - 12n. \end{aligned}$$

Thus, both approaches yield the same expression for the Gutman index. □

IV. Examples and Verifications

To illustrate and verify the formulas obtained in the previous section, we consider two generalized quaternion groups, namely Q_8 and Q_{12}. For each example, we construct the corresponding power graph explicitly, then directly compute the relevant topological indices. We then compare the results with the general formulas obtained in Section 3.

4.1. Example 1: The Power Graph of Q_8

Consider the generalized quaternion group Q_8, which corresponds to $n = 2$. The group can be represented as

$$Q_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\},$$

where

$$a^4 = e, \qquad a^2 = b^2, \qquad b^{-1}ab = a^{-1}.$$

The vertex set of the power graph Γ_{Q_8} is therefore

$$V(\Gamma_{Q_8}) = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$

The vertices e and a^2 are adjacent to every other vertex. Furthermore, the remaining vertices form three pairs,

$$\{a, a^3\}, \qquad \{b, a^2b\}, \qquad \{ab, a^3b\},$$

where the two vertices in each pair are adjacent. Consequently, the power graph has the structure.

$$\Gamma_{Q_8} = K_2 \vee (3K_2).$$

The graph Γ_{Q_8} is illustrated in Figure 1..

From Figure 1., the degree sequence of Γ_{Q_8} is

$$(7, 7, 3, 3, 3, 3, 3, 3).$$

Hence,

$$\sum_{v \in V(\Gamma_{Q_8})} d_v = 2(7) + 6(3) = 32.$$

By the Handshake Lemma,

$$|E(\Gamma_{Q_8})| = \frac{32}{2} = 16.$$

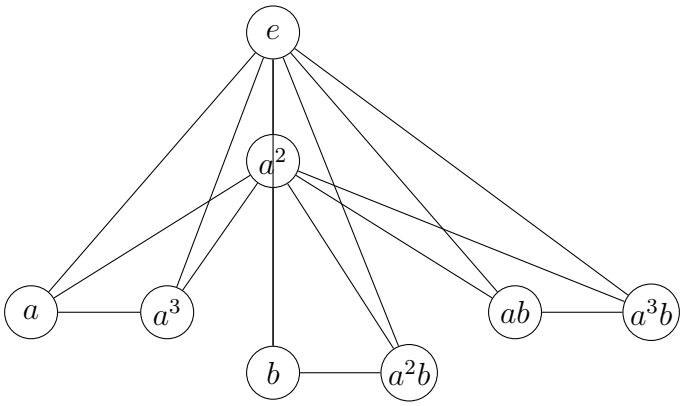


Figure 1. The power graph Γ_{Q_8}.

This agrees with Lemma 1, since

$$|E(\Gamma_{Q_8})| = 2n^2 + 4n = 2(2)^2 + 4(2) = 16.$$

Since Γ_{Q_8} has eight vertices, the total number of unordered pairs of vertices is

$$\binom{8}{2} = 28.$$

There are 16 adjacent pairs, each having distance one, and

$$28 - 16 = 12$$

non-adjacent pairs, each having distance two.

Therefore, the Wiener index can be calculated directly as

$$\begin{aligned} W(\Gamma_{Q_8}) &= 16(1) + 12(2) \\ &= 40. \end{aligned}$$

On the other hand, substituting $n = 2$ into Theorem 1 gives

$$W(\Gamma_{Q_8}) = 14(2)^2 - 8(2) = 40.$$

Thus, the direct calculation agrees with the general formula.

For the Hyper-Wiener index, we obtain

$$\begin{aligned} WW(\Gamma_{Q_8}) &= \frac{1}{2}\left[16(1 + 1^2) + 12(2 + 2^2)\right] \\ &= \frac{1}{2}\left[32 + 72\right] \\ &= 52. \end{aligned}$$

Using Theorem 2,

$$WW(\Gamma_{Q_8}) = 20(2)^2 - 14(2) = 52.$$

Hence, the direct calculation agrees with the general formula.

Similarly, the Harary index is

$$\begin{aligned} H(\Gamma_{Q_8}) &= 16(1) + 12\left(\frac{1}{2}\right) \\ &= 22. \end{aligned}$$

Using Theorem 3,

$$H(\Gamma_{Q_8}) = 5(2)^2 + 2 = 22.$$

Therefore, both approaches give the same value.

To verify the Gutman index, first consider the first Zagreb index. From the degree sequence,

$$\begin{aligned} M_1(\Gamma_{Q_8}) &= 2(7^2) + 6(3^2) \\ &= 98 + 54 \\ &= 152. \end{aligned}$$

This agrees with Lemma 2, since

$$M_1(\Gamma_{Q_8}) = 8(2)^3 + 16(2)^2 + 12(2) = 152.$$

For the second Zagreb index, there is one edge joining two vertices of degree 7, twelve edges joining vertices of degrees 7 and 3, and three edges joining two vertices of degree 3. Hence,

$$\begin{aligned} M_2(\Gamma_{Q_8}) &= 1(7)(7) + 12(7)(3) + 3(3)(3) \\ &= 49 + 252 + 27 \\ &= 328. \end{aligned}$$

This agrees with Lemma 2, since

$$M_2(\Gamma_{Q_8}) = 8(2)^4 + 4(2)^3 + 42(2)^2 = 328.$$

Consequently, the Gutman index is

$$\begin{aligned} Gut(\Gamma_{Q_8}) &= 4|E(\Gamma_{Q_8})|^2 - M_1(\Gamma_{Q_8}) - M_2(\Gamma_{Q_8}) \\ &= 4(16)^2 - 152 - 328 \\ &= 544. \end{aligned}$$

Using Theorem 4, we obtain

$$\begin{aligned} Gut(\Gamma_{Q_8}) &= 8(2)^4 + 52(2)^3 + 6(2)^2 - 12(2) \\ &= 544. \end{aligned}$$

Thus, the direct calculation and the general formula give the same value.

4.2. Example 2: The Power Graph of Q_{12}

As a second verification, consider the generalized quaternion group Q_{12}, corresponding to $n = 3$. The group can be represented as

$$Q_{12} = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\},$$

where

$$a^6 = e, \qquad a^3 = b^2, \qquad b^{-1}ab = a^{-1}.$$

The vertex set of $\Gamma_{Q_{12}}$ is therefore

$$V(\Gamma_{Q_{12}}) = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}.$$

The two vertices e and a^3 are adjacent to every vertex. The vertices

$$\{a, a^2, a^4, a^5\}$$

form a complete graph K_4, while the remaining six vertices form three disjoint pairs,

$$\{b, a^3b\}, \qquad \{ab, a^4b\}, \qquad \{a^2b, a^5b\}.$$

Consequently,

$$\Gamma_{Q_{12}} = K_2 \vee (K_4 \cup 3K_2).$$

The graph $\Gamma_{Q_{12}}$ is illustrated in Figure 2..

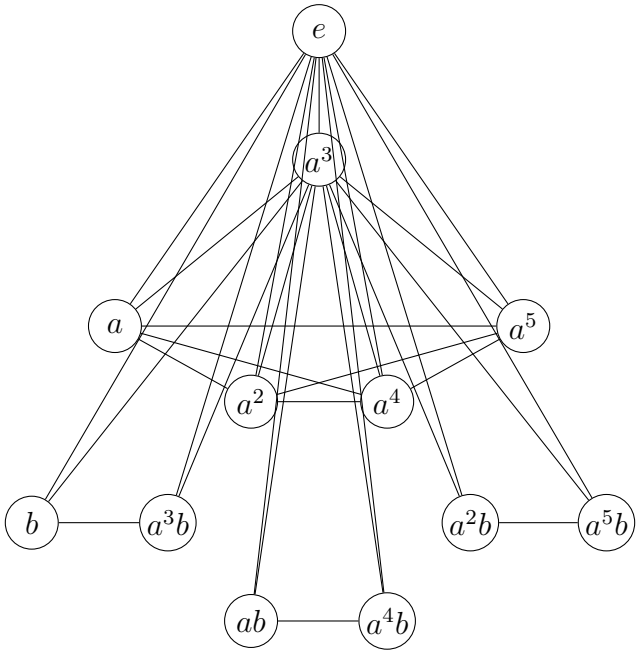


Figure 2. The power graph $\Gamma_{Q_{12}}$.

From Figure 2., the degree sequence of $\Gamma_{Q_{12}}$ is

$$(11, 11, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3).$$

Thus,

$$\begin{aligned}\sum_{v\in V(\Gamma_{Q_{12}})} d_v &= 2(11) + 4(5) + 6(3)\\ &= 60.\end{aligned}$$

By the Handshake Lemma,

$$|E(\Gamma_{Q_{12}})| = \frac{60}{2} = 30.$$

This agrees with Lemma 1, since

$$|E(\Gamma_{Q_{12}})| = 2(3)^2 + 4(3) = 30.$$

There are

$$\binom{12}{2} = 66$$

unordered pairs of vertices. Since 30 pairs are adjacent, the remaining

$$66 - 30 = 36$$

pairs are non-adjacent. Therefore,

$$\begin{aligned}W(\Gamma_{Q_{12}}) &= 30(1) + 36(2)\\ &= 102.\end{aligned}$$

Using Theorem 1,

$$W(\Gamma_{Q_{12}}) = 14(3)^2 - 8(3) = 102.$$

Hence, the direct calculation agrees with the general formula.

For the Hyper-Wiener index,

$$\begin{aligned}WW(\Gamma_{Q_{12}}) &= \frac{1}{2}\left[30(1+1^2) + 36(2+2^2)\right]\\ &= \frac{1}{2}(60 + 216)\\ &= 138.\end{aligned}$$

Using Theorem 2,

$$WW(\Gamma_{Q_{12}}) = 20(3)^2 - 14(3) = 138.$$

Thus, the direct calculation agrees with the general formula.

For the Harary index,

$$\begin{aligned}H(\Gamma_{Q_{12}}) &= 30(1) + 36\left(\frac{1}{2}\right)\\ &= 48.\end{aligned}$$

Using Theorem 3,

$$H(\Gamma_{Q_{12}}) = 5(3)^2 + 3 = 48.$$

Therefore, the direct calculation agrees with the general formula.

The first Zagreb index is

$$\begin{aligned} M_1(\Gamma_{Q_{12}}) &= 2(11^2) + 4(5^2) + 6(3^2) \\ &= 242 + 100 + 54 \\ &= 396. \end{aligned}$$

This agrees with Lemma 2, since

$$M_1(\Gamma_{Q_{12}}) = 8(3)^3 + 16(3)^2 + 12(3) = 396.$$

For the second Zagreb index, classify the edges by the degrees of their endpoints. There is one edge joining the two vertices of degree 11, eight edges joining vertices of degrees 11 and 5, twelve edges joining vertices of degrees 11 and 3, six edges joining two vertices of degree 5, and three edges joining two vertices of degree 3. Therefore,

$$\begin{aligned} M_2(\Gamma_{Q_{12}}) &= 1(11)(11) + 8(11)(5) + 12(11)(3) \\ &\quad + 6(5)(5) + 3(3)(3) \\ &= 121 + 440 + 396 + 150 + 27 \\ &= 1134. \end{aligned}$$

This agrees with Lemma 2, since

$$M_2(\Gamma_{Q_{12}}) = 8(3)^4 + 4(3)^3 + 42(3)^2 = 1134.$$

Consequently, the Gutman index is

$$\begin{aligned} Gut(\Gamma_{Q_{12}}) &= 4|E(\Gamma_{Q_{12}})|^2 - M_1(\Gamma_{Q_{12}}) - M_2(\Gamma_{Q_{12}}) \\ &= 4(30)^2 - 396 - 1134 \\ &= 2070. \end{aligned}$$

Using Theorem 4,

$$\begin{aligned} Gut(\Gamma_{Q_{12}}) &= 8(3)^4 + 52(3)^3 + 6(3)^2 - 12(3) \\ &= 2070. \end{aligned}$$

Thus, the direct calculation and the general formula give the same value.

For completeness, the results obtained from the two examples are summarized in Table 1..

Table 1. Verification of the topological indices for Γ_{Q_8} and $\Gamma_{Q_{12}}$.

Topological index	General formula	Q_8 $(n = 2)$	Q_{12} $(n = 3)$
$\|E\|$	$2n^2 + 4n$	16	30
M_1	$8n^3 + 16n^2 + 12n$	152	396
M_2	$8n^4 + 4n^3 + 42n^2$	328	1134
W	$14n^2 - 8n$	40	102
WW	$20n^2 - 14n$	52	138
H	$5n^2 + n$	22	48
Gut	$8n^4 + 52n^3 + 6n^2 - 12n$	544	2070

These two examples directly verify the formulas obtained in Section 3. In both cases, the values calculated directly from the corresponding power graphs agree exactly with the values obtained by substituting the respective values of n into the general formulas. This agreement supports the validity of the structural approach used to derive the topological indices considered.

V. Conclusions

In this paper, we investigated several topological indices of the power graph associated with the generalized quaternion group Q_{4n}, where $n = p^k$ for a prime number p. Using the structural properties of the power graph, particularly its degree classification and diameter-two property, we developed a unified approach to determine the Wiener, Hyper-Wiener, Harary, and Gutman indices. We first obtained the number of edges of the power graph from its degree structure. This result then provided a common basis for determining the distance-based indices. Since the diameter of the power graph is two, the Wiener, Hyper-Wiener, and Harary indices can be derived directly from the number of vertices and edges. We then determined the Gutman index using the edge count together with the first and second Zagreb indices. The resulting expressions are $W(\Gamma_{Q_{4n}}) = 14n^2 - 8n$, $WW(\Gamma_{Q_{4n}}) = 20n^2 - 14n$, $H(\Gamma_{Q_{4n}}) = 5n^2 + n$, and $Gut(\Gamma_{Q_{4n}}) = 8n^4 + 52n^3 + 6n^2 - 12n$. In addition, the corresponding Zagreb indices were obtained as $M_1(\Gamma_{Q_{4n}}) = 8n^3 + 16n^2 + 12n$ and $M_2(\Gamma_{Q_{4n}}) = 8n^4 + 4n^3 + 42n^2$.

To support the validity of the formulas, we considered two explicit examples using the power graphs associated with Q_8 and Q_{12}. In each case, we constructed the graph explicitly, and calculated the relevant topological indices directly from the vertices, degrees, edges, and distances. The values obtained from these direct calculations agree with the values obtained by substituting the corresponding values of n into the general formulas. These examples therefore explicitly verify the derived results and show how the structural properties of the power graph can simplify the computation of several topological indices. Overall, the results provide a systematic treatment of several topological indices for the power graphs of generalized quaternion groups and demonstrate that their common structural properties can be exploited to obtain the indices within a unified framework. The approach presented here may also be useful for investigating related topological indices and other families of power graphs arising from finite non-abelian groups.

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