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OPTIMAL CONTROL MODEL OF DIVORCE DYNAMICS BY POST-MARRIAGE EDUCATION AND COUNSELING

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Abstract. Divorce is a complex social phenomenon with various causes, including incompatibility between partners, poor communication, economic issues, and infidelity. This research was conducted by modifying the SMBD model with the addition of the Healthy Marriage (H) and Unhealthy Marriage (U_h) subpopulations, adapted from the MHUCKD model, and considering controls in the form of post-marriage education and counseling, resulting in the SMHU_hD model. The purpose of implementing controls is to minimize the Unhealthy Marriage (U_h) and Divorce (D) subpopulations, as well as the intervention costs correlated with post-marriage education and counseling. The optimal control condition is obtained by applying the Pontryagin's Maximum Principle. Next, a numerical simulation was conducted using the 4th-order Runge-Kutta method with the Forward-Backward Sweep Method approach, where the state equations were solved using a forward step and the costate equations were solved using a backward step. The research results show that educational control and counseling, when given together, can minimize the occurrence of divorce.

Keywords: Mathematical Model, Divorce, Optimal Control, Post-marital Education, Counseling.

I. INTRODUCTION

Divorce is a complex issue that involves not only legal aspects but is also closely related to social and cultural issues, which continue to increase and have far-reaching impacts on family stability, society, and even the systems governing it [1]. In Islamic law, divorce is referred to as *thalaq*, which means the dissolution of a bond (*hall al-Qaid*), namely the termination of a marital relationship through prescribed words [2]. In Islam, there are two types of divorce, namely divorce sued by wife and talaq divorce. Divorce sued by wife refers to a divorce initiated by the wife (the applicant) by filing a petition with the Religious Court, with the husband (the respondent) giving his consent [3]. Meanwhile, talaq refers to the dissolution of a marriage resulting from a talaq pronounced by the husband or through a divorce petition, as stipulated in Article 114 of the Compilation of Islamic Law (KHI) [4].

According to data from the Statistics Indonesia (BPS) of Central Java Province in 2023, the number of divorce cases in Indonesia, particularly in Central Java Province, has increased in recent years. The factors contributing to the increase in divorce cases are complex, ranging from infidelity, persistent marital disputes, interpersonal incompatibility, domestic violence, to economic problems [5]. Although divorce is permitted under Islamic law, Islamic teachings encourage the resolution of marital conflicts through deliberation and family mediation before reaching a final decision, as stated in Q.S. An-Nisa' verse 35.

Advances in technology and mathematical sciences have facilitated the simulation of complex problems arising in everyday life and the quantification of various phenomena occurring in the natural world [6]. Mathematical models provide an approach for understanding and addressing social problems [7], including divorce-related issues. One example of a social phenomenon that can be mathematically modeled is divorce dynamics.

One mathematical modeling approach that is relevant to this problem is optimal control.

Optimal control is an optimization framework that aims to determine the best strategy for a given process in order to achieve the desired outcome [8]. Several previous studies have investigated mathematical models using an optimal control approach, including those by [9], [10], and [11]. These studies demonstrate that control interventions can reduce the adverse effects of the modeled problems, thereby providing an important basis for the development of the model presented in this study.

Intervention strategies for preventing divorce have been extensively investigated, including educational and psychological approaches such as counseling. Post-marital education and individual counseling are two distinct forms of intervention aimed at maintaining marital stability. Post-marital education is preventive in nature and is provided to married couples after marriage, with the aim of equipping them with knowledge of family roles, effective communication, conflict management, and child-rearing practices [12]. In contrast, individual counseling is curative in nature and is directed toward an individual experiencing marital difficulties, such as psychological distress, disharmony, or emotional conflict. Counseling is more personalized and focuses on improving the individual's psychological well-being and addressing marital problems gradually [13]. Therefore, the implementation of post-marital education and individual counseling interventions is expected to minimize the number of divorces.

Several divorce dynamics models have been developed in previous studies. The MHUCKD model (*Marriage – Healthy Marriage – Unhealthy Marriage – Counseling – Complexity Life – Divorce*) developed by [14] demonstrates that the provision of counseling can increase the number of healthy marriages while reducing marital complexity and the divorce rate. Meanwhile, the SMBD model (*Single – Marriage – Broken – Divorce*) proposed by [15] suggests reducing interactions between married and divorced individuals and providing education to couples who are separated but have not yet divorced (*Broken*) to discourage divorce and encourage reconciliation within the marriage.

Based on the studies by [14] and [15], we modify the SMBD model by incorporating the concepts of the *Healthy Marriage* (H) and *Unhealthy Marriage* (U_h) subpopulations adopted from the MHUCKD model. The model developed in this study divides the population into five subpopulations: individuals who are unmarried, at least 19 years old, and have never been married (S); individuals who decide to enter into marriage (M); individuals in healthy marriages, characterized by good communication, mutual respect, and the maintenance of trust (H); individuals in unhealthy marriages, characterized by frequent disputes and verbal mistreatment (U_h); and divorced individuals who have ended their marital relationship because no satisfactory solution to the problems encountered during the marriage could be found (D).

In this study, the authors also interested in adding two control variables, namely post-marital education, which aims to provide married individuals with an understanding of their rights, responsibilities, and roles within marriage in order to prevent conflicts between spouses [16], and counseling, which aims to assist individuals in identifying, determining, and making decisions regarding their current family life [17].

II. RESEARCH METHOD

This research uses a literature study approach to develop and analyze a mathematical model of divorce dynamics based on various relevant sources, such as books, journal articles, and theses. The stages carried out include literature review, model construction by modifying

previous models, determination of the objective function to minimize the subpopulations of Unhealthy Marriages (U_h) and divorces (D) as well as their intervention costs, formation of the Hamiltonian function based on the Pontryagin's Maximum Principle, and derivation of the stationary conditions, state equations, and costate equations. The model was solved numerically using the 4th order Runge-Kutta method with a forward-backward sweep approach, where the state equations were solved forward and the costate equations backward. Next, the results of the numerical simulation are interpreted to analyze the effectiveness of post-marriage education interventions and counseling in reducing divorce rates.

III. MATHEMATICAL MODEL

The mathematical model of divorce dynamics in this study is built based on a number of assumptions. The population is assumed to be closed, and factors such as sex, race, and social status do not influence the decision to divorce [14]. To facilitate the representation of the proposed model, the following assumptions are imposed in the SMHU_hD divorce dynamics model:

1. Factors such as gender, race, and social status are assumed not to influence an individual's decision to divorce.
2. Individuals who reach the age of 19 years are assumed to have no previous history of divorce and enter the Single (S) subpopulation at a rate of π per unit time.
3. Individuals experience natural mortality at a rate of μ per unit time and mortality associated with divorce at a rate of ε per unit time.
4. Following the model proposed by [15], this study modifies the model by replacing the Broken (B) subpopulation with the Healthy Marriage (H) and Unhealthy Marriage (U_h) subpopulations, following the concepts introduced by [14].
5. The population is divided into five compartments, namely the Single (S), Marriage (M), Healthy Marriage (H), Unhealthy Marriage (U_h), and Divorce (D) subpopulations.
6. Based on Law No. 16 of 2019 of the Republic of Indonesia, individuals in the Single (S) subpopulation are defined as those who are at least 19 years old.
7. Married individuals are classified into two groups: Healthy Marriage (H) and Unhealthy Marriage (U_h).
8. The Divorce (D) subpopulation can influence the Marriage (M) subpopulation through environmental factors, represented by a contact rate β per unit time.
9. The parameter α represents the proportion of individuals who transition into the Unhealthy Marriage (U_h) subpopulation.
10. Under the control u_2, representing the provision of counseling, the Divorce (D) subpopulation can be reduced with an effectiveness rate of δ.
11. The Healthy Marriage (H) subpopulation is assumed to consist of couples who are able to resolve conflicts without domestic violence.

12. The Unhealthy Marriage (U_h) subpopulation is assumed to consist of couples who are unable to resolve conflicts, resulting in persistent disputes that may lead to domestic violence.

Based on the assumptions that have been made, a compartment diagram of the SMHU_hD model can be formed as follows.

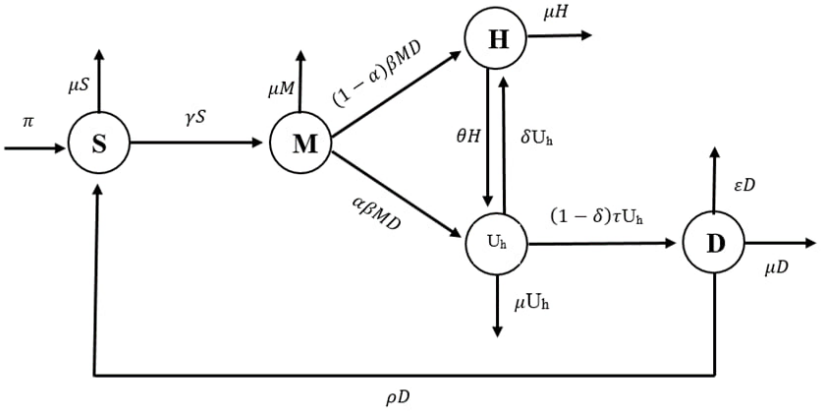


Figure 1. SMHU_hD (Single - Marriage - Healthy Marriage - Unhealthy Marriage - Divorce) Model Compartment Diagram

The following is a mathematical model of divorce dynamics in the form of a System of Differential Equations.

$$\begin{aligned}
\frac{dS}{dt} &= \pi N + \rho D - \gamma S - \mu S \\
\frac{dM}{dt} &= \gamma S - \beta M D - \mu M \\
\frac{dH}{dt} &= (1-\alpha)\beta M D + \delta U_h - \theta H - \mu H \\
\frac{dU_h}{dt} &= \alpha\beta M D + \theta H - (1-\delta)\tau U_h - \delta U_h - \mu U_h \\
\frac{dD}{dt} &= (1-\delta)\tau U_h - (\rho + \varepsilon + \mu) D
\end{aligned} \tag{1}$$

The variables and parameters used in the mathematical model of divorce dynamics in the explanation above are as follows.

Table 1. Variables and Parameters of the SMHU_hD (Single - Marriage - Healthy Marraige - Unhealthy Marriage - Divorce) Model

Parameters	Description
$N(t)$	Total population at time t.
$S(t)$	Subpopulation Single or individuals aged 19 years.
$M(t)$	Subpopulation Marriage or individuals who decide to get married.
$H(t)$	Subpopulation Healthy Marriage or individuals whose marriages are healthy.
$U_h(t)$	Subpopulation Unhealthy Marriage or individuals whose marital relationships are unhealthy.
$D(t)$	Subpopulation Divorce or individuals who decide to get divorced.
π	The rate of increase in individuals who are 19 years old.
μ	Natural death rate.
ρ	The rate of transition for individuals who have divorced to become single again.
γ	The rate at which individuals decide to get married.
α	Individual factors that lead to an unhealthy marriage.
β	The interaction rate between the Marriage (M) and Divorce (D) subpopulations due to several environmental factors.
τ	The average number of individuals moving from subpopulation U_h to subpopulation D.
θ	The transition rate of individuals from subpopulation H to subpopulation U_h.
δ	Effectiveness of counseling provision.
ε	The rate of death due to divorce.
u_1	The proportion of the effects of post-marriage education.
u_2	The proportion of the effects of implementing counseling.

IV. OPTIMAL CONTROL ANALYSIS

The system in equation (1) was then modified by adding two control variables, namely u_1 in the form of post-marriage education and u_2 in the form of counseling provision. The solution to the optimal control problem is carried out using the Pontryagin's Maximum Principle, resulting in the SMHUU_hD model with the following control:

$$\begin{aligned}
\frac{dS}{dt} &= \pi N + \rho D - \gamma S - \mu S \\
\frac{dM}{dt} &= \gamma S - (1-u_1)\beta M D - \mu M \\
\frac{dH}{dt} &= (1-u_1)(1-\alpha)\beta M D + \delta u_2 U_h - \theta H - \mu H \\
\frac{dU_h}{dt} &= (1-u_1)\alpha\beta M D + \theta H - (1-\delta u_2)\tau U_h - \delta u_2 U_h - \mu U_h \\
\frac{dD}{dt} &= (1-\delta u_2)\tau U_h - (\rho + \varepsilon + \mu) D
\end{aligned} \tag{2}$$

4.1. Objective Function

The objective function of equation (2) is formulated in the following mathematical formula.

$$J(u_1, u_2) = \int_0^t U_h + D + \frac{1}{2}Au_1^2 + \frac{1}{2}Bu_2^2 \ \ dt \tag{3}$$

with A being the weight correlated with the cost of preventing interactions between Marriage (M) subpopulation and the Divorce (D) subpopulation. Meanwhile, B is the weight correlated with the cost of providing counseling. Next, find the optimal controls u_1^* and u_2^*, such that:

$$J(u_1^*, u_2^*) = \min\{J(u_1, u_2) \mid u_1\, u_2 \in U\},$$

with $U = \{(u_1, u_2) \mid 0 \leq u_1 \leq 1, 0 \leq u_2 \leq 1\}$. The optimal control problem is satisfied if it meets the conditions of the Pontryagin's maximum principle.

Before determining the optimal conditions, the existence of optimal control must first be proven. In this context, the existence of optimal control is important to ensure that there is a control pair u that can minimize the objective function $J(u_1, u_2)$ for the divorce dynamics model differential equation system. Based on the proof by Fleming and Rishel (1975), the following are the conditions that guarantee the existence of optimal control.

Theorem 1 *Given an optimal control problem. There exists an optimal control $u^* \in U$ that minimizes the objective function $J(u_1^*, u_2^*)$, such that*

$$J(u_1^*, u_2^*) = minJ(u_1, u_2)|(u_1, u_2) \in U$$

Proof. To prove the existence of control, we follow the given assumption [18]. Since

1. The set of control and state variables is non-empty.
2. The set of control is convex and closed.
3. The right-hand side of the state system (2) is bounded by a linear function of the state and control.
4. The integrand of the objective function (3) is convex on U.
5. The exist constant $C_1, C_2 > 0$ and $\beta > 1$ such that the integrand of the objective functional satisfies $L(x, u) = -c + \frac{\alpha}{2}\bar{u}^2$.

Based on the proof, it has been demonstrated that optimal control in the divorce dynamics model is guaranteed to exist, allowing for the implementation of education and counseling control in the SMHU_hD model. □

V. CONDITION OF OPTIMAL

To find the optimal solution, a Hamiltonian associated with our optimal control problem will be formed.

5.1. Hamiltonian

The optimal control problem is satisfied if it meets the Pontryagin's Maximum Principle, with the initial step being to determine the Hamiltonian function corresponding to equation (2), resulting in:

$$H_a = U_h + D + \frac{1}{2}Au_1^2 + \frac{1}{2}Bu_2^2 + \sum_{\lambda=1}^{5} \lambda_i g_i \tag{4}$$

so it will result in the following Hamiltonian equation.

$$\begin{aligned} H_a(t,x,u,\lambda) = U_h + D + \frac{1}{2}Au_1^2 + \frac{1}{2}Bu_2^2 \\ + \lambda_1[\pi N + \rho D - \gamma S - \mu S] \\ + \lambda_2[\gamma S - (1-u_1)\beta MD - \mu M] \\ + \lambda_3[(1-u_1)(1-\alpha)\beta MD + \delta u_2 U_h - \theta H - \mu H] \\ + \lambda_4[(1-u_1)\alpha\beta MD + \theta H - (1-\delta u_2)\tau U_h - \delta u_2 U_h - \mu U_h] \\ + \lambda_5[(1-\delta u_2)\tau U_h - (\rho+\varepsilon+\mu)D] \end{aligned}$$

To obtain the optimal condition of the Hamiltonian function, it must satisfy the conditions of *stationer*, *state*, and *costate*.

5.2. Stationary Condition

Stationary conditions require the optimal controls u_1 and u_2 to minimize the Hamiltonian function. Therefore, the conditions that must be met are that the first derivatives of the Hamiltonian with respect to u_1 and u_2 are each equal to zero.

$$\begin{aligned} &\frac{\partial H_a}{\partial u_1} = 0 \\ &\leftrightarrow Au_1 + \lambda_2\beta MD + \lambda_3(\alpha-1)\beta MD - \lambda_4\alpha\beta MD = 0 \\ &\leftrightarrow Au_1 = \alpha\beta MD\lambda_4 - \beta MD\lambda_2 - \lambda_3(\alpha-1)\beta MD \\ &\leftrightarrow u_1 = \frac{\beta MD(\alpha\lambda_4 - \lambda_2 - (\alpha-1)\lambda_3)}{A} \end{aligned}$$

$$\begin{aligned} &\frac{\partial H_a}{\partial u_2} = 0 \\ &\leftrightarrow Bu_2 - \lambda_4(\delta U_h - \tau\delta U_h) + \lambda_3\delta U_h - \lambda 5\tau\delta U_h = 0 \\ &\leftrightarrow Bu_2 = \delta U_h\lambda_4 - \tau\delta U_h\lambda_4 - \delta U_h\lambda_3 + \lambda_5\tau\delta U_h \\ &\leftrightarrow u_2 = \frac{\delta U_h(\lambda_4 - \lambda_3) - \tau\delta U_h(\lambda_4 - \lambda_5)}{B} \end{aligned}$$

The control is restricted in $U = \{(u_1, u_2) | 0 \le u_1 \le 1,\ \ 0 \le u_2 \le 1\}$, the optimal control values u_1^* and u_2^* are obtained as follows.

$$u_1^* = \min\left\{\max\left(0, \frac{\beta MD(\alpha\lambda_4 - \lambda_2 - (\alpha-1)\lambda_3)}{A}\right), 1\right\}$$

$$u_2^* = \min\left\{\max\left(0, \frac{\delta U_h(\lambda_4 - \lambda_3) - \tau\delta U_h(\lambda_4 - \lambda_5)}{B}\right), 1\right\}$$

In optimal control theory, the state equations describe the dynamics of the system, namely the evolution of the state variables over time as a consequence of the initial conditions, model parameters, and applied controls. Thus, the state equations serve as dynamic constraints in the optimization process. Meanwhile, the costate (adjoint) equations describe how variations in the state variables affect the objective functional and play an important role in determining the optimality conditions for the control variables. According to the Pontryagin Maximum Principle, the costate equations are obtained from the negative partial derivatives of the Hamiltonian function with respect to the state variables, given by

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$

where H denotes the Hamiltonian function, x represents the state variables, and λ denotes the corresponding costate variables. Therefore, the state and costate equations are solved simultaneously to obtain the optimal control that minimizes or maximizes the objective functional, with the state equations generally solved forward in time from the initial conditions and the costate equations solved backward in time from the terminal conditions.

5.3. State Equation

The divorce dynamics model with control is formulated through the following state equation or constraint functions.

$$\begin{aligned}
\frac{\partial H_a}{\partial \lambda_1} &= \frac{dS}{dt} && = \pi N + \rho D - \gamma S - \mu S \\
\frac{\partial H_a}{\partial \lambda_2} &= \frac{dM}{dt} && = \gamma S - (1-u_1)\beta M D - \mu M \\
\frac{\partial H_a}{\partial \lambda_3} &= \frac{dH}{dt} && = (1-u_1)(1-\alpha)\beta M D + \delta u_2 U_h - \theta H - \mu H \\
\frac{\partial H_a}{\partial \lambda_4} &= \frac{dU_h}{dt} && = (1-u_1)\alpha\beta M D + \theta H - (1-\delta u_2)\tau U_h - \delta u_2 U_h - \mu U_h \\
\frac{\partial H_a}{\partial \lambda_5} &= \frac{dD}{dt} && = (1-\delta u_2)\tau U_h - (\rho + \varepsilon + \mu) D
\end{aligned}$$

with initial conditions, namely:

$$S(0) = S_0, \quad M(0) = M_0, \quad H(0) = H_0, \quad U_h(0) = U_{h_0}, \quad D(0) = D_0.$$

5.4. Costate Equation

Based on the Pontryagin's Maximum Principle, the derivative of the Hamiltonian with respect to the state variable yields the following costate equations:

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -\frac{\partial H_a}{\partial S} && = \lambda_1(\gamma+\mu) - \lambda_2\gamma \\
\frac{d\lambda_2}{dt} &= -\frac{\partial H_a}{\partial M} && = \lambda_2(\mu - \beta D(u_1-1)) - \lambda_3\beta D(\alpha-1)(u_1-1) + \lambda_4\alpha\beta D(u_1-1) \\
\frac{d\lambda_3}{dt} &= -\frac{\partial H_a}{\partial H} && = \lambda_3(\theta+\mu) - \lambda_4\theta \\
\frac{d\lambda_4}{dt} &= -\frac{\partial H_a}{\partial U_h} && = \lambda_4(\mu + \delta u_2 - \tau(\delta u_2 - 1)) - \lambda_3\delta u_2 + \lambda_5\tau(\delta u_2 - 1) - 1 \\
\frac{d\lambda_5}{dt} &= -\frac{\partial H_a}{\partial D} && = \lambda_5(\rho+\epsilon+\mu) - \lambda_1\rho - \lambda_2\beta M(u_1-1) - \lambda_3\beta M(\alpha-1)(u_1-1) \\
& && \quad + \lambda_4\alpha\beta M(u_1-1) - 1
\end{aligned}$$

with the transversal conditions, namely $\lambda_1(t) = \lambda_2(t) = \lambda_3(t) = \lambda_4(t) = \lambda_5(t) = 0$.

VI. NUMERICAL SIMULATIONS

The numerical simulation is obtained using the forward-backward sweep method, where the state equations are solved with a forward step and the costate equations are solved with a backward step. The initial values used are $S(0) = 1000, \quad M(0) = 20, \quad H(0) = 200, \quad U_h(0) = 100, \quad D(0) = 30$. Below are the values for each parameter in the model that will be used in the numerical simulation.

Table 2. Parameter Values in the SMHU_hD Model

Parameter	Value	Source
π	0.016	Assumed
μ	0.015	[15]
γ	0.08	[15]
α	0.1	[14]
β	0.00004	Assumed
θ	0.002	Assumed
δ	0.3	[15]
τ	0.05	[15]
ε	0.3	[15]
ρ	0.1	Assumed

The parameter values used in the numerical simulations of the SMHU_hD model were determined based on a combination of previous studies and data obtained from Statistics Indonesia. Parameters for which empirical statistical data were unavailable were estimated using

demographic proportions derived from Statistics Indonesia data or assigned based on reasonable assumptions to ensure the feasibility and numerical stability of the system.

The parameter π is calculated using a demographic ratio approach based on the adolescent age group obtained from Statistics Indonesia (BPS) data, according to the following equation:

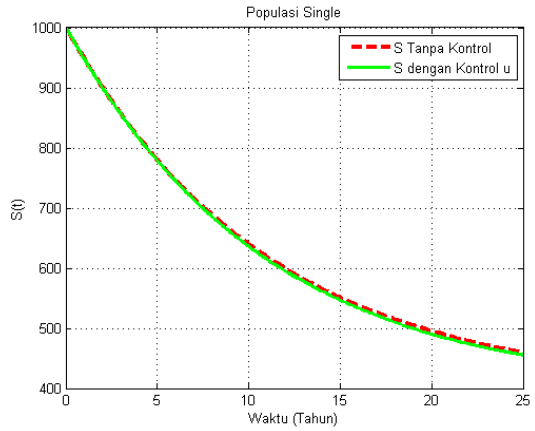
$$\pi = \frac{1}{5}\left(\frac{N_{15-19}}{N}\right)$$

where N_{15-19} denotes the population aged 15–19 years, and N denotes the total population.

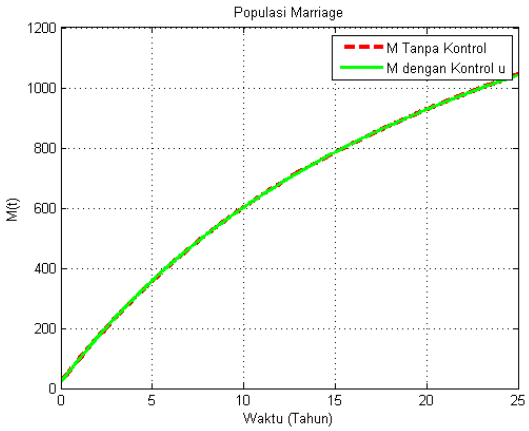
The parameters β and θ represent the external social interaction rate associated with the term (βMD) and the internal deterioration rate of healthy marital relationships ($H \rightarrow U_h$), respectively. The parameter β is assigned a relatively small value as a scaling factor for the bilinear interaction term to maintain the stability of the M subpopulation. Accordingly, $\theta = 0.002$ represents an internal deterioration rate of 0.2% per year. This value is set relatively small so that the primary drivers of marital deterioration in the model remain the external interaction dynamics and the effectiveness of the control interventions.

The parameter ρ represents the transition rate of individuals from the Divorce (D) subpopulation back to the Single (S) subpopulation. This parameter is calibrated using historical Statistics Indonesia (BPS) data for the period 2020–2022, which recorded a total of 1,261,764 divorce cases, corresponding to an average of 420,588 cases per year. Considering a remarriage rate of 33.33%, the annual transition rate of divorced individuals who become eligible to remarry is set at $\rho = 0.1$ relative to the accumulated divorced population.

The numerical simulations were performed using the forward-backward sweep method, in which the state equations were solved forward in time and the costate equations were solved backward in time using the 4th order Runge-Kutta method in MATLAB R2013a.



(a)



(b)

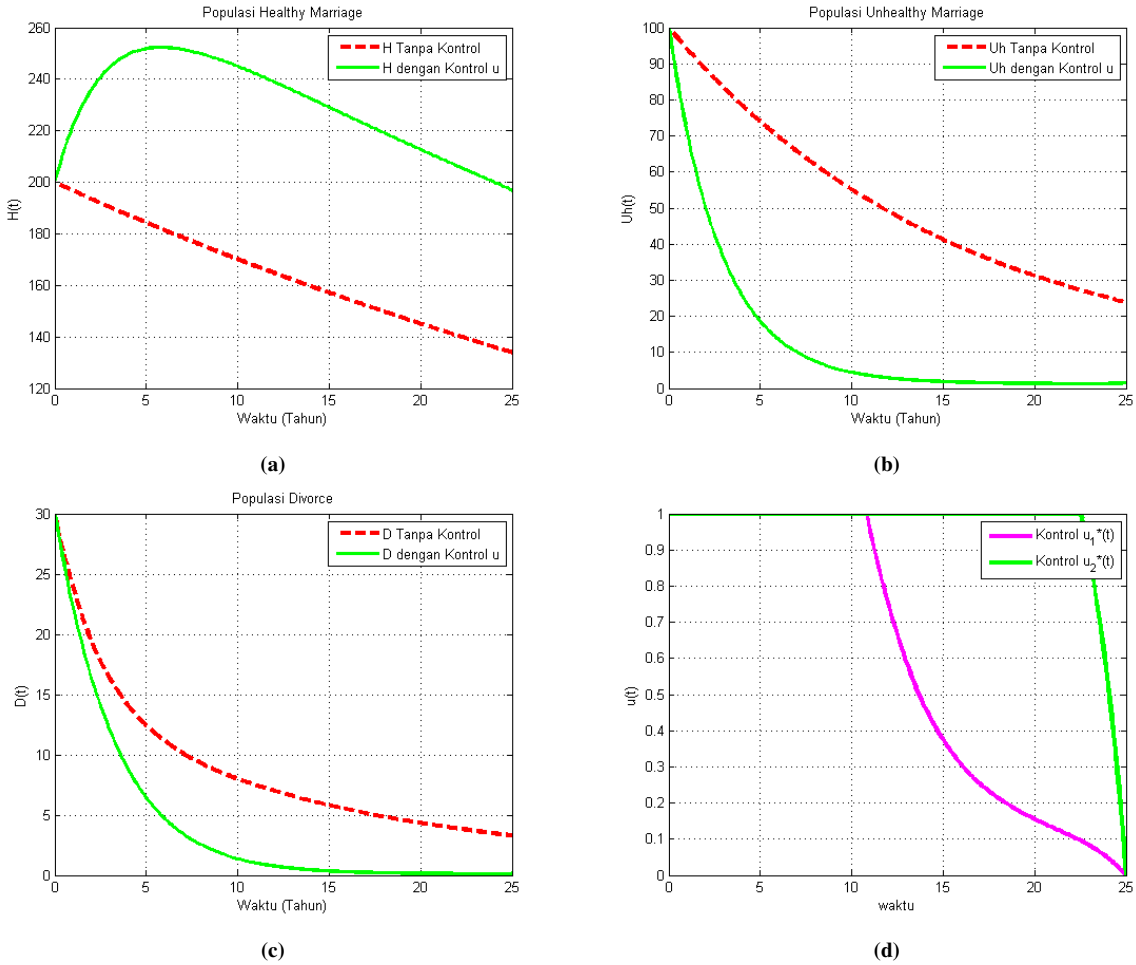


Figure 3. Simulation Results Before and After Control

The numerical simulations presented in Figure 2 illustrate the changes in the dynamics of each subpopulation in the SMHU_hD model before and after the implementation of the control measures. The simulation results indicate that the simultaneous implementation of post-marital education and counseling can reduce the number of couples in Unhealthy Marriages and decrease the number of Divorces. These results demonstrate that interventions through educational and counseling approaches play an important role in reducing the number of divorces.

VII. CONCLUSIONS

Based on the discussion presented above, the SMHU_hD model incorporates two control variables, namely post-marital education and counseling, which aim to minimize the sizes of the *Unhealthy Marriage* and *Divorce* subpopulations, as well as the intervention costs associated with the education and counseling controls. The numerical simulation results demonstrate that the combination of educational and counseling controls is effective in reducing the number of divorces. For future research, parameter sensitivity analysis can be conducted to assess the influence of model parameters on the system dynamics, and the model can be further extended by incorporating more complex interactions among the compartments.

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