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STABILITY ANALYSIS OF THE BULLYING MODEL WITH THE FACTOR OF CHANGING THE NATURE OF THE VICTIM TO BECOME A BULLIES

Marshellino Marshellino[1*], Abyan Daffa Akbar[2]

[1]*Bunda Mulia University, Jakarta, Indonesia*
[2]*Department of Mathematics, Universitas Airlangga, Surabaya, Indonesia*
Email: [1]marshellino@bundamulia.ac.id , [2]abyan.daffa.akbar-2022@fst.unair.ac.id
*Corresponding author

Abstract. Bullying cases in Indonesia have increased over time. One of the places where the most bullying cases occur is in the educational environment, namely a form of abusive behaviour when students show aggressive behaviour that lasts a long time and is carried out repeatedly against their peers. In 2024, 527 cases of bullying occurred in educational environments in Indonesia. In this article, a mathematical model of a bullying case is formulated in non-dimensional form, taking into account the possibility that bullying victims can become bullies. Based on the model analysis, we obtained an equilibrium point. The equilibrium point is locally asymptotically stable according to the Routh-Hurwitz criterion. Next, the numerical simulation of the model is demonstrated to support the analytical results.
Keywords: Stability Analysis, Bullying Model, Non-dimensional Form.

I. INTRODUCTION

Bullying cases in Indonesia have increased over time. Bullying is the act of someone using their power to hurt or intimidate people who are weaker than them [1]. Bullying behavior usually occurs due to several factors, namely family, friends, and the media [2]. Usually, bullies will attack weaker people and give inappropriate words or actions for various reasons, such as to cause fear, as a form of entertainment, increase self-esteem, or make others obey under pressure. The victim of bullying is someone who has less power than the bully, someone who has a quiet nature and does not fight back when being bullied. Victims of bullying will usually not tell anyone about the actions they have encountered. This happens because the victim of bullying gets threats from the bully. As a result, victims of bullying will feel hurt by the actions they receive and become a wound that will never heal, causing depression to death [3].

Bullying cases are often found in the world of education, especially in school. Bullying in schools is a form of abusive behavior when students show aggressive behavior that lasts a long time and is carried out repeatedly against their peers. This behavior is carried out with full awareness of the students who cannot defend themselves because there is an imbalance of power against the bully [4]. According to the Indonesian Education Monitoring Network (JPPI), the number of bullying cases in the educational environment in Indonesia has continued to increase from 2020 to 2024. In 2020, there were 91 cases of bullying. This number increased to 142 cases in 2021, 194 cases in 2022, 285 cases in 2023, and 573 cases in 2024 [5]. If cases of bullying in schools are allowed to continue, it will have an impact on student achievement. In some cases, students who experience bullying experience health problems, dropout of school, and death [3]. In addition, victims of bullying can also take revenge against the bullies in a more extreme form. Thus, victims of bullying will become bullies [6].

Several psychological mechanisms have been proposed to explain this phenomenon. Victims of bullying often experience emotional distress, fear of negative evaluation, and reduced self-esteem, which may contribute to the development of aggressive responses in future social interactions [7]. In addition, research on adolescent behavioral profiles has consistently identified distinct groups of students involved in bullying dynamics, including bullies, victims, and bully-victims, with the latter group displaying the highest levels of psychological maladjustment and aggressive tendencies [8]. The existence of these behavioral profiles indicates that bullying participation is not limited to static categories but rather reflects evolving social roles influenced by psychological and environmental factors [9]. Furthermore, large-scale studies have demonstrated that victimization is strongly associated with mental health problems such as anxiety, depression, and emotional dysregulation, which may further increase the likelihood of aggressive behavior toward others [7]. These findings highlight the importance of understanding bullying as a dynamic social process in which individuals may transition between different roles over time [10].

Bullying has also been widely recognized as a complex social and public health problem that affects adolescents in various environments, including schools, families, and online communities. Studies have shown that bullying behavior is influenced not only by individual psychological characteristics but also by peer interactions, school climate, and broader social contexts [11]. In recent years, the development of digital communication technologies has further expanded bullying behavior into cyberspace, giving rise to cyberbullying, which allows aggressive interactions to occur continuously beyond physical school environments [12]. The prevalence of both traditional bullying and cyberbullying among adolescents has been widely documented across different countries, indicating that bullying remains a persistent global concern [13].

From a psychological perspective, bullying involvement is strongly associated with emotional and behavioral difficulties among adolescents. Victims of bullying frequently experience psychological distress, anxiety, depression, and reduced emotional regulation abilities, which can negatively affect their social development and academic performance [14]. Longitudinal studies have also reported that exposure to repeated victimization during adolescence may produce long-term psychological consequences that persist into adulthood, including increased risk of mental health disorders and social maladjustment [15]. In addition, emotional dysregulation and social frustration experienced by victims may increase the probability that they adopt aggressive coping strategies when interacting with peers [16]. This psychological mechanism partly explains why some individuals who initially experience victimization may later become involved in bullying perpetration.

Furthermore, bullying behavior is closely related to the structure of peer relationships in adolescent social networks. Within peer groups, bullying may function as a strategy for gaining social dominance, recognition, or status among classmates [17]. Empirical studies have shown that adolescents who experience difficulties in peer relationships or who are exposed to aggressive peer norms are more likely to engage in bullying behavior [18]. These social dynamics suggest that bullying cannot be understood solely as an individual behavior but rather as a collective phenomenon shaped by interactions among students within a social system.

From a public health perspective, bullying has also been identified as a major risk factor affecting adolescent well-being worldwide. International reports indicate that a substantial proportion of students experience bullying during their school years, making it one of the most prevalent forms of youth violence globally [19]. Exposure to bullying has been associated with multiple negative outcomes, including physical health complaints, academic difficulties, social withdrawal, and increased risk of self-harm behaviors [20]. Moreover, global health studies highlight that preventing bullying is essential for improving adolescent mental health and promoting safe educational environments [21].

Taken together, these findings emphasize that bullying is a multidimensional phenomenon influenced by psychological, social, technological, and environmental factors. Consequently, understanding the dynamic interactions among individuals involved in bullying behavior is essential for developing effective prevention strategies. One approach that has been increasingly used to study such complex social interactions is mathematical modeling, which allows researchers to analyze how bullying behaviors evolve over time within a population and how different intervention strategies may influence these dynamics.

To find out the dynamics of bullying cases that occur in Indonesia, we can use mathematical modeling. Mathematical modeling is a field of mathematics that is used to formulate a real problem into a mathematical model [22]. Some studies related to bullying cases using mathematical modeling are [3] who constructed a model of bullying cases in schools with the assumption that the rate of bullies is directly proportional to the victims of bullying. [23] constructed a model of bullying cases in schools by separating individual victims of bullying who fought back and did not fight back and considering individuals who stopped being perpetrators and victims of bullying. [24] constructed a model of bullying cases in schools by considering the presence or absence of education in the family.

Based on the above, we propose a mathematical model of bullying cases referring to [3]. We also modified the basic model of [3] by considering the possibility that the victim of bullying can also be a bully. Furthermore, this article is organized in the following way: Section 2 discusses the development and explanation of the mathematical model used. Next, we modify the model in Section 3 to discuss the existence of solutions and the uniqueness of the solutions of the developed model. In Section 4, we calculate the non-endemic and endemic equilibria points and the basic reproduction number based on the obtained model. Furthermore, we analyze the local stability at each of the equilibria points. In Section 5, numerical simulations are conducted to validate the model. Finally, Section 6 concludes the discussion related to the analysis of mathematical models of bullying cases.

II. MODEL FORMULATION

A mathematical model of a bullying case is constructed under the following assumptions, each reflecting important real-world considerations and sociological insights.

1. Every student deserves the opportunity to receive an education.
   This fundamental principle underlines the ethical foundation of the model, emphasizing that maintaining student well-being and academic access is a priority. It motivates the need to understand and control bullying dynamics, which can jeopardize students' ability to learn in a safe environment.
2. The rate of students who graduate or transfer to other schools for various reasons is proportional to the number of students in the school.
   This assumption reflects natural turnover within the school population. The departure of students through graduation or transfer is modeled as a flow that depends directly on the

current size of the student body, a common simplification in population-based modeling.

3. The rate of students becoming victims of bullying is directly proportional to the number of victims and bullies.
   Here, bullying is treated as a dynamic interaction process: the more bullies and the more vulnerable students (i.e., potential victims), the higher the likelihood of new bullying incidents. This mirrors interaction-driven phenomena often modeled in epidemiology and social contagion theory.
4. Students who are victims of bullying have the potential to become bullies.
   This assumption introduces a critical feedback mechanism into the model. It acknowledges the documented psychological tendency for victimized individuals to adopt aggressive behavior as a coping or adaptive response, thereby perpetuating the cycle of bullying.
5. All parameters have positive values.
   This technical assumption ensures that all modeled rates and coefficients represent meaningful, real-world processes such as transition, interaction, or conversion rates. It also guarantees that the system of equations remains mathematically well-posed and interpretable within a biological or sociological context.

In this model, the variable t represents time, which is measured in *months*, since bullying dynamics in schools are typically observed over periods such as semesters or academic years. The state variable $x(t)$ denotes the number of students who are victims of bullying at time t. Therefore, the unit of $x(t)$ is *students*, representing individuals in the school population who experience bullying. Similarly, the variable $y(t)$ represents the number of students who act as bullies at time t, and its unit is also *students*, describing individuals who engage in bullying behavior toward others. The parameters of the model describe various rates that govern the dynamics of the system. The parameter Λ represents the rate at which students enter the school population, for example through new enrollment or student transfers, and thus its unit is *students per month*. The parameter α denotes the natural rate at which students leave the school system, such as through graduation or relocation to another school. Since it represents a proportional rate, its unit is *per month* (month^{-1}). The parameter β represents the rate at which victims leave the school due to bullying experiences, and therefore it also has the unit *per student per month* $(\text{student}^{-1}\text{month}^{-1})$. The parameter δ describes the transition rate at which victims may become bullies, reflecting behavioral changes that may arise from psychological or social factors, and its unit is likewise *per month* (month^{-1}). Furthermore, the parameter γ represents the rate at which bullies are removed from the school population due to disciplinary actions, such as suspension or expulsion, and its unit is *per month* (month^{-1}). Finally, the parameter μ denotes the interaction rate between bullies and other students that may generate new victims through bullying encounters. Since this parameter is associated with interactions between individuals, its unit is *per student per month* $(\text{student}^{-1}\text{month}^{-1})$. All parameters are assumed to be positive constants throughout the analysis.

A description of the variables and parameters in the mathematical model of bullying case can be seen in Table 1 and Table 2 as follows:

Table 1. Description of the variables

Variable	Description	Dimension
t	time	[time period]
x	the number of victims of bullying in school at time t	[individuals]
y	the number of bullies in school at time t	[individuals]

Table 2. Description of the parameters

Parameter	Description	Dimension
Λ	Rate of students enrolling in school	$\frac{[\text{individuals}]}{[\text{time}]}$
α	Rate of students graduating or transferring to a school in the area	$\frac{1}{[\text{time}]}$
β	Rate of students leaving school due to bullying by bullies	$\frac{1}{[\text{individuals}][\text{time}]}$
δ	Rate of victims of bullying becomes bullies	$\frac{1}{[\text{time}]}$
γ	Rate of bullies being expelled from school	$\frac{1}{[\text{time}]}$
μ	Rate of bullies againts other students	$\frac{1}{[\text{individuals}][\text{time}]}$

Based on these assumptions, variables, and parameters, the compartment diagram of the bullying model can be formed in Figure 1 as follows:

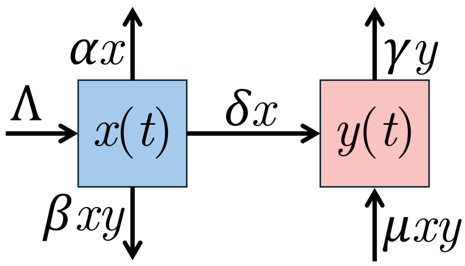


Figure 1. Compartment Diagram for mathematical model of bullying case

Based on Figure 1, the mathematical model of the bullying case can be formed as follows:

$$\begin{aligned} \frac{dx}{dt} &= \Lambda - (\alpha+\delta)x - \beta xy \\ \frac{dy}{dt} &= \delta x - \gamma y + \mu xy \end{aligned} \tag{1}$$

As a standard procedure in dynamical systems analysis, we next nondimensionalize the model to uncover its essential structure. Let $x = \frac{\alpha+\delta}{\mu}X$, $y = \frac{\alpha+\delta}{\beta}Y$, and $t = \frac{1}{\alpha+\delta}\tau$. Using the chain rule processes, we get a non-dimensional form of system (1) as

$$\begin{aligned} \frac{dX}{d\tau} &= \rho - X(1+Y) \\ \frac{dY}{d\tau} &= \theta X - Y(\eta - X) \end{aligned} \tag{2}$$

where $\rho = \frac{\Lambda\mu}{(\alpha+\delta)^2}$, $\theta = \frac{\beta\delta}{\mu(\alpha+\delta)}$, and $\eta = \frac{\gamma}{\alpha+\delta}$. Note that each variable and parameter in (2) is a dimensionless form.

III. MODEL ANALYSIS

3.1. Positiveness And Boundedness Of Solution

We begin by verifying whether the solutions produced by model (2) remain positive for all time, as formally stated in Theorem 1 below. This step is crucial to ensure that the model reflects meaningful and physically relevant behavior, especially in contexts where negative values would be nonphysical or nonsensical.

Theorem 1 *If the initial conditions satisfy $X(0) > 0$ and $Y(0) > 0$, then the solution of the system* (2) *remains positive for all $\tau \geq 0$, that is $X(\tau) > 0$ and $Y(\tau) > 0$.*

Proof. First, we show the positivity of the solution for the individuals who are victims of bullying. ($X(\tau) \equiv X$). Using the integration factor method, we have

$$\begin{aligned} \frac{dX}{d\tau} + X(1+Y) &= \rho. \\ \frac{d\left(Xe^{\tau+\int_0^\tau Y(u)du}\right)}{d\tau} &= \rho\, e^{\tau+\int_0^\tau Y(u)du} \\ Xe^{\tau+\int_0^\tau Y(u)du} &= \rho \int_0^\tau e^{v+\int_0^v Y(u)du} dv + C_1 \\ X &= \left(\rho \int_0^\tau e^{v+\int_0^v Y(u)du} dv + C_1\right) e^{-\tau-\int_0^\tau Y(u)du} \end{aligned}$$

Where C_1 is a constant. When $\tau = 0$, then $X = X(0)$, such that we obtain

$$X(\tau) = \left(\rho \int_0^\tau e^{v+\int_0^v Y(u)du} dv + X(0)\right) e^{-\tau-\int_0^\tau Y(u)du}$$

Because the exponential function is a positive function, so that $X(\tau) > 0$ for all $\tau > 0$.

Second, ee show the positivity of the solution for the individuals who are perpetrators of bullying ($Y(\tau) \equiv Y$). Using separation of variables, we obtained

$$\begin{aligned} \frac{dY}{d\tau} &= \theta X - Y(\eta - X) \\ \frac{dY}{d\tau} &\geq -Y(\eta - X). \\ \int \frac{dY}{Y} &\geq -\int (\eta - X) d\tau \\ \ln Y &\geq -\left(\eta\tau - \int_0^\tau X(u)du\right) + C_2 \\ Y &\geq C_3 e^{-\left(\eta\tau - \int_0^\tau X(u)du\right)} \end{aligned}$$

where C_3 is constant. When $\tau = 0$, then $Y = Y(0)$, such that we obtain

$$Y(\tau) \geq Y(0)\, e^{-\left(\eta\tau - \int_0^\tau X(u)du\right)}$$

Because the exponential function is a positive function, so that $Y(\tau) > 0$ for all $\tau > 0$. □

Theorem 2 *All solutions of system* (2)*, with initial conditions* $(X(0), Y(0)) \in \mathbb{R}_+^2$*, are remain bounded for all* $\tau \geq 0$.

Proof. To establish boundedness, consider the auxiliary function $\Phi(\tau) = X(\tau) + Y(\tau)$. This function represents the total population of the two interacting variables. Differentiating with respect to τ gives

$$\frac{d\Phi}{d\tau} = \rho + (\theta - 1)X - \eta Y$$

Observe that $-\eta Y = -\eta(X + Y) + \eta X$, we get

$$\frac{d\Phi}{d\tau} \leq \rho + (\theta + \eta - 1)X - \eta(X + Y).$$

Since $X(\tau) \geq 0$ dan $Y(\tau) \geq 0$, it follows that $X(\tau) \leq X(\tau) + Y(\tau)$, hence $X(\tau) \leq \Phi(\tau)$. Using this inequality, we obtain

$$(\theta + \eta - 1)X \leq (\theta + \eta - 1)\Phi. \tag{3}$$

Therefore,

$$\frac{d\Phi}{d\tau} \leq \rho + (\theta - 1)\Phi,$$

Using integrating factor, we obtain

$$\Phi(\tau) \leq \Phi(0)e^{(\theta-1)\tau} + \frac{\rho}{\theta - 1}\left(e^{(\theta-1)\tau} - 1\right) = \phi(\tau).$$

where $\theta \neq 1$. Thus, $\Phi(\tau) = X(\tau) + Y(\tau) \leq \phi(\tau)$. Therefore, the solutions $X(\tau)$ and $Y(\tau)$ remain bounded for all $\tau \geq 0$. □

3.2. Equilibrium Points

Subsequently, we analyze the equilibrium points of model (2), and the results of this analysis are encapsulated in the following theorem

Theorem 3 *The model* (1)*, which is equivalent to the nondimensionalized system* (2)*, admits a single positive equilibrium point, i.e.* (X_*, Y_*)*, where*

$$X_* = \frac{\rho}{1 + Y_*}$$

and Y_* *satisfy the equation* $\rho\theta - (\eta - \rho)Y - \eta Y^2 = 0$.

Proof. To determine the equilibrium points, we solve the system by setting the right-hand sides of the differential equations to zero, that is

$$\frac{dX}{d\tau} = \mu - X(1 + Y) = 0 \quad \text{and} \quad \frac{dY}{d\tau} = \theta X - Y(\eta - X) = 0 \tag{4}$$

From equation (4), we derive the following expression

$$X = \frac{\rho}{1+Y} \quad \text{and} \quad X = \frac{\eta Y}{\theta + Y} \tag{5}$$

Next, by substituting the two expressions for X obtained in (5), we derive the following equation in terms of Y,

$$\frac{\rho\theta - (\eta - \rho)Y - \eta Y^2}{(1+Y)(\theta+Y)} = 0 \tag{6}$$

Since $1+Y \neq 0$ or $\theta + Y \neq 0$, then $\eta Y^2 + (\eta-\rho)Y - \rho\theta = 0$. Therefore $Y = Y_*$ which satisfy the equation $a_1Y^2 + a_2Y + a_3 = 0$ are

$$Y_*^{(1)} = \frac{-(\eta-\rho) + \sqrt{(\eta-\rho)^2 + 4\eta\rho\theta}}{2\eta} \quad \text{and} \quad Y_*^{(2)} = \frac{-(\eta-\rho) - \sqrt{(\eta-\rho)^2 + 4\eta\rho\theta}}{2\eta},$$

where the value of $Y_*^{(1)} > 0$ and $Y_*^{(2)} < 0$, Hence, the positivity condition for the equilibrium is fulfilled only by $Y_*^{(1)}$. □

The equilibrium point established in Theorem 3 plays a fundamental role in understanding the long-term behavior of the system. However, identifying the equilibrium alone is not sufficient; its stability must also be analyzed to determine whether nearby trajectories converge to or diverge from this point. The notion of local stability used in this study follows the standard definition in dynamical systems theory that used in [25]. Theorem 4 provides a rigorous characterization of the stability properties of this equilibrium, offering insight into the system's local dynamics in its vicinity.

Theorem 4 *The equilibrium point* $(\frac{\rho}{1+Y_*}, Y_*)$ *is locally asymptotically stable.*

Proof. Determine the Jacobian matrix for system (2) that evaluate at (X_*, Y_*):
Let $f(X,Y) = \rho - X(1+Y)$ and $g(X,Y) = \theta X - Y(\eta - X)$, then

$$J_* = J(X_*, Y_*) = \begin{bmatrix} D_X f(X_*,Y_*) & D_Y f(X_*,Y_*) \\ D_X g(X_*,Y_*) & D_Y g(X_*,Y_*) \end{bmatrix} = \begin{bmatrix} -(1+Y_*) & -X_* \\ \theta + Y_* & X_* - \eta \end{bmatrix}$$

The Jacobian matrix $J(X_*, Y_*)$ has eigenvalues with the characteristic equation given by $\lambda^2 - b_1\lambda + b_2 = 0$, where $b_1 = \text{tr}(J_*) = -1 - Y_* + X_* - \eta$ and $b_2 = \det(J_*) = X_*(\theta - 1) + \eta(1+Y_*)$. According to the Routh-Hurwitz criterion, the following aspects can be assessed.

1) Verify whether $\det[-b_1] > 0$.

$$H_1 = \det[-b_1] = -b_1 = 1 + Y_* - X_* + \eta = \frac{\rho\theta + Y_* + 2Y_*^2 + Y_*^3}{Y_*(1+Y_*)} > 0$$

obtained $b_1 > 0$.

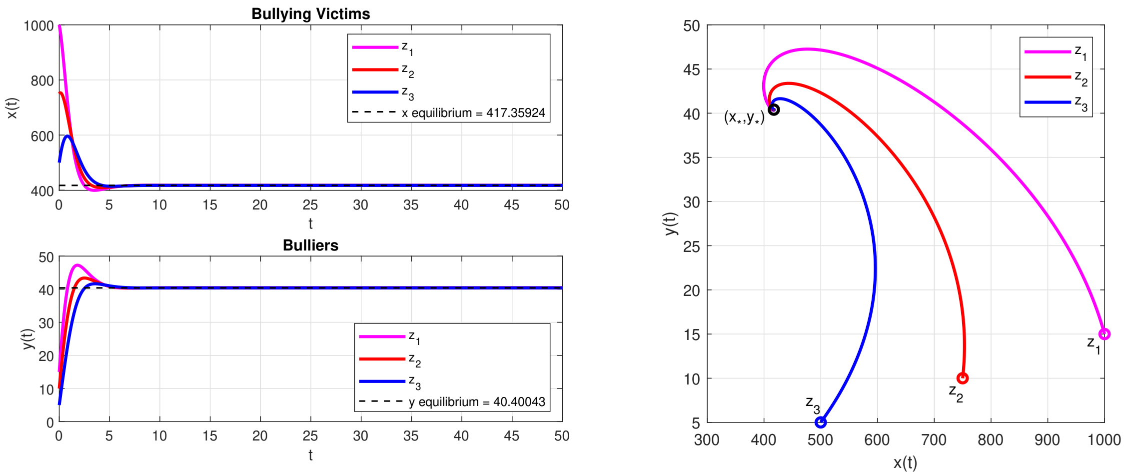
2) Verify whether $\det \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix} > 0$.

$$H_2 = \det \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix} = b_2 b_2 = \left(\frac{\rho\theta + Y_* + 2Y_*^2 + Y_*^3}{Y_*(1+Y_*)} \right) \left(\frac{\rho\theta + \rho\theta Y_* + \eta Y_*^2 + \eta Y_*^3}{Y_*(1+Y_*)} \right) > 0$$

Since $H_1 > 0$ and $H_2 > 0$, according to the Routh-Hurwitz criterion, all the real parts of the eigenvalues λ obtained from the characteristic equation are negative. Therefore, the equilibrium point (X_*, Y_*) is locally asymptotically stable. □

IV. SIMULATION

For the numerical simulations, the parameter values $\Lambda = 500$, $\alpha = 0.34$, $\beta = 0.02$, $\gamma = 0.6$, and $\mu = 0.0002$ are taken from the study of [3], while the parameter δ is varied in order to investigate its effect on the model dynamics. Furthermore, three different initial conditions $(x(0), y(0))$ are considered, namely $z_1 = (1000, 15)$, $z_2 = (750, 10)$, and $z_3 = (500, 5)$.



(a) **(b)**

Figure 2. Plot of numerical solutions using $\delta = 0.05$ (a) $x(t)$ and $y(t)$ of the school bullying model (2) versus time (months) and (b) the phase portrait for three initial conditions

Where the function $x(t) = \frac{\alpha+\delta}{\mu} X(t)$ and $y(t) = \frac{\alpha+\delta}{\beta} Y(t)$.

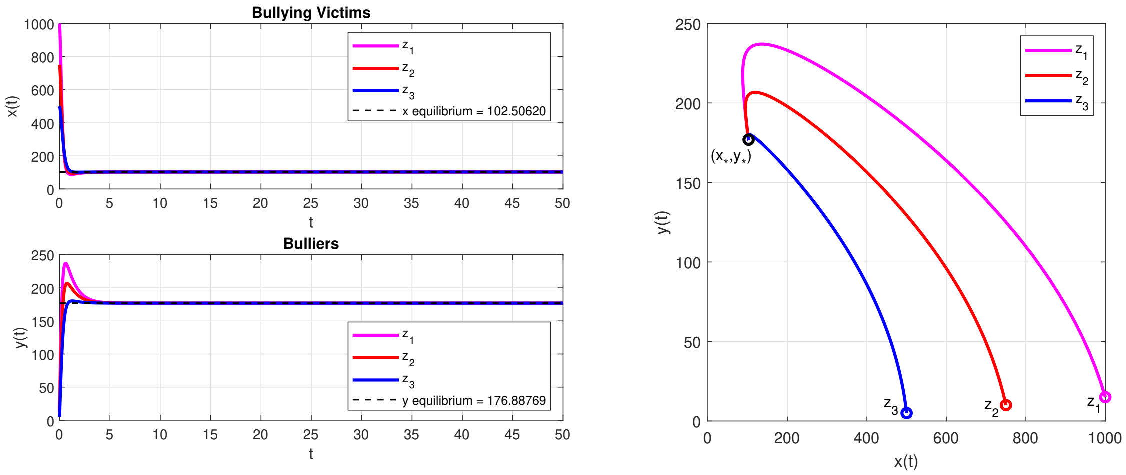
From Figure 2a, it is evident that the bully population converges to values that are more than the victim population. In other words, the victims are dominate. In contrast, Figure 3a shows the opposite situation, where bullies dominate. Between these two regimes lies a critical value δ_{cr} at which the two steady states coincide. This value marks the threshold beyond which the bully population persists at higher levels than the victim population over long times. Mathematically, δ_{cr} is obtained by solving $x_*(\delta) = y_*(\delta)$ or $X_*(\delta) = \frac{\mu}{\beta} Y_*(\delta)$, and we get

$$\delta_{cr} = \frac{(\alpha\mu + 2\beta\gamma - \gamma\mu) - \mu\sqrt{(\alpha\beta + \gamma\mu)^2 + 4\Lambda(\beta - \mu)}}{2(\beta - \mu)}$$

With this definition, $\delta > \delta_{cr}$ implies that $x_*(\delta) > y_*(\delta)$ which bullies is dominates, and for $\delta < \delta_{cr}$ implies that $x_*(\delta) < y_*(\delta)$ which victims is dominates.

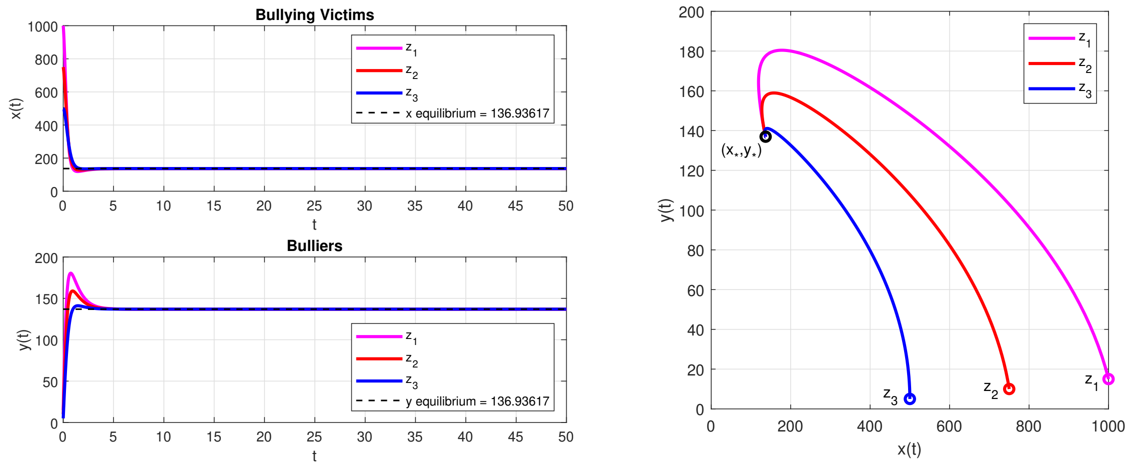
Based on the values of parameters, we have $d_{cr} \approx 0.57261$. And we get the threshold equilibrium is $(x_{cr}, y_{cr}) \approx (136, 136)$.

The parameter δ represents the rate at which susceptible individuals become involved in bullying dynamics through social exposure or interaction with individuals already engaged in



(a) **(b)**

Figure 3. Plot of numerical solutions using $\delta = 1$ (a) $x(t)$ and $y(t)$ of the school bullying model (2) versus time (months) and (b) the phase portrait for three initial conditions



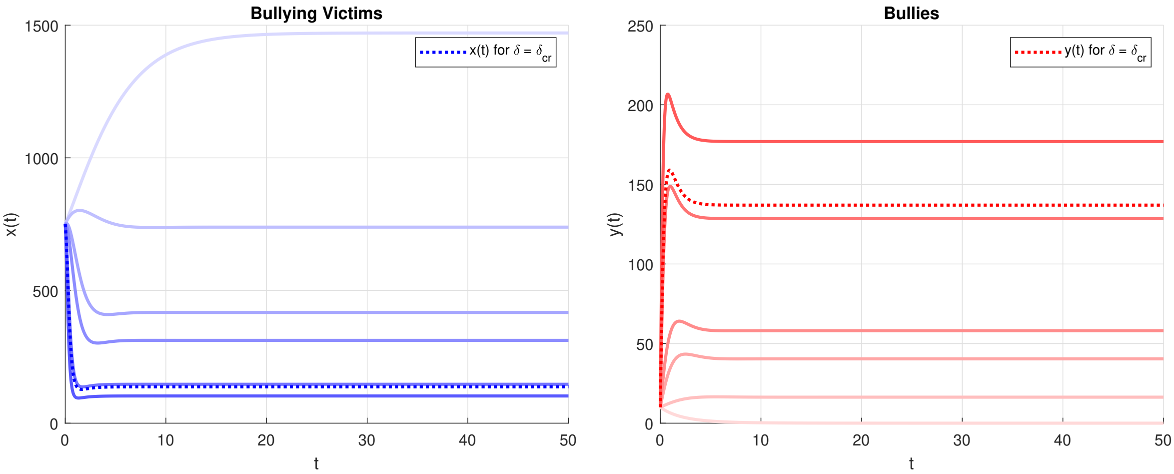
(a) **(b)**

Figure 4. Plot of numerical solutions using $\delta = \delta_{cr} \approx 0.57261$ (a) $x(t)$ and $y(t)$ of the school bullying model (2) versus time (months) and (b) the phase portrait for three initial conditions

bullying behavior. In the context of this model, δ captures the influence of social environments, peer pressure, and behavioral imitation that may encourage previously uninvolved individuals to participate in bullying-related interactions. We can see in the Figure 5.

From the Figure 5 above, the most solid line is the most value of δ

A higher value of δ indicates that susceptible individuals are more easily influenced by their surroundings, leading to a faster transition into the bullying interaction group and consequently increasing the prevalence of bullying behavior in the population. Conversely, a smaller value of δ implies that individuals are less affected by such social influences, resulting in a slower transition from the susceptible class to the bullying interaction class. Therefore, the parameter δ plays an important role in determining the intensity and speed at which bullying behavior spreads within the population and significantly influences the long-term dynamics of the system.



(a) **(b)**

Figure 5. Plot of numerical solutions using varying δ (a) for $x(t)$ and (b) for $y(t)$

V. CONCLUSION AND FUTURE RESEARCH

In this article, we have investigated a mathematical model of bullying dynamics expressed in a non-dimensional form. The basic framework introduced in [3] was extended by incorporating a new parameter, δ, which represents the rate at which individuals who are initially victims of bullying transition into becoming bullies. This modification provides a more realistic representation of the cyclical nature of bullying behavior.

We reformulated the system into its non-dimensional form and carried out a rigorous mathematical analysis. In particular, we identified and examined the equilibrium points of the system, establishing their stability using the Routh–Hurwitz criterion. The results show that the equilibrium point is locally asymptotically stable under specific parameter conditions, thereby confirming the theoretical robustness of the model.

To complement the analytical findings, numerical simulations were performed for various values of the parameter δ. The simulations demonstrate a clear threshold behavior: when $\delta > \delta_{cr}$, the bully population dominates in the long term, whereas when $\delta < \delta_{cr}$, the victim population prevails. These results not only validate the theoretical analysis but also highlight the crucial role of the transition rate δ in shaping the long-term dynamics of bullying interactions.

Looking ahead, several directions for future research can be pursued. First, the model can be enriched by incorporating fractional-order derivatives, which may better capture memory effects and the complex dynamics inherent in social interactions. Second, the inclusion of control variables, such as preventive measures or intervention strategies, could provide insights into effective ways of reducing the prevalence of bullying. Finally, parameter estimation based on real-world data remains an important avenue, as it would allow the model to be calibrated and validated against empirical observations, thereby enhancing its relevance and applicability in educational and social contexts.

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