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ON PROPERTIES OF PROJECTIVE SPACE DETERMINED BY QUOTIENT MAP

Edi Kurniadi[1*], Badrulfalah[2]

[1,2]*Departemen Matematika FMIPA Unpad*
Email : [1]edi.kurniadi@unpad.ac.id, [2]badrulfalah@unpad.ac.id
*Corresponding Author

Abstract. The state of a system in quantum theory is not always described by an element of a Hilbert space but by an element of projective space. The research aims to prove that the real projective space consisting of one-dimensional linear subspaces is a smooth manifold which is constructed by a quotient map. It is shown that a projective space is a Hausdorff space, second countable, and n-dimensional locally Euclidean. It is also proved that the n-dimensional real a projective space is homeomorphic to the quotient topology $\mathbb{R}^{n+1} - \{0\}$. The proof involves a quotient map which is defined by a quotient topology.
Keywords: Projective space, Quotient map, Quotient topology, Subspaces, Smooth manifold

I. INTRODUCTION

The Euclidean space contributes in a fundamental of mathematics. Recently, this space can be generalized in naturally extension sense to a projective space. Fundamentally, a projective space has important role in many concepts of modern mathematics, in particular in geometry. Roughly speaking, a projective space consists of one-dimensional subspaces of a vector space. In the other words, it is identified by lines through the origin. It is mentiond here, at least there are three perspectives why a projective space is significant in many areas of both pure mathematics and applied mathematics. Firstly, in algebraic view, a projective space gives a fundamental theory in studying algebraic varieties and homogeneous polynomials. Secondly, in quantum mechanics view, a projective space relates to the space of states of a quantum system, and thirdly, in topology and geometry, a projective space admits a rich of property of a smooth manifold. The latter view, it gives motivations of this research.

In many recent researches, many fields corresponds to projective spaces, for example, it is considered on Ulrich bundles on double covers of projective spaces [1], projective space containing symplectic leaves [2], and permutation polynomials [3]. Subsequently, it is also discussed the stability of the quartenion projective space [4], biharmonic submanifolds of the quarternionic projective space [5], distances class of projective space [6], Hodge Cohomology class [7], and Ulrich bundles on double covers [8]. In other words, these new discoveries involve the notion of a projective space. In [9], Lee stated that projective space is a smooth manifold and he proved a projective space as an n-dimensional locally Euclidean. But the proof is not complete yet, including how to apply a quotient topology. Using a quotient topology, here, it is showed that there exists an open subset of a projective space. Moreover, it is applied a quotient map to prove that a projective space is an n-dimensional locally Euclidean. However, the notion of a projective space as a Hausdorff and second countable spaces still needs to be proven in detail. Moreover, the notion of smoothness in a projective space is also proven. The smooth notion equal to the notion of infinitely differentiable which is nothing but $\mathbf{C}^{\infty}$. Furthermore, it is also discussed one-dimensional complex projective space corresponding to the concept of projective representations ([10], [11], [12], [13]).

The paper is organized as follows. In introduction, it is stated literature reviews, a gap of research, and significance of research. In preliminaries, the important materials are proposed: quotient topology, quotient map, topological manifold, smoothness or infinitely differentiable. In results, it shown that a projective space is a smooth manifold through a quotient map. It is also shown a projective space in concept of representation theory, particularly in one-dimensional complex projectve space.

II. PRELIMINARIES

Some important materials such as quotient topology, quotient map, open equivalnce relation, openness of a quotient map are proposed briefly.

Definition 1 [9]. *A topological space $\mathfrak{M}$ is said to be an n-dimensional manifold it it admits the following conditions*:

1. $\mathfrak{M}$ *is second countable*
2. $\mathfrak{M}$ *is a Hausdorff space*
3. *for each $m \in \mathfrak{M}$, there exist an open set $\mathfrak{U} \subseteq \mathfrak{M}$ containing m such that the following map is a homeomorphism*

$$\psi: \mathfrak{U} \to \widetilde{U} := \psi(U) \subseteq \mathbb{R}^n. \tag{1}$$

Example 1. The set $\mathbb{R}^n$ and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \ ; |x| = 1\} \subseteq \mathbb{R}^n$ are topological spaces because they admit second coutable bases, they are Hausdorff spaces and locally Euclideans of dimension n. The set $\mathbb{R}^n$ is Hausdorff space because for given point $x, y \in \mathbb{R}^n$ with $x \neq y$, it can be chosen $r = \frac{1}{2}|x - y|$, then both the open balls $\mathfrak{B}(x, r)$ and $\mathfrak{B}(y, r)$ shall be disjoint. In the other hand, since $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \ ; |x| = 1\}$ subspace of $\mathbb{R}^n$ then $\mathbb{S}^{n-1}$ is also Hausdorff space. Moreover, the collection of all open balls in $\mathbb{R}^n$ with rational radii and centers forms a basis for $\mathbb{R}^n$. In addition, this basis is countable. Thus, $\mathbb{R}^n$ is second countable. Second countable for $\mathbb{S}^{n-1}$ is inherited from second countable of $\mathbb{R}^n$.

Example 2. The set $\mathrm{O}(n) = \{A \in \mathrm{M}_n(\mathbb{R}) \ ; AA^t = I\}$ and $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) \ ; |A| = 1\}$ are topogical manifolds. We note here that $\mathrm{SO}(n) \simeq \mathbb{S}^{n-1}$ which consists of rotation matrices. In particular, for $n = 2$, it can be shown the map defined by

$$\tau: \mathrm{SO}(2) \ni \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto e^{i\theta} \in \mathbb{S}^1. \tag{2}$$

is bijective map.

New surfaces can be created by gluing the edges of malleable square [14]. The gluing is called a quotient process. In this stage, the final space obtained by a quotient space is not manifold generally eventhough the original space is a manifold.

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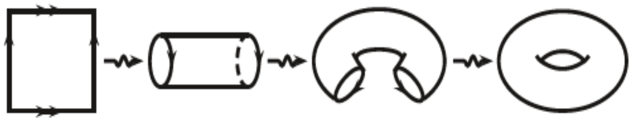

Figure 1. Gluing process of a square (source: [14])

Definition 2[14]. *Let A be a topological space and B be any set. Let $\xi: A \to B$ be a surjective map. The quotient topology on B considered by ξ is a subset $V \subseteq B$ such that the necessary and sufficient conditions for V be open in B are $\xi^{-1}(V) \subseteq A$ open. In the other words, V is open if and if $\xi^{-1}(V)$ is open in A.*

Definition 3[14]. *Let A and B be topological spaces. A map $\xi: A \to B$ is said to be a quotient map if B admits a quotient topology and a map ξ is continuous and surjective.*

A relation R is called an equivalence on a set H if R is reflexive, symmetric, and transitive. Let $a, b \in H$, a in relation to b is denoted by aRb. The equivalence class of $a \in H$ is denoted by $[a]$ and is defined by

$$[a] = \{b \in H \ ; aRb\}. \tag{3}$$

The set of all equivalence classes of H is denoted by H/R, namely, $H/R = \{ [a] \, ; a \in H\}$ and it is called the quotient of H generated by an equivalence realtion R.

Fact 1. The map defined by

$$\zeta: H \ni a \mapsto [a] \in H/R \tag{4}$$

is a natural projection map.

Example 3. In this case, H/R is topology space because it admits a topology. Sets $\emptyset, H/R$, are open sets, a union of family of open sets is open and a finite intersection of open sets is open. It is called a quotient topology as mentioned before.

Fact 2. The natural projection map defined in eq. (4) is continuous. To see this, in fact, it is known that $A \subseteq H/R$ is open if and only if $\zeta^{-1}(A) \subseteq H$ is open. Thus, The natural projection map is continuous.

Fact 3. Indeed, using analogy of the fist isomorphism theorem, for every map $\pi: H \to B$ there exist a map $\beta: H/R \to B$ such that $\pi = \beta \circ \zeta$. Namely, for every $a \in H$, then $\pi(a) = \beta \circ \zeta(a) = \beta(\zeta(a)) = \beta([a])$. In the other words, the map π **induces** the map β.

Fact 4. The continuity of β implies the continuity of π and vice versa. To see this, let β be the continuous map. Since the natural projection map is continuous and $\pi = \beta \circ \zeta$, then π is the continuous map. Conversely, let π be a continuous map. To see this, let U be open set in B. This implies that $\pi^{-1}(U) = \zeta^{-1}(\beta^{-1}(U))$ is open in H. But since H/R is a quotient topology then $\beta^{-1}(U)$ is open H/R . Thus, the map β is continuous.

III. RESULT AND DISCUSSION

First of all, it shall be discussed how to derive the real projective space using a quotient topolgy as discussed in previous. Let $\mathbb{RP}^n$ be the real projective space. Moreover, it shall proved the remain structures of $\mathbb{RP}^n$, namely a Hausdorff space and second countable. The topological manifold and a smooth structure of $\mathbb{RP}^n$ can be observed in ([9], pp. 6-7&21) . Furthermore, then by definition, It can be viewed that $\mathbb{RP}^n$ consists of all one-dimensional subspaces of $\mathbb{R}^{n+1}$, then elements of the projective space $\mathbb{RP}^n$ can be seen by an identification of a subset to a point.

To obtain the real projectictive space $\mathbb{RP}^n$, let a and b be points in $\mathbb{R}^{n+1} - \{0\}$. It is said that aRb if there exists $0 \neq \alpha \in \mathbb{R}$ such that $y = \alpha x$. It can be observed that R ia equivalence relation on $\mathbb{R}^{n+1} - \{0\}$. Equivalence classes of $a \in \mathbb{R}^{n+1} - \{0\}$ is given by

$$[a] = \{b \in \mathbb{R}^{n+1} - \{0\} \ ; \exists 0 \neq \alpha \in \mathbb{R} \text{ such that } b = \alpha a \}. \tag{5}$$

The quotient space $\mathbb{R}^{n+1} - \{0\}/R$ is called the real projective space and is denoted by $\mathbb{RP}^n$ as claimed before.

Example 4. Let $\bar{x}_1 = (x_1, y_1), \bar{x}_2 = (x_2, y_2)$ be non zero elements of $\mathbb{R}^2$. Let $\bar{x}_1 R \bar{x}_2$ if there exists $0 \neq \alpha \in \mathbb{R}$ such that $\bar{x}_2 = \alpha \bar{x}_1$. Then $\mathbb{R}^2 - \{0\}/R$ is nothing but the one-dimensional projective plane $\mathbb{RP}$. The elements of $\mathbb{RP}$ can be representated as follows:

$$[(x_1, y_1)] = \{(x_2, y_2) \in \mathbb{R}^2 - \{0\} \ ; \exists 0 \neq \alpha \in \mathbb{R} \text{ such that } (x_2, y_2) = \alpha(x_1, y_1) \}. \tag{6}$$

Geometrically, the elements of $\mathbb{RP} \simeq \mathbb{R}^2 - \{0\}/R$ can be viewed as points lie on the same line through the origin.

In the other words, from the equation (5) and it is confirmed by the equation (6), then the n-dimensional projective space $\mathbb{RP}^n \simeq \mathbb{R}^{n+1} - \{0\}/R$ can be interpretated as the set of all lines in $\mathbb{R}^{n+1}$ through the origin.

The following results confirm questions regarding the real projective space (see [9], pp 6-7 and [14]) . The results of the n-dimensional projective space $\mathbb{RP}^n \simeq \mathbb{R}^{n+1} - \{0\}/R$ are written in the following propositions.

Proposition 1. Let $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \ ; |x| = 1\}$ be torus. Let $x, y \in \mathbb{S}^n$ and let $xR'y$ if and only if $x = \pm y$. Then $\mathbb{RP}^n \simeq \mathbb{R}^{n+1} - \{0\}/R$ is a homeomorphic to $\mathbb{S}^n/R'$.

Proof. Firstly, It will be shown that the map given by

$$\pi: \mathbb{R}^{n+1} - \{0\} \ni x \mapsto \frac{x}{|x|} \in \mathbb{S}^n$$

induces the map

$$\beta: \mathbb{RP}^{\,n} \simeq \mathbb{R}^{n+1} - \{0\}/R \to \mathbb{S}^n/R'.$$

Let us observe the following commutative diagram.

$$\begin{tikzcd}
\mathbb{R}^{n+1} \arrow[r, "\pi"] \arrow[d, "\tau"'] & \mathbb{S}^n \arrow[d, "\psi"] \\
\mathbb{RP}^n \arrow[r, "\beta"] & \mathbb{S}^n/R'
\end{tikzcd}$$

Figure 2. Commutative diagram of a quotient map $\mathbb{RP}^n \simeq \mathbb{S}^n/R$.

The map β can be written as $\beta(x) = \psi \circ \pi \circ \tau^{-1}(x)$ for all $x \in \mathbb{RP}^n$. Thus, β is induced by the map π. Let us define the map

$$\beta\big(\tau(x)\big) = \beta([x]_R) = \psi\big(\pi(x)\big) = \psi\left(\frac{x}{|x|}\right) = \left[\frac{x}{|x|}\right]_{R'} \tag{7}$$

for every $x \in \mathbb{R}^{n+1} - \{0\}$. It is noticed that the equivalence classe of $\left[\frac{x}{|x|}\right]_{R'}$ are given by $\left\{\frac{x}{|x|}, -\frac{x}{|x|}\right\}$. Since for all $[x]_R$ and $[x']_R$ with $\beta([x]_R) = \beta([x']_R)$ then $\frac{x}{|x|} = \frac{x'}{|x'|}$ or $\frac{x}{|x|} = -\frac{x'}{|x'|}$. Therefore, there exist $\lambda = \pm 1$ such that $x = \lambda x'$. In the other words, $[x]_R = [x']_R$. Thus, β is one-one. Let $[x'']_{R'}$ be any element in $\mathbb{S}^n/R'$. Therefore, elements of $[x'']_{R'}$ can be chosen from $\{x'', -x''\} \subseteq \mathbb{S}^n \subseteq \mathbb{R}^{n+1} - \{0\}$. These elements span one-dimensional subspaces of $\mathbb{R}^{n+1} - \{0\}$ such that $[x'']_{R'} = \beta([x'']_R)$. Thus, β is onto. The map β is continuous since the maps π, τ, and ψ are continuous.

Now, it remains to show that β^{-1} is continuous. To see this, let $A \subseteq \mathbb{RP}^n$ be an open set. It shall be established that $\beta(A) \subseteq \mathbb{S}^n$ is open, namely, $\psi \circ \pi \circ \tau^{-1}(A)$ is open. Since $A \subseteq \mathbb{RP}^n$ is open, then by the quotient map of $\mathbb{RP}^n$ in Definition 3, the set $\tau^{-1}(A) \subseteq \mathbb{R}^{n+1} - \{0\}$ is also open. Notice that $\tau^{-1}(A) \cap \mathbb{S}^n \subseteq \mathbb{S}^n$ is open since:

$$\begin{aligned}
\tau^{-1}(A) \cap \mathbb{S}^n &= \{x \in \mathbb{R}^{n+1} - \{0\} \,; \tau(x) \in A\} \cap \{x \in \mathbb{R}^{n+1} - \{0\} \,; |x| = 1\} \\
&= \{x \in \mathbb{R}^{n+1} - \{0\} \,; [x]_R \in A\} \cap \{x \in \mathbb{R}^{n+1} - \{0\} \,; |x| = 1\} \\
&= \left\{x \in \mathbb{R}^{n+1} - \{0\} \,; \frac{x}{|x|} = \pi\right\} = \pi\big(\tau^{-1}(A)\big) = \pi \circ \tau^{-1}(A) \subseteq \mathbb{S}^n.
\end{aligned}$$

Let x be any element in $\tau^{-1}(A) \cap \mathbb{S}^n$. It shall be we shown that if $x \in \pi \circ \tau^{-1}(A)$ then $[x]_{R'} \subseteq \pi \circ \tau^{-1}(A) \subseteq \mathbb{S}^n$. By definition of R' then

$$[x]_{R'} = \{y \in \mathbb{S}^n \,; x = \pm y\} \subseteq \mathbb{R}^{n+1} - \{0\}.$$

Let $y \in [x]_{R'}$, then $x = \pm y$. Using inclusion map $\iota: \mathbb{S}^n \ni y \mapsto \iota(y) = y \in \mathbb{R}^{n+1} - \{0\}$, $\iota(x) R' \iota(y)$ in $\mathbb{R}^{n+1} - \{0\}$. Moreover, $\tau^{-1}(\{y\}) \subseteq \tau^{-1}(U)$. In other words, $\pi \circ \iota^{-1}(\{y\}) \subseteq \pi \circ \tau^{-1}(A)$, namely, $\pi \circ \iota^{-1}(y) \in \pi \circ \tau^{-1}(A)$. Thus, $\psi \circ \pi \circ \tau^{-1}(A) = \beta(A) \subseteq \mathbb{S}^n/R'$ is open.

Therefore, $\mathbb{RP}^n \simeq \mathbb{R}^{n+1} - \{0\}/R$ is a homeomorphic to $\mathbb{S}^n/R'$.

■

In addition, the second result, it shall discussed that the real projective space has Hausdorf space and second countable structures.

Proposition 2. Let $\mathbb{RP}^n$ be a real projective space. Then $\mathbb{RP}^n$ is a Hausdorff space.

Proof. Let us define the relation on $\mathbb{R}^{n+1} - \{0\}$ which is given by aRb if and if there exists a $t \in \mathbb{R} - \{0\}$ such that $b = ta$, for every $a, b \in \mathbb{R}^{n+1} - \{0\}$. Moreover, thet set $K = \{(a,b) \in \mathbb{R}^{n+1} - \{0\} \times \mathbb{R}^{n+1} - \{0\}\,; b = ta,\ t \in \mathbb{R} - \{0\}\}$ is characterized by an $(n+1) \times 2$ following matrix

$$T = \begin{pmatrix} a_1 & ta_1 \\ a_2 & ta_2 \\ \vdots & \vdots \\ a_n & ta_n \\ a_{n+1} & ta_{n+1} \end{pmatrix}. \tag{8}$$

The reduced row echelon matrix of the equation (8) is given by

$$S = \begin{pmatrix} a_1 & ta_1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

Therefore, the rank of the matrix $T = 1$ if $a_1 \neq 0$ and 0 if $a_1 = 0$. Namely, $\text{rank}\,(T) \leq 1$. Recalling that the relation R is an open equivalance relation if the projection map $\pi: \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}/R$ is open (see **Fact 2**). Firstly, it shall be proved that K is closed subset of $\mathbb{R}^{n+1} - \{0\} \times \mathbb{R}^{n+1} - \{0\}$ and secondly, it shall be shown that R is an open equivalance relation in $\mathbb{R}^{n+1} - \{0\}$.

- Firstly, it shall be shown that K is closed subset of $\mathbb{R}^{n+1} - \{0\} \times \mathbb{R}^{n+1} - \{0\}$. Since K contains of the intersection Zariski-closed sets, then K is closed subset of $\mathbb{R}^{n+1} - \{0\} \times \mathbb{R}^{n+1} - \{0\}$.

- Secondly, to see that R is an open equivalance relation in $\mathbb{R}^{n+1} - \{0\}$, let A be an open subset of $\mathbb{R}^{n+1} - \{0\}$. It shall be proved that $\pi^{-1}(\pi(A)) \subseteq \mathbb{R}^{n+1} - \{0\}$ is open. Because $t \in \mathbb{R} - \{0\} \underbrace{\simeq}_{\text{homeomorphic}} \mathbb{R}^{n+1} - \{0\}$ and $A \subseteq \mathbb{R}^{n+1} - \{0\}$ is open, then tA is also open. But, since $\pi^{-1}(\pi(A)) = \bigcup_{t \in \mathbb{R} - \{0\}} tA$, then $\pi^{-1}(\pi(A))$ is also open in $\mathbb{R}^{n+1} - \{0\}$. Therefore, R is an open equivalence relation in $\mathbb{RP}^n$.

Thus, $\mathbb{R}^{n+1} - \{0\}/R$ is a Hausforff space. But it is known that $\mathbb{R}^{n+1} - \{0\}/R \simeq \mathbb{RP}^n$, then $\mathbb{RP}^n$ is also Hausdorff space.

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Proposition 3. Let $\mathbb{RP}^n$ be a real projective space. Then $\mathbb{RP}^n$ is second countable.

Proof. Let aRb iff $b = ta, a, b \in \mathbb{R}^{n+1} - \{0\}$, $t \in \mathbb{R} - \{0\}$. Let $\mathfrak{B}$ be basis for $\mathbb{R}^{n+1} - \{0\}$ and let $\pi: \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}/R$ be a projection map. Since the relation R is open equivalence relation on a second countable $\mathbb{R}^{n+1} - \{0\}$, then $\mathbb{R}^{n+1} - \{0\}/R$ is second countable. But $\mathbb{R}^{n+1} - \{0\}/R \simeq \mathbb{RP}^n$, then $\mathbb{RP}^n$ is second countable.

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For future research, many areas involving projective space should be identified. Algebraic K-theory and index theory where the properties of $\mathbb{RP}^n$ play a critical, unresolved role, such as generalized index theorems on $\mathbb{RP}^n$. The concept of connections between $\mathbb{RP}^n$ and related mathematical objects, such as flag varieties, quadrics, or generalizations to other types of projective spaces $\mathbb{CP}^n$, $\mathbb{HP}^n$, looking for unifying theories, are still open problem for future research. Indeed, differential geometry is an essential problem corresponding to the projective space $\mathbb{RP}^n$.

IV. CONCLUSION

It is proved that the n-dimensional real projective space $\mathbb{RP}^n$ has smooth manifold structure. The proof is given through a quotient map of a quotien topology $\mathbb{R}^{n+1} - \{0\}/R$. However, the proof $\mathbb{R}^{n+1} - \{0\}/R \simeq \mathbb{RP}^n \simeq \mathbb{S}^n/R'$ can be still constructed by using the compacness of $\mathbb{RP}^n$ to Hausdorf space on a continuous bijection map.

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