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A NOTE ON OUTER-CONNECTED HOP ROMAN DOMINATING FUNCTION IN GRAPHS

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Abstract. Let $G = (V(G), E(G))$ be a simple, connected, and finite graph with vertex set $V(G)$ and edge set $E(G)$. Let $\varphi: V(G) \to \{0, 1, 2\}$ be an HRDF on G, and for each $i \in \{0, 1, 2\}$, let $V_i = \{u \in V(G): \varphi(u) = i\}$. A function $\varphi = (V_0, V_1, V_2)$ is an outer-connected hop Roman dominating function (OcHRDF) on G if for every $v \in V_0$, there exists $u \in V_2$ such that $d_G(v, u) = 2$ and either $V_1 = V(G)$ or the sub-graph $\langle V_0 \rangle$ is connected. The weigth of OcHRDF φ denoted by $\widetilde{\omega}_G^{chR}(\varphi)$ and defined by $\widetilde{\omega}_G^{chR}(\varphi) = \sum_{v \in V(G)} \varphi(v) = |V_1| + 2|V_2|$. The outer-connected hop Roman domination number of $G$is denoted by $\tilde{\gamma}_{chR}(G)$ and is defined by $\tilde{\gamma}_{chR}(G) = min\{\widetilde{\omega}_G^{chR}(\varphi): \varphi \text{ is an OcHRDF on } G\}$. Moreover, any OcHRDF φ on G with $\tilde{\gamma}_{chR}(G) = \widetilde{\omega}_G^{chR}(\varphi)$ is called $\tilde{\gamma}_{chR}$-function on G. In this paper, a new restricted parameter of a hop Roman domination in graphs is introduced, and some combinatorial properties are discussed..
Keywords: Outer-connected domination, hop Roman domination, connected subgraph

I. INTRODUCTION

Graph theory is one of the progressive branches of mathematics considering its variety of interesting topics that discrete mathematicians are devoting to publish new results [1]. Dominating sets in graphs is one of the intriguing topics in graph theory that is fast growing due to its interesting theoretic structures and application to other sciences [2]. Among the parameters of domination in graphs, the Roman dominating function has captured the interests of many mathematicians. Cockayne et al. [3] introduced the concept of *Roman dominating set* in connection to a historical problem of defending the Roman empire as represented in [4]. Let G be a connected and finite graph of order $n \geq 1$. A *Roman dominating function* (RDF) φ on a G is a mapping from V(G) to $\{0, 1, 2\}$, that is, $\varphi: V(G) \to \{0, 1, 2\}$ such that for every vertex $v \in V(G)$ for which $\varphi(v) = 0$ is adjacent to at least one vertex $u \in V(G)$ for which $\varphi(u) = 2$. The weight of an RDF function φon G denoted by $\omega_G(\varphi)$ is defined by $\omega_G(\varphi) = \sum_{v \in V(G)} \varphi(v)$ and the *Roman domination number* on G denoted by $\gamma_R(G)$ is defined by $\gamma_R(G) = min\{\omega_G(\varphi): \varphi \textit{ is an RDF on } G\}$, that is, the minimum weight of an RDF on graph G. So, any RDF φ on G with $\gamma_R(G) = \omega_G(\varphi)$ is called $\gamma_R(G)$-function on G. Apparently, there are several research studies has emerged on the topic of Roman domination in graphs which can be found in [5], [6], [7], [8]. A subset $H \subseteq V(G)$ is called a *hop dominating set* of G if, for every vertex in $u \in V(G) \backslash H$, there exists $v \in H$ such that $d_G(u, v)=0$ or $d_G(u, v)=2$. The smallest cardinality of a hop dominating set of G is called the *hop domination number* denoted by $\gamma_h(G)$. A hop dominating set of cardinality $\gamma_h(G)$ is called γ_h-set in G [9].

Let $\varphi: V(G) \to \{0,1,2\}$ be a function on G, and for each $i \in \{0,1,2\}$, let $V_i = \{u \in V(G): \varphi(u) = i\}$. A *hop Roman dominating function* (HRDF) on G is a function $\varphi = (V_0, V_1, V_2)$ in which for every $v \in V_0$, there exists $v \in V_0$ such that $d_G(u, v)$=2 [10]. The weight of an HRDF φ is the sum $\omega_G^{hR}(\varphi) = \sum_{v \in V(G)} \varphi(v) = |V_1| + 2|V_2|$. The *hop Roman domination number* of a graph G denoted by $\gamma_{hR}(G)$ is the minimum weight of an HRDF on G, that is, $\gamma_{hR}(G) = min\{\omega_G^{hR}(\varphi): \varphi\ is\ a\ HRDF\ on\ G\}$. Thus, every HRDF f on graph G with $\omega_G^{hR}(\varphi) = \gamma_{hR}(G)$ is called a γ_{hR}-function on G. Some interesting studies of *hop Roman domination* can be found in [11], [12].

A subset $O_c \subseteq V(G)$ is called an *outer-connected dominating set* of G provided that for $O_c = V(G)$ or $\langle V(G) \backslash S \rangle$ is connected. The smallest cardinality of an outer-connected dominating set of G is called the *outer-connected domination number* denoted by $\tilde{\gamma}_c(G)$. An outer-connected dominating set of cardinality $\tilde{\gamma}_c(G)$ is called $\tilde{\gamma}_c$-set in G [13]. A subset $H \subseteq V(G)$ is called a *hop dominating* set of G if, for every vertex in $u \in V(G) \backslash H$, there exists $v \in H$ such that $d_G(u, v)$=0 or $d_G(u, v)$=2. The smallest cardinality of a hop dominating set of G is called the *hop domination number* denoted by $\gamma_h(G)$. A hop dominating set of cardinality $\gamma_h(G)$ is called γ_h-set in G [14]. A subset $O_h \subseteq V(G)$ is called an *outer-connected hop dominating set* of G if for every vertex in $u \in V(G) \backslash H$, there exists $v \in H$ such that $d_G(u, v)$=0 or $d_G(u, v)$=2, and either $O_h = V(G)$ or the subgraph $\langle V(G) \backslash O_h \rangle$ is connected. The smallest cardinality of an outer-connected hop dominating set of G is called the *outer-connected hop domination number* denoted by $\tilde{\gamma}_{ch}(G)$. An outer-connected hop dominating set of cardinality $\tilde{\gamma}_{ch}(G)$ is called $\tilde{\gamma}_{ch}$-set in G [15]. In that case, our working definition for this study is constructed. A function $\varphi = (V_0, V_1, V_2)$ is an *outer-connected hop Roman dominating function* (OcHRDF) on G if, for every $v \in V_0$, there exists $u \in V_2$ such that $d_G(v, u) = 2$ and either $V_1 = V(G)$ or the sub-graph $\langle V_0 \rangle$ is connected. The weigth of OcHRDF φ denoted by $\tilde{\omega}_G^{chR}(\varphi)$ is defined by $\tilde{\omega}_G^{chR}(\varphi) = \sum_{v \in V(G)} \varphi(v) = |V_1| + 2|V_2|$. The *outer-connected hop Roman domination number* of G is denoted by $\tilde{\gamma}_{chR}(G)$ and is defined by $\tilde{\gamma}_{chR}(G) = min\{\tilde{\omega}_G^{chR}(\varphi): \varphi\ is\ an$ OcHRDF on $G\}$. In this case, any OcHRDF φ on G with $\tilde{\gamma}_{chR}(G) = \tilde{\omega}_G^{chR}(\varphi)$ is called $\tilde{\gamma}_{chR}$-function on G. Consider the graph G of order 5 in the figure below (Figure 1). Let $\varphi = (V_0, V_1, V_2)$ be an HRDF on G such that $V_0 = \{v_3, v_4, v_5\}$, $V_1 = \{v_2\}$, and $V_2 = \{v_1\}$. Note that the subgraph $\langle V_0 \rangle$ is connected. By construction, we have φ is an OcHRDF on G. Thus, $\tilde{\gamma}_{chR}(G) = |V_1| + 2|V_2| = 3$.

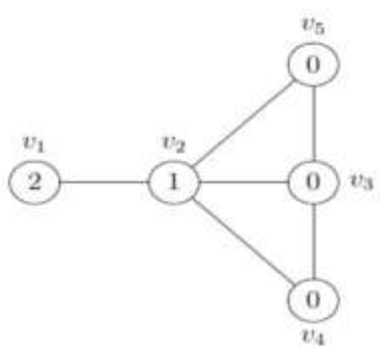

Figure 1. A graph G with $\tilde{\gamma}_{chR}(G) = 3$.

In this study, we need the following definitions of terms needed for the results. Let $G = (V(G), E(G))$ be a simple, connected, and finite graph with vertex set $V(G)$ and edge set $E(G)$. The order of graph G is given by $|V(G)|$ and the size is given by $|E(G)|$. An *open neighborhood* of a vertex $u \in V(G)$ is defined as a set $N_G(u) = N(u) = \{v \in V(G): uv \in E(G)\}$ and a *closed neighborhood* is defined as $N_G[u] = N[u] = \{u\} \cup N(u)$. In general, if we let $S \subseteq V(G)$, then the *open neighborhood* of set S is defined as $N_G(S) = N(S) = \bigcup_{u \in S} N_G(u)$, and the *closed neighborhood* of set S is defined as $N_G[S] = N[S] = S \cup N(S)$. We define the distance between two the vertices u and v in graph G by the length of the shortest path between u and v, denoted by $d_G(u,v)$. In addition, the degree of a vertex v in G is defined as the number of incident edges which is denoted by $deg_G(v)$. A path of order $n \geq 1$ is a graph denoted by P_n and can be described as a finite sequence of vertices that joins a sequence of edges. A cycle graph denoted by C_n is a graph that consists of a single cycle in which the number of vertices is connected in a closed chain. A complete graph denoted by K_n and defined as every pair of distinct vertices is connected by a unique edge. A complete bipartite graph denoted by $K_{m,n}$ where $m, n \geq 2$ is a special kind of bipartite graph and is defined as every vertex of the first set is connected to every vertices of the second set. The star graph denoted by S_n of order $\mathrm{n} + 1$ is obtained from $K_1 + \overline{K}_n$. The fan graph denoted by F_n is obtained from $K_1 + P_n$ where K_1 is a complete graph of order 1 and P_n is a path graph of order n. The order of fan graph F_n is $\mathrm{n} + 1$. The wheel graph denoted by W_n is of order $\mathrm{n} + 1$ and is obtained from $K_1 + C_n$. More definitions in graph theory can be found in [16], [17], [18], [19], [20], [21], [22], [23]. In this paper, we introduced a new restricted parameter of hop Roman domination in graphs and obtained some mathematical theoretic results. Moreover, the exact values of outer-connected hop Roman domination number for some classes of graphs were determined and some characterizations were obtained.

II. RESULTS

In this section, we present some interesting results of the outer-connected hop Roman dominating function on a connected graph G of order $n \geq 1$.

Proposition 2.1. *Let G be a connected graph. If $\varphi = (V_0, V_1, V_2)$ is a $\tilde{\gamma}_{chR}$-function on G, then $V_1 \cup V_2$ is an outer-connected hop dominating set on G.*

Proof: Assume that $\varphi = (\mathrm{V_0}, \mathrm{V_1}, \mathrm{V_2})$ is a $\tilde{\gamma}_{chR}$-function on graph G with order $n \geq 1$. Then φ is an OcHRDF on G. This implies that for each $v \in V_0$ there exists $u \in V_2$ such that $d_G(u,v) = 2$ and either $V_1 = V(G)$ or the sub-graph $\langle V_0 \rangle$ is connected. Therefore, it follows that $V_1 \cup V_2$ is an outer-connected hop dominating set in G. This completes the proof. □

Remark 2.2. *Let G be a connected graph of order n. If $\varphi = (V_0, V_1, V_2)$ is an OcHRDF on G with $|V_0| = |V_2|$, then $\tilde{\gamma}_{chR}(G) = n$.*

Proof: Suppose that $\varphi = (V_0, V_1, V_2)$ is an OcHRDF on G for which $|V_0| = |V_2|$. In that case, we obtain $\tilde{\gamma}_{chR}(G) = |V_1| + 2|V_2| = |V_1| + |V_2| + |V_0| = |V(G)| = n$. This completes the proof. □

Theorem 2.3. *Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on any connected graph G with order n. Then $\tilde{\gamma}_{chR}(G) < n$ if and only if $|V_2| < |V_0|$.*

Proof: Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on $G$with $|V(G)| = n \geq 1$. Assume that $\tilde{\gamma}_{chR}(G) < n$. By Remark 2.2, we have $|V_2| \neq |V_0|$. Now, suppose that $|V_2|>|V_0|$. Then, it follows that $\tilde{\gamma}_{chR}(G) = \widetilde{\omega}_G^{chR}(\varphi) = |V_1| + 2|V_2| > |V_1| + |V_2| + |V_0| = |V(G)| = n$, a contradiction. Thus, it suffices to conclude that $|V_2| < |V_0|$. Conversely, assume that $|V_2| < |V_0|$. Then, it implies that $\tilde{\gamma}_{chR}(G) = \widetilde{\omega}_G^{chR}(\varphi) = |V_1| + 2|V_2| < |V_1| + |V_2| + |V_0| = |V(G)| = n$. This completes the proof. □

Theorem 2.4. *Let G be a connected graph with $|V(G)| = n \geq 1$ and $\varphi = (V_0, V_1, V_2)$ be an OcHRDF on G. If $V_1 \cup V_2$ is a $\tilde{\gamma}_{ch}$-set on G and $|V_2|$ is minimal, then $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G.*

Proof: Let $\varphi = (V_0, V_1, V_2)$ be a OcHRDF on G with $|V(G)| = n \geq 1$. Suppose that $V_1 \cup V_2$ is a $\tilde{\gamma}_{ch}$-set on G and $|V_2|$ is minimal. Seeking for contradiction. Assume for a moment that $\varphi = (V_0, V_1, V_2)$ is not a $\tilde{\gamma}_{chR}$-function on G. Then, there exists a $\tilde{\gamma}_{chR}$-function $\sigma = (W_0, W_1, W_2)$ on G such that $\tilde{\gamma}_{chR}(G) = \widetilde{\omega}_G^{chR}(\sigma) = |W_1| + 2|W_2| < |V_1| + 2|V_2| = \widetilde{\omega}_G^{chR}(\varphi)$ where $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ such that $|V_2|$ is minimal. Consequently, $|W_1| \leq |V_1|$ and $|W_2| \leq |V_2|$. Since $W_1 \cap W_2 = \emptyset$ and $V_1 \cap V_2 = \emptyset$, it follows that $|W_1 \cup W_2| < |V_1 \cup V_2|$. And it implies that $|W_1| + |W_2| < |V_1| + |V_2|$, a contradiction since $V_1 \cup V_2$ is a $\tilde{\gamma}_{chR}$-set on G. Hence, it suffices to say that $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G. This completes the proof. □

Theorem 2.5. *Let G be a connected graph of order $n \geq 1$ and $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G. If $V_1 = \emptyset$, then $V_2 \neq \emptyset$ is a $\tilde{\gamma}_{ch}$-set and $\tilde{\gamma}_{chR}(G) = 2\tilde{\gamma}_{ch}(G)$.*

Proof: Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G of order $n \geq 1$. Assume that $V_1 = \emptyset$. Then by Theorem 2.1., $V_1 \cup V_2 = V_2$ is an outer-connected hop dominating set on G. Seeking for a contradiction. Assume for a moment that V_2 is not a $\tilde{\gamma}_{ch}$-set on G. Let $\widetilde{V_2}$ be a $\tilde{\gamma}_{ch}$-set on G. Then it follows that $\widetilde{V_2} \subset V_2$. Define a function $\sigma = (W_0, W_1, W_2)$ on G for which $W_0 = V(G) \setminus \widetilde{V_2}$, $W_1 = \emptyset$, and $W_2 = \widetilde{V_2}$. It implies that a mapping $\sigma = (W_0, W_1, W_2)$ is an OcHRDF on G and it follows that $\widetilde{\omega}_G^{chR}(\sigma) = 2|W_2| < 2|V_2| = \widetilde{\omega}_G^{chR}(\varphi) = \tilde{\gamma}_{chR}(G)$. This is a contradiction since $\varphi = (V_0, V_1, V_2)$ is a $\tilde{\gamma}_{chR}$-function on G. Therefore, it suffices to conclude that V_2 is a $\tilde{\gamma}_{ch}$-set on G and so, $|V_2| = \tilde{\gamma}_{ch}(G)$. Moreover, we end up with $\tilde{\gamma}_{chR}(G) = |V_1| + 2|V_2| = 2|V_2| = 2\tilde{\gamma}_{ch}(G)$. This completes the proof. □

Theorem 2.6. *Let G be a connected graph of order $n \geq 1$ and $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G. Then $V_0 = \emptyset$ if and only if $V_2 = \emptyset$ and $\tilde{\gamma}_{chR}(G) = n$.*

Proof: Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G with $|V(G)| = n \geq 1$. Suppose that $V_0 = \emptyset$. Seeking for contradiction. Assume for a moment that $V_2 \neq \emptyset$. Let $u \in V_2$. Also, let $W_0 = V_0$, $W_1 = V_1 \cup \{u\}$, and $W_2 = V_2 \setminus \{u\}$. This implies that $\sigma = (W_0, W_1, W_2)$ is an OcHRDF on G. It is worth noting that $\widetilde{\omega}_G^{chR}(\sigma) = |W_1| + 2|W_2| = (|V_1| + 1) + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 \leq \widetilde{\omega}_G^{chR}(\varphi) = \tilde{\gamma}_{chR}(G)$. This is a contradiction since $\varphi = (V_0, V_1, V_2)$ is a $\tilde{\gamma}_{chR}$-function on G. Hence, it suffices to conclude that $V_2 = \emptyset$. Moreover, $\tilde{\gamma}_{chR}(G) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. Conversely, let $V_2 = \emptyset$. In that case, it is easy to see that $V_0 = \emptyset$. This completes the proof. □

The following results are the exact values of the outer-connected hop Roman domination number of some special graphs.

Proposition 2.7. *Let* $G = P_n$ *be a path of order* $n \geq 1$. *Then* $\tilde{\gamma}_{chR}(G) = n$.

Proof: Assume that $G = P_n$ where $n \geq 1$. Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G. Clearly, if $n = 1$ or 2, then $\tilde{\gamma}_{chR}(G) = |V_1| = |V(G)| = n$ and so, $\tilde{\gamma}_{chR}(G) = 1$ or 2, respectively. Let $n = 3$. Then, there exists $v \in V_2$ and $u \in V_0$ such that $d_G(v, u) = 2$. In that case, the remaining vertex w with $d_G(v, w) = 1 = d_G(w, u)$ implies that $w \in V_1$. Since the subgraph induced by V_0 is a trivial connected graph, i.e., $\langle V_0 \rangle = K_1$, it simply follows that $\tilde{\gamma}_{chR}(G) = 3$. Now, let $n \geq 4$. Seeking for contradiction. Assume for a moment that $\tilde{\gamma}_{chR}(G) < n$. Then, by Remark 2.2, we have $|V_0| \neq |V_2|$. Suppose that $|V_0| < |V_2|$. Then, we get $\tilde{\gamma}_{chR}(G) = |V_1| + 2|V_2| > |V_1| + |V_2| + |V_0| = |V(G)| = n$, a contradiction. On the other hand, suppose that $|V_0| > |V_2|$. Then, there exists $x \in V_2$ such that $|N_G^2(x) \cap V_0| = 2$. Since G is a path graph, then $x \in V_2$ is a cut vertex and so the sub-graph induced by V_0 is disconnected, a contradiction. Hence, it suffices to conclude that $\tilde{\gamma}_{chR}(G) = n$. This completes the proof. ☐

Proposition 2.8. *Let* $G = C_n$ *be a cycle of order* $n \geq 3$. *Then,* $\tilde{\gamma}_{chR}(G) = n$.

Proof: Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on $G = C_n$ with order $n \geq 3$. In view of Proposition 2.7, it is clear that $\tilde{\gamma}_{chR}(G) = |V_1| = |V(G)| = n$. This completes the proof. ☐

Proposition 2.9. *Let* $G = S_n$ *with* $n \geq 3$. *Then,* $\tilde{\gamma}_{chR}(G) = n + 1$.

Proof: Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on $G = S_n$ with order $n \geq 3$. Then we have $G = K_1 + \overline{K_n}$ and $|V(G)| = n + 1$. Since $\langle V(G) \setminus V(K_1) \rangle = \overline{K_n}$, it is easy to check that $V_2 = \emptyset$. Therefore, $V(G) = V_1$ and so, $\tilde{\gamma}_{chR}(G) = |V_1| = |V(G)| = n + 1$. This completes the proof. ☐

The following results gives an outer-connected hop Roman domination number lesser than the order of some special graphs.

Proposition 2.10. *Let* $G = K_{m,n}$ *with* $m, n \geq 2$. *Then,* $\tilde{\gamma}_{chR}(G) = 4$.

Proof: Let $G = K_{m,n} = (U, V, E)$ be a complete bipartite graph where U and V denote the partition, that is, $|U| = m$ and $|V| = n$, and E is the edge set of G. Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G. Then, choose any vertex $u \in U$ and $v \in V$ such that $V_2 = \{u, v\}$. Since $U \setminus \{u\} \subseteq N_G^2(u)$ and $V \setminus \{v\} \subseteq N_G^2(v)$, it follows that $V_1 = \emptyset$ and $V_0 = (U \cup V) \setminus \{u, v\}$. In this case, the subgraph induced by V_0 is connected. Now, since any removal of vertex u or v in $V_1 \cup V_2$ indicates a non-hop dominating set in G, it follows that by construction $\tilde{\gamma}_{chR}(G) = |V_1| + 2|V_2| = 2|\{u, v\}| = 2(2) = 4$. This completes the proof. ☐

Proposition 2.11. *If* $G = F_n$ *with* $n \geq 3$, *then* $\tilde{\gamma}_{chR}(G) = 4$.

Proof: Assume that $G = F_n$ for which $n \geq 3$. Then, $|V(G)| = n + 1$ and $G = K_1 + P_n$. In that case, there are 2 vertices of degree 2. Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G. Then, choose an arbitrary vertex $v \in V(G)$ with $deg_G(v) = 2$ to be $v \in V_2$. Then, it follows that $|N_G^2(v) \cap V(G)| = n - 2$. It implies that for all $u \in V(G)$ with $d_G(v, u) = 1$, $u \in V_1$. Since $deg_G(v) = 2$, it means that $|V_1| = 2$. Note that for any $w \in V_1 \cup V_2$, $(V_1 \cup V_2) \setminus \{w\}$ is no longer an outer-connected hop dominating set, hence $V_1 \cup V_2$ is already minimum. Hence, we end up with $\tilde{\gamma}_{chR}(G) = |V_1| + 2|V_2| = 2 + 2(1) = 4$. This completes the proof. □

Proposition 2.12. *If $G = W_n$ with $n \geq 3$, then $\tilde{\gamma}_{chR}(G) = 5$.*

Proof: Assume that $G = W_n$ for which $n \geq 3$. Then, $|V(G)| = n + 1$ and $V(G) = V(K_1) \cup V(C_n)$. Choose an arbitrary vertex $v \in V(C_n)$. Then, it follows that $deg_G(v)$ =3 and $|N_G^2(v) \cap V(G)| = |V(C_n)| - 3$. Now, let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G. This means that for each $u \in V(G)$ with $d_G(v, u) = 1$ implies that $u \in V_1$ and so, $|V_1| = 3$, $|V_2| = 1$. It is worth noting that for any $w \in V_1 \cup V_2$, $(V_1 \cup V_2) \setminus \{w\}$ is not an outer-connected hop dominating set on G, hence $V_1 \cup V_2$ is the minimum outer-connected hop dominating set. Therefore, we obtain $\tilde{\gamma}_{chR}(G) = |V_1| + 2|V_2| = 3 + 2(1) = 5$. This completes the proof. □

One of the graphs that does not contain a distance of at least two on its vertices is a complete graph. Hence, the following Theorem is immediate.

Proposition 2.13. *If $G = K_n$ with $n \geq 1$, then $\tilde{\gamma}_{chR}(G) = n$.*

Proof: Suppose that $G = K_n$ for which $n \geq 3$. Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G. Seeking for contradiction. Assume for a moment that $\tilde{\gamma}_{chR}(G) < n$. Then, it follows that $V_0 \neq \emptyset$ and hence, $V_2 \neq \emptyset$. Let $v \in V_0$. Then, there exists $u \in V_2$ such that $d_G(v, u)$ =2. This is a contradiction since $G = K_n$. Therefore, $\tilde{\gamma}_{chR}(G) = n$. This completes the proof. □

The next result is a characterization of outer-connected hop Roman domination numbers with small values.

Theorem 2.14. *Let G be a connected graph with $|V(G)| = n$. Then the following holds:*

i. $\tilde{\gamma}_{chR}(G) = 1$ *if and only if* $G = K_1$; *and*

ii. $\tilde{\gamma}_{chR}(G) = 2$ *if and only if* $G = K_2$.

Proof: Assume that G is a connected graph with $|V(G)| = n$. Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G. Suppose that $\tilde{\gamma}_{chR}(G) = 1$. Seeking for contradiction. Assume for a moment that $G \neq K_1$. Then $|V(G)| > 1$. Now, if $|V_0| \geq 1$, then $|V_2| \geq 1$. In this case, we have $\tilde{\gamma}_{chR}(G) \geq 2|V_2| > 1$, a contradiction. Moreover, if $|V_0| = 0$, then $|V_2| = 2$. This follows that $\tilde{\gamma}_{chR}(G) = |V_1| = |V(G)| > 1$, contrary to the assumption. The converse is easy. Hence, (*i.*) holds. On the other hand, suppose that $\tilde{\gamma}_{chR}(G) = 2$. Then we have $|V_1| + 2|V_2| = 2$ and so, $|V_2| \leq 1$. Then consider the following cases:

Case 1. Let $|V_2| = 0$.

In this case, $|V_0| = 0$ and it follows that $\tilde{\gamma}_{chR}(G) = |V_1| = |V(G)| = 2$. Since G is connected, it follows that $G = K_2$.

Case 2. Let $|V_2| = 1$.

Then $|V_1| = 0$. Let $V_2 = \{u\}$. Then there exists $v \in V_0$ such that $d_G(u,v) = 2$. Now, let $w \in N_G(u) \cap N_G(v)$. Since $|V_1| = 0$, it means that $V_2 = \{u\}$ is not a hop dominating set in G, a contradiction. Hence, $|V_2| = 1$ is not possible when $\tilde{\gamma}_{chR}(G) = 2$.

And the converse is clear. Hence, $(ii.)$ holds. □

The following remark is useful for our next result.

Remark 2.15. *Let G be a connected graph. Then $\gamma_R(G) \leq \tilde{\gamma}_{chR}(G)$.*

The next theorem determines the lower and upper bound of the outer-connected hop Roman domination number of a graph.

Theorem 2.16. *Let G be a connected graph with order n. Then*

$$max\{\tilde{\gamma}_{ch}(G), \gamma_R(G)\} \leq \tilde{\gamma}_{chR}(G) \leq min\{n, 2\tilde{\gamma}_{ch}(G)\}$$

Proof: Assume that G is a connected graph with $|V(G)| = n$. Let $\varphi = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{chR}$-function on G. By Proposition 2.1., $V_1 \cup V_2$ is an outer-connected hop dominating set on G. Thus, it follows that $\tilde{\gamma}_{ch}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \tilde{\omega}_G^{chR}(\varphi) = \tilde{\gamma}_{chR}(G)$. By Remark 2.15., we get $\gamma_R(G) \leq \tilde{\gamma}_{chR}(G)$. Thus, $max\{\tilde{\gamma}_{ch}(G), \gamma_R(G)\} \leq \tilde{\gamma}_{chR}(G)$. On the other hand, $\varphi = (\emptyset, V(G), \emptyset)$ is a OcHRDF on G. And so, $\tilde{\gamma}_{chR}(G) \leq \tilde{\omega}_G^{chR}(\varphi) = |V_1| + 2|V_2| = |V_1| \leq |V(G)| = n$. Now, if $\varphi = (V_0, \emptyset, V_2)$ is a $\tilde{\gamma}_{chR}$-function G, then by Theorem 2.5., V_2 is a $\tilde{\gamma}_{ch}$-set on G, that is, $|V_2| = \tilde{\gamma}_{ch}(G)$. So, it follows that $\tilde{\gamma}_{chR}(G) \leq \tilde{\omega}_G^{chR}(\varphi) = |V_1| + 2|V_2| = 2|V_2| = 2\tilde{\gamma}_{ch}(G)$. Therefore, $\tilde{\gamma}_{chR}(G) \leq min\{n, 2\tilde{\gamma}_{ch}(G)\}$ and so, $max\{\tilde{\gamma}_{ch}(G), \gamma_R(G)\} \leq \tilde{\gamma}_{chR}(G) \leq min\{n, 2\tilde{\gamma}_{ch}(G)\}$. This completes the proof. □

The next result is a characterization of an OcHRDF in the join of two connected graphs.

Theorem 2.17. *Let G and H be connected graphs. Then $\varphi = (V_0, V_1, V_2)$* is an OcHRDF on $G + H$ if and only if $\varphi|_G$ and $\varphi|_H$ are OcHRDF on G and H, respectively.

Proof: Assume that $\varphi = (V_0, V_1, V_2)$ is an OcHRDF on $G + H$. Let $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$ for each $i \in \{0,1,2\}$. Then, $\varphi|_G = \{V_0^G, V_1^G, V_2^G\}$ and $\varphi|_H = \{V_0^H, V_1^H, V_2^H\}$. Let $x \in V_0^G$. Then $x \in V_0$. Since φ is an OcHRDF on $G + H$, there exists $y \in V_2$ such that $d_{G+H}(x,y) = 2$ and $\langle V_0 \rangle$ is connected. Since $d_{G+H}(x,w) = 1$ for all $w \in V(H)$, it follows that $y \in V_2^G$ and $\langle V_0^G \rangle$ is connected. Hence, $\varphi|_G$ is an OcHRDF on G. By similar argument, it is also concluded that $\varphi|_H$ is an OcHRDF on H. Conversely, assume that $\varphi|_G$ and $\varphi|_H$ are OcHRDF on G and H, respectively. For each $j \in \{0,1,2\}$, let $V_j = V_j^G \cup V_j^H$. Then $\varphi = (V_0, V_1, V_2)$ is a function on G. Let $a \in V_0$. Then $a \in V_0^G$ or $a \in V_0^H$. Without loss of generality, consider $a \in V_0^G$. Since $\varphi|_G$ is an OcHRDF on G, there exists $b \in V_2^G$ such that $d_G(a,b) = 2$ and $\langle V_0^G \rangle$ is connected. Now, since $V_j^G \subseteq V_j$ for all $j \in \{0,1,2\}$, it implies that $b \in V_2$ and $\langle V_0 \rangle$ is connected. Therefore, it suffices to conclude that φ is an OcHRDF on $G + H$. This completes the proof. □

The following corollary and remark are direct consequence of Theorem 2.17.

Corollary 2.18. *Let G and H be connected graphs with $|V(G)| = n$ and $|V(H)| = m$, respectively. Then $\tilde{\gamma}_{chR}(G + H) = \tilde{\gamma}_{chR}(G) + \tilde{\gamma}_{chR}(H)$.*

Remark 2.19. *Let G and H be complete graphs with $|V(G)| = n$ and $|V(H)| = m$, respectively. Then $\tilde{\gamma}_{chR}(G + H) = n + m$.*

III. CONCLUSION

This paper has introduced a new parameter variation of the hop Roman dominating function on a graph namely the outer-connected hop Roman dominating function. It is depicted that if $\varphi = (V_0, V_1, V_2)$ is a $\tilde{\gamma}_{chR}$-function on graph G, then $V_1 \cup V_2$ is an outer-connected hop dominating set on G. It is concluded that if $\tilde{\gamma}_{chR}(G) < n$, then $|V_2| < |V_0|$ and the converse is also true. The outer-connected hop Roman domination number has been characterized with respect to small values, e.g. 1 or 2. In addition, the outer-connected hop Roman domination number of some special graphs has been determined and provided detailed proof. Moreover, the bounds of outer-connected hop Roman domination numbers on a graph have been investigated. For future research, it is interesting to explore the combinatorial properties of an outer-connected hop Roman dominating function under some product operations in graphs.

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