

Estrada Index and Laplacian Estrada Index on the Non-Coprime Graph of the Dihedral Group of Prime Power Order

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Abstract. This study focuses on the non-coprime graph of the dihedral group D_{2n} with $n = p^k$, where p is a prime number and $k \in \mathbb{Z}^+$, by analyzing two fundamental spectral parameters, namely the Estrada index and the Laplacian Estrada index, which are defined in terms of the eigenvalues of the adjacency and Laplacian matrices. In this paper, the derivation of explicit general formulas for these indices is presented. The obtained results contribute to the advancement of algebraic graph theory through the spectral analysis of non-coprime graphs associated with algebraic structures.

Keywords: Non-Coprime Graph, Estrada Index, Laplacian Estrada Index, Dihedral Group, Spectral Graph Theory.

I. INTRODUCTION

Graphs constitute one of the central objects of study in discrete mathematics, serving as mathematical structures that represent relationships between objects in the form of ordered pairs [1]. Since their introduction by Leonhard Euler through the classical Königsberg Bridges problem [2][3], graph theory has undergone significant development and has found applications in various disciplines, including applied mathematics, operations research, electrical engineering, computer science, biological systems, and chemistry [3][4][5].

One notable development in graph theory is the non-coprime graph, which was introduced by Mansoori et al. [6] in their study entitled “*Non-coprime graph of a finite group*”. This graph is constructed from the structure of a finite group by considering the non-coprimality relation between the orders of its elements. Specifically, the non-coprime graph of a group G is defined as a graph with vertex set $G \setminus \{e\}$, where e denotes the identity element of G , and two distinct vertices are adjacent if and only if the orders of the corresponding group elements are not relatively prime [7]. Subsequent studies have extended the investigation of non-coprime graphs to various classes of groups. For instance, the non-coprime graph of the upper unitriangular matrix group was studied in [8], while the structure of the non-coprime graph of the dihedral group, particularly when the group order is a power of a prime number, was examined in [9]. Moreover, a generalization of non-coprime graphs for finite groups was proposed in [10], indicating that these graphs may belong to several well-known graph classes, such as complete graphs, complete bipartite graphs, triangle-free graphs, paths, stars, and others.

In the study of graph structures, spectral parameters play an important role in characterizing both structural and connectivity properties. One such parameter is the Estrada index, which

was originally introduced to measure subgraph centrality and has been widely applied in studies related to protein folding and the structural analysis of complex networks [11][12][13]. The Estrada index is defined in terms of the eigenvalues of the adjacency matrix and provides insight into the spectral properties of a graph. In addition, a related invariant known as the Laplacian Estrada index has been developed by utilizing the eigenvalues of the Laplacian matrix [14]. Both indices have been shown to be effective in the analysis of topological characteristics of various graph families.

Motivated by these observations, this study aims to investigate the Estrada index and the Laplacian Estrada index of non-coprime graphs constructed from dihedral groups of order p^k , where p is a prime number and $k \in \mathbb{Z}^+$. This research not only broadens the scope of non-coprime graph studies in the context of dihedral groups but also provides a systematic investigation of their spectral properties through the application of the Estrada and Laplacian Estrada indices as analytical tools in algebraic graph theory.

II. PRELIMINARIES

We begin by introducing the dihedral group, which forms the foundational algebraic structure for the graphs considered in this study.

Definition 1 [15, 16] *The dihedral group, denoted by D_{2n} , is the group generated by two elements a and b , written as*

$$D_{2n} = \langle a, b \mid a^n = e, b^2 = e, bab^{-1} = a^{-1} \rangle, \quad (1)$$

where e denotes the identity element of the group, $\langle a, b \rangle$ denotes the group generated by a and b , a represents a rotation of order n , and b represents a reflection of order 2, with $n \geq 3$.

Based on this group structure, we construct a graph by utilizing a number-theoretic relation among its elements. The resulting graph is referred to as the non-coprime graph, which serves as the main object of investigation in this article.

Definition 2 [6] *Let G be a finite group. The non-coprime graph of G , denoted as $\overline{\Gamma}_G$, is a graph with vertices $G \setminus \{e\}$. Two distinct vertices x and y in $\overline{\Gamma}_G$ are adjacent if*

$$(|x|, |y|) \neq 1. \quad (2)$$

In the specific case of the dihedral group of order p^k , the structure of the resulting non-coprime graph exhibits a characteristic connectivity pattern. Depending on the factorization properties of p^k , the graph can be recognized either as a complete graph when $p = 2$, or can be partitioned into two disjoint complete graphs when p is an odd prime. This difference arises from the distribution of element orders in the group, which determines the nature of non-coprimality among its elements.

Theorem 1 [9] *Let D_{2n} be a dihedral group. If $n = 2^m$ for some $m \in \mathbb{N}$, then $\overline{\Gamma}_{D_{2n}}$ is a complete graph.*

In contrast, when p is an odd prime, the resulting graph is no longer complete, but rather

two disjoint complete graphs, one for rotations and one for reflections, with no edges between the subgraphs.

Theorem 2 [9] *Let D_{2n} be a dihedral group. Suppose $n = p^m$ for some $m \in \mathbb{N}$, where p is an odd prime. Then the graph $\overline{\Gamma}_{D_{2n}}$ can be partitioned into two disjoint complete graphs.*

To proceed with the spectral analysis of the constructed non-coprime graphs, we introduce the adjacency matrix, which provides a useful algebraic representation of the graph's structure and connections, and is essential for the calculation of both the Estrada index and the Laplacian Estrada index.

Lemma 1 [17] *Let $\overline{\Gamma}$ be a graph. The adjacency matrix of $\overline{\Gamma}$, denoted as $AM(\overline{\Gamma}) = [a_{ij}]$, where:*

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j, \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

with v_i, v_j are the vertices of the graph $\overline{\Gamma}$.

The adjacency matrix is a fundamental representation used to capture vertex connections in a graph. To incorporate information about both vertex degrees and adjacencies, we next introduce the Laplacian matrix.

Definition 3 [18] *Let $\overline{\Gamma}$ be a graph with vertices $V(\overline{\Gamma})$. The Laplacian matrix of $\overline{\Gamma}$, denoted as $LM(\overline{\Gamma})$, is defined as:*

$$LM(\overline{\Gamma}) = DM(\overline{\Gamma}) - AM(\overline{\Gamma}), \quad (4)$$

where $DM(\overline{\Gamma})$ is the degree matrix and $AM(\overline{\Gamma})$ is the adjacency matrix. The degree matrix $DM(\overline{\Gamma})$ is a diagonal matrix whose i -th diagonal entry corresponds to the degree d_i of vertex i , where $d_i = |\mathcal{N}_i|$, and \mathcal{N}_i denotes the set of all neighbors of vertex i .

As previously described, the non-coprime graph of $D_{2(2^k)}$ forms a complete graph, while that of $D_{2(p^k)}$ with odd prime p forms two disjoint complete graphs. To support the forthcoming derivations related to the Estrada index and Laplacian Estrada index, we now recall two useful lemmas from linear algebra. These results are essential for obtaining the explicit forms of the eigenvalues corresponding to the symmetric matrix representations of the graph.

Lemma 2 [19]

Let $q \in \mathbb{N}$, and $\alpha, \beta \in \mathbb{R}$. Let A be a symmetric matrix of order q where all diagonal entries are α , and all off-diagonal entries are β . If the matrix has this form, then its determinant is given by:

$$\begin{vmatrix} \alpha & \beta & \cdots & \beta \\ \beta & \alpha & \cdots & \beta \\ \vdots & \vdots & \ddots & \vdots \\ \beta & \beta & \cdots & \alpha \end{vmatrix} = (\alpha - \beta)^{q-1} [\alpha - (q - 1)\beta]. \quad (5)$$

Lemma 3 [20] *Let M be a matrix of size $m \times m$, composed of submatrices A, B, O arranged in a block-diagonal form $M = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$, then*

$$\det(M) = \det(A) \cdot \det(B), \quad (6)$$

with A and B being square matrices.

Based on these spectral considerations, we now introduce the Estrada index, defined in terms of the eigenvalues of the adjacency matrix.

Definition 4 [21] *The Estrada index of a graph $\bar{\Gamma}$ is a spectrum-based invariant defined as:*

$$EE(\bar{\Gamma}) = \sum_{i=1}^n e^{\lambda_i} \quad (7)$$

where λ_i are the eigenvalues of the adjacency matrix $AM(\bar{\Gamma})$.

In addition to the Estrada index, a related spectral invariant based on the eigenvalues of the Laplacian matrix, known as the Laplacian Estrada index, is also considered in this study.

Definition 5 [22] *The Laplacian Estrada index of a graph $\bar{\Gamma}$ is defined as:*

$$LEE(\bar{\Gamma}) = \sum_{i=1}^n e^{(\mu_i - \frac{2r}{q})}, \quad (8)$$

where μ_i are the eigenvalues of the Laplacian matrix $LM(\bar{\Gamma})$, q is the number of vertices, and r is the number of edges in the graph $\bar{\Gamma}$.

III. RESULTS

Based on the preliminaries established in the previous section, this section presents the main results on the Estrada index and the Laplacian Estrada index of non-coprime graphs associated with dihedral groups.

3.1. Estrada Index

An explicit formula for the Estrada index of the complete non-coprime graph associated with a dihedral group is established for the case $n = 2^k$.

Theorem 3 *Let $\overline{\Gamma_{D_{2^n}}}$ be dihedral group with $n = 2^k$, $k \in \mathbb{Z}^+$. Then, the Estrada index of $\overline{\Gamma_{D_{2^n}}}$ is given by:*

$$EE(\overline{\Gamma_{D_{2(2^k)}}}) = (2(2^k) - 2)e^{-1} + e^{-(2(2^k)-2)}.$$

Proof. When $n = 2^k$, the non-coprime graph of the dihedral group D_{2^n} is a complete graph. The group $D_{2(2^k)}$ has $|D_{2(2^k)}| = 2(2^k)$ elements, consisting of 2^k rotations and 2^k reflections. In the non-coprime graph, all elements except the identity e are considered as vertices, since

$\gcd(|e|, |x|) = |x|$, and thus e is not adjacent to any other element. Therefore, the number of vertices is: $|D_{2(2^k)} \setminus \{e\}| = 2(2^k) - 1$. Since the graph is complete, each vertex is adjacent to all other vertices, and the adjacency matrix is a square matrix of size: $(2(2^k) - 1) \times (2(2^k) - 1)$. By the definition of the adjacency matrix, the matrix $AM(\overline{\Gamma_{D_{2(2^k)}}})$ is given by :

$$AM(\overline{\Gamma_{D_{2(2^k)}}}) = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$

The characteristic equation is given by:

$$|AM(\overline{\Gamma_{D_{2(2^k)}}}) - \lambda I| = |\lambda I - AM(\overline{\Gamma_{D_{2(2^k)}}})| = 0. \quad (9)$$

Expanding,

$$|AM(\overline{\Gamma_{D_{2(2^k)}}})| = \begin{vmatrix} \lambda & -1 & \cdots & -1 \\ -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda \end{vmatrix} = 0$$

By using **Lemma 2**, we consider the characteristic polynomial of the matrix:

$$\begin{vmatrix} \lambda & -1 & \cdots & -1 \\ -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda \end{vmatrix} = (\lambda + 1)^{(2(2^k)-1)-1} [\lambda - ((2(2^k) - 1) - 1)(-1)] = 0$$

Simplifying the expression gives:

$$(\lambda + 1)^{2(2^k)-2} [\lambda + ((2(2^k) - 2)] = 0$$

Solving for eigenvalues, we get:

$$(\lambda + 1)^{2(2^k)-2} = 0 \quad \text{or} \quad \lambda + (2(2^k) - 2) = 0.$$

Thus, we obtain $\lambda_1 = -1$ with multiplicity $2(2^k) - 2$ and $\lambda_2 = -(2(2^k) - 2)$. By summing the exponentials of the eigenvalues, the Estrada index of the graph $(\overline{\Gamma_{D_{2(2^k)}}})$ is obtained:

$$\begin{aligned} EE(\overline{\Gamma_{D_{2(2^k)}}}) &= \sum_{i=1}^n e^{\lambda_i} \\ &= (2(2^k) - 2)e^{-1} + e^{-(2(2^k)-2)}. \end{aligned}$$

□

The following result determines the Estrada index of the non-coprime graph associated with a dihedral group whose graph decomposes into two disjoint complete graphs, corresponding to the case where the group order is p^k with p an odd prime.

Theorem 4 Let $\overline{\Gamma}$ be the non-coprime graph of the dihedral group of order p^k , where p is an odd prime number and $k \in \mathbb{Z}^+$. Then, the Estrada index of $\overline{\Gamma}_{D_{2(p^k)}}$ is given by:

$$EE(\overline{\Gamma}_{D_{2(p^k)}}) = (2p^k - 3)e^{-1} + e^{-(p^k-2)} + e^{-(p^k-1)}.$$

Proof. Let $D_{2(p^k)}$ be a dihedral group of order $2p^k$, consisting of p^k rotations and p^k reflections. In the non-coprime graph $\overline{\Gamma}_{D_{2(p^k)}}$, all elements except the identity e are considered as vertices, giving $2p^k - 1$ in total. Since e has order 1 and is not adjacent to any other element, it is excluded. This leaves $p^k - 1$ non-identity rotations. By **Theorem 2**, the graph $\overline{\Gamma}_{D_{2(p^k)}}$ consists of two disjoint complete graphs: one on the $p^k - 1$ rotations and one on the p^k reflections. As $\gcd(p^k, 2) = 1$, rotation and reflection elements are not adjacent in the graph $\overline{\Gamma}$. Thus, the adjacency submatrix A is of order $(p^k - 1) \times (p^k - 1)$, the submatrix B is $p^k \times p^k$, and by **Lemma 1**, the adjacency matrix of $D_{2(p^k)}$ follows accordingly.

$$AM(\overline{\Gamma}_{D_{2(p^k)}}) = \begin{bmatrix} 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

The eigenvalues of the adjacency matrix are determined by solving the corresponding characteristic equation:

$$|AM(\overline{\Gamma}_{D_{2(p^k)}}) - \lambda I| = |\lambda I - AM(\overline{\Gamma}_{D_{2(p^k)}})| = 0. \quad (10)$$

$$\begin{vmatrix} \lambda & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ -1 & \lambda & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \lambda & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & \lambda \end{vmatrix} = 0.$$

Using **Lemma 3**, we consider the following matrix determinant:

$$\underbrace{\begin{vmatrix} \lambda & -1 & \cdots & -1 \\ -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda \end{vmatrix}}_{\det(A)} \underbrace{\begin{vmatrix} \lambda & -1 & \cdots & -1 \\ -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda \end{vmatrix}}_{\det(B)} = 0 \quad (11)$$

According to **Lemma 3**, the characteristic equation becomes:

$$[(\lambda - (-1))^{(p^k-1)-1}[\lambda - ((p^k - 1) - 1)(-1)]][(\lambda - (-1))^{p^k-1}[\lambda - (p^k - 1)(-1)]] = 0. \quad (12)$$

This simplifies to:

$$[(\lambda + 1)^{p^k-2}(\lambda + (p^k - 2))][(\lambda + 1)^{p^k-1}(\lambda + (p^k - 1))] = 0. \quad (13)$$

Therefore, the eigenvalues λ are given by:

$$(\lambda + 1)^{p^k-2} = 0, \quad \lambda + (p^k - 2) = 0, \quad (\lambda + 1)^{p^k-1} = 0, \quad \text{or} \quad \lambda + (p^k - 1) = 0.$$

As a result, we obtain the following eigenvalues for the adjacency matrix of the non-coprime graph $\overline{\Gamma_{D_{2(p^k)}}}$: $\lambda = -1$, with multiplicity $p^k - 2$; $\lambda = -(p^k - 2)$, with multiplicity 1; $\lambda = -1$, with multiplicity $p^k - 1$; or $\lambda = -(p^k - 1)$, with multiplicity 1.

For the non-coprime graph $\Gamma_{D_{2(p^k)}}$, the Estrada index is determined as follows:

$$\begin{aligned} EE(\overline{\Gamma_{D_{2(p^k)}}}) &= \sum_{i=1}^n e^{\lambda_i} \\ &= (p^k - 2)e^{-1} + e^{-(p^k-2)} + (p^k - 1)e^{-1} + e^{-(p^k-1)} \\ &= (p^k - 2 + p^k - 1)e^{-1} + e^{-(p^k-2)} + e^{-(p^k-1)} \\ &= (2p^k - 3)e^{-1} + e^{-(p^k-2)} + e^{-(p^k-1)}. \end{aligned}$$

□

3.2. Laplacian Estrada Index

An explicit formula for the Laplacian Estrada index of the complete non-coprime graph associated with a dihedral group is established for the case $n = 2^k$.

Theorem 5 Let $\overline{\Gamma_{D_{2n}}}$ be the non-coprime graph of the dihedral group, where $n = 2^k$, $k \in \mathbb{Z}^+$, then the Laplacian Estrada index of $\overline{\Gamma_{D_{2(2^k)}}}$ is given by:

$$LEE(\overline{\Gamma_{D_{2(2^k)}}}) = (2(2^k) - 2) + e^{2(2^k)-2}.$$

Proof. When $n = 2^k$, the non-coprime graph associated with the dihedral group D_{2n} forms

a complete graph on $2(2^k) - 1$ vertices, as the identity element is not included as a vertex. Since the graph is complete, each vertex has degree $2(2^k) - 2$, meaning it is connected to all other vertices. Therefore, the degree matrix $DM(\overline{\Gamma})$ is a diagonal matrix of size $(2(2^k) - 1) \times (2(2^k) - 1)$, with each diagonal entry equal to $2(2^k) - 2$. Meanwhile, the adjacency matrix $AM(\overline{\Gamma})$ has a value of 1 in all positions except for the main diagonal, where the entries are 0.

Based on **Definition 3**, the explicit form of the Laplacian matrix $L(\overline{\Gamma}_{D_{2(2^k)}})$ is as follows:

$$\begin{aligned}
 LM(\overline{\Gamma}_{D_{2(2^k)}}) &= DM(\overline{\Gamma}_{D_{2(2^k)}}) - AM(\overline{\Gamma}_{D_{2(2^k)}}) \\
 &= \begin{bmatrix} 2(2^k) - 2 & 0 & \cdots & 0 \\ 0 & 2(2^k) - 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2(2^k) - 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 2(2^k) - 2 & -1 & \cdots & -1 \\ -1 & 2(2^k) - 2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 2(2^k) - 2 \end{bmatrix}
 \end{aligned}$$

The characteristic equation is given by:

$$|LM(\overline{\Gamma}_{D_{2(2^k)}}) - \mu I| = |\mu I - LM(\overline{\Gamma}_{D_{2(2^k)}})| = 0.$$

Expanding the determinant, the following expression is obtained:

$$\left| LM(\overline{\Gamma}_{D_{2(2^k)}}) \right| = \begin{vmatrix} \mu - 2(2^k) + 2 & 1 & \cdots & 1 \\ 1 & \mu - 2(2^k) + 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \mu - 2(2^k) + 2 \end{vmatrix} = 0.$$

Applying **Lemma 2**, the characteristic polynomial of the adjacency matrix above is obtained, namely:

$$\begin{aligned}
 ((\mu - 2(2^k) + 2) - 1)^{(2(2^k)-1)-1} [\mu - 2(2^k) + 2 - ((2(2^k) - 1) - 1)(1)] &= 0 \\
 (\mu - 2(2^k) + 1)^{(2(2^k)-2)} [\mu - 2(2^k) + 2 - (2(2^k) - 2)] &= 0 \\
 (\mu - 2(2^k) + 1)^{(2(2^k)-2)} [\mu - 4(2^k) + 4] &= 0
 \end{aligned}$$

Therefore, the eigenvalues μ are given by:

$$(\mu - 2(2^k) + 1)^{(2(2^k)-2)} = 0 \quad \text{or} \quad \mu - 4(2^k) + 4 = 0$$

Thus, we obtain $\mu = 2(2^k) - 1$ with multiplicity $2(2^k) - 2$, or $\mu = 4(2^k) - 4$ with multiplicity 1. By summing the exponentials of these eigenvalues and the number of edges among the vertices

of a complete graph with $2(2^k) - 1$ vertices is $C_2^{2(2^k)-1} = (2(2^k) - 1)(2^k - 1)$. The Laplacian Estrada index of the graph $\overline{D_{2(2^k)}}$ is obtained as follows:

$$\begin{aligned} LEE(\overline{\Gamma_{D_{2(2^k)}}}) &= \sum_{i=1}^n e^{(\mu_i - \frac{2r}{q})} \\ &= (2(2^k) - 2)e^{(2(2^k)-1) - \frac{2((2(2^k)-1)(2^k-1))}{2(2^k)-1}} + e^{(2(2^k)-1) - \frac{2((2(2^k)-1)(2^k-1))}{2(2^k)-1}} \\ &= (2(2^k) - 2)e^{(2(2^k)-1) - (2(2^k)-1)} + e^{(4(2^k)-4) - (2(2^k)-1)} \\ &= (2(2^k) - 2) + e^{2(2^k)-2}. \end{aligned}$$

□

The next result establishes the Laplacian Estrada index for the non-coprime graph associated with a dihedral group that decomposes into two disjoint complete graphs, arising when the group order is p^k with p an odd prime.

Theorem 6 Let $\overline{\Gamma_{D_{2n}}}$ be the non-coprime graph of the dihedral group of order p^k , where p is an odd prime number and $k \in \mathbb{Z}^+$, then the Laplacian Estrada index of $\overline{\Gamma_{D_{2(p^k)}}}$ is given by:

$$LEE(\overline{\Gamma_{D_{2(p^k)}}}) = (p^k - 2)e^{\left(\frac{p^k-1}{2p^k-1}\right)} + e^{\left(\frac{2p^{2k}-6p^k+2}{2p^k-1}\right)} + (p^k - 1)e^{\left(\frac{3p^k-2}{2p^k-1}\right)} + e^{\left(\frac{2p^k(p^k-1)}{2p^k-1}\right)}.$$

Proof. Let $D_{2(p^k)}$ be a dihedral group of order $2p^k$, consisting of p^k rotations and p^k reflections. In the non-coprime graph $\overline{\Gamma_{D_{2(p^k)}}}$, all elements except the identity e are considered as vertices, giving $2p^k - 1$ in total. Since e has order 1 and is not adjacent to any other element, it is excluded. This leaves $p^k - 1$ non-identity rotations. By **Theorem 2**, the graph $\overline{\Gamma_{D_{2(p^k)}}}$ consists of two disjoint complete graphs: one on the $p^k - 1$ rotations and one on the p^k reflections. As $\gcd(p^k, 2) = 1$, rotation and reflection elements are not adjacent in the graph $\overline{\Gamma}$, forming two disjoint complete graphs. Thus, the adjacency submatrix A is of order $(p^k - 1) \times (p^k - 1)$, the submatrix B is $p^k \times p^k$, and by **Lemma 1**, the adjacency matrix of $D_{2(p^k)}$ follows accordingly.

$$AM(\overline{\Gamma_{D_{2(p^k)}}}) = \begin{bmatrix} 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

By the definition of the Laplacian matrix, the explicit form of $LM(\overline{\Gamma_{D_{2(p^k)}}})$ is given by:

$$LM(\overline{\Gamma_{D_{2(p^k)}}}) = DM(\overline{\Gamma_{D_{2(p^k)}}}) - AM(\overline{\Gamma_{D_{2(p^k)}}})$$

$$LM(\overline{\Gamma_{D_{2(p^k)}}}) = \begin{bmatrix} p^k - 2 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ -1 & p^k - 2 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & p^k - 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & p^k - 1 & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 & -1 & p^k - 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & p^k - 1 \end{bmatrix}$$

The characteristic equation is given by:

$$|LM(\overline{\Gamma_{D_{2(p^k)}}}) - \mu I| = |\mu I - LM(\overline{\Gamma_{D_{2(p^k)}}})| = 0.$$

$$\begin{vmatrix} \mu - p^k + 2 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & \mu - p^k + 2 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \mu - p^k + 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \mu - p^k + 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & \mu - p^k + 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & \mu - p^k + 1 \end{vmatrix} = 0.$$

Applying **Lemma 3**, the determinant of the following block matrix is obtained, leading to the characteristic equation

$$\underbrace{\begin{bmatrix} \mu - p^k + 2 & 1 & \cdots & 1 \\ 1 & \mu - p^k + 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \mu - p^k + 2 \end{bmatrix}}_{\det(A)} \underbrace{\begin{bmatrix} \mu - p^k + 1 & 1 & \cdots & 1 \\ 1 & \mu - p^k + 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \mu - p^k + 1 \end{bmatrix}}_{\det(B)} = 0 \quad (14)$$

Applying the equation in **Lemma 3**, the characteristic polynomial of the adjacency matrix above is obtained, namely:

$$\begin{aligned} & ((\mu - p^k + 2) - 1)^{(p^k - 1) - 1} [(\mu - p^k + 2) - ((p^k - 1) - 1)(1)] \\ & ((\mu - p^k + 1) - 1)^{(p^k - 1) - 1} [(\mu - p^k + 1) - ((2p^k - 1) - 1)(1)] = 0 \end{aligned}$$

$$(\mu - p^k + 1)^{(p^k - 2)} [(\mu - p^k + 2) - (p^k - 2)] (\mu - p^k)^{(p^k - 1)} [(\mu - p^k + 1) - (p^k - 1)] = 0$$

$$(\mu - p^k + 1)^{(p^k-2)}[\mu - 2(p^k - 2)](\mu - p^k)^{(p^k-1)}[\mu - 2(p^k + 1)] = 0$$

Therefore, the eigenvalues μ are given by:

$$(\mu - p^k + 1)^{(p^k-2)} = 0, \quad \mu - 2(p^k - 2) = 0, \quad (\mu - p^k)^{(p^k-1)} = 0, \quad \text{or} \quad \mu - 2(p^k + 1) = 0$$

Thus, we obtain $\mu = p^k - 1$ with multiplicity $p^k - 2$, $\mu = 2(p^k - 2)$ with multiplicity 1, $\mu = p^k$ with multiplicity $p^k - 1$, or $\mu = 2(p^k + 1)$ with multiplicity 1.

Subsequently, it is essential to determine the total number of *vertices* and *edges* in the graph, as these quantities are critical for the calculation of the Laplacian Estrada index. Specifically, the total number of edges in the graph must be computed to proceed with the Laplacian Estrada index calculation.

To determine the number of edges r , we consider the adjacency matrices A and B . The edges in matrix A , which correspond to the rotations, are represented by the binomial coefficient $C(2, p^k - 1)$, which gives the number of edges between the $p^k - 1$ rotations. Similarly, the edges in matrix B , which correspond to the reflections, are given by the binomial coefficient $C(2, p^k)$, representing the number of edges between the p^k reflections. Thus, the total number of edges r is:

$$\begin{aligned} r &= C_2^{p^k-1} + C_2^{p^k} \\ &= \frac{(p^k - 1)(p^k - 2)}{2} + \frac{p^k(p^k - 1)}{2} \\ &= (p^k - 1)(p^k - 1) \\ &= p^{2k} - 2p^k + 1. \end{aligned}$$

By summing the exponentials of these eigenvalues, the total number of vertices and edges, the Laplacian Estrada index of the graph $\overline{D}_{2(p^k)}$ is obtained as follows:

$$\begin{aligned} LEE(\overline{\Gamma}_{D_{2(p^k)}}) &= \sum_{i=1}^n e^{(\mu_i - \frac{2r}{q})} \\ &= (p^k - 2)e^{(p^k-1) - \frac{2(p^{2k}-2p^k+1)}{2p^k-1}} + e^{(2p^k-4) - \frac{2(p^{2k}-2p^k+1)}{2p^k-1}} + (p^k - 1)e^{(p^k) - \frac{2(p^{2k}-2p^k+1)}{2p^k-1}} \\ &\quad + e^{(2p^k-2) - \frac{2(p^{2k}-2p^k+1)}{2p^k-1}} \\ &= (p^k - 2)e^{\left(\frac{p^k-1}{2p^k-1}\right)} + e^{\left(\frac{2p^{2k}-6p^k+2}{2p^k-1}\right)} + (p^k - 1)e^{\left(\frac{3p^k-2}{2p^k-1}\right)} + e^{\left(\frac{2p^k(p^k-1)}{2p^k-1}\right)}. \\ &= (p^k - 2)e^{\left(\frac{p^k-1}{2p^k-1}\right)} + e^{\left(\frac{2p^{2k}-6p^k+2}{2p^k-1}\right)} + (p^k - 1)e^{\left(\frac{3p^k-2}{2p^k-1}\right)} + e^{\left(\frac{2p^k(p^k-1)}{2p^k-1}\right)}. \end{aligned}$$

□

IV. CONCLUSIONS AND FUTURE RESEARCH DIRECTION

This study establishes explicit general expressions for the Estrada index and the Laplacian Estrada index of the non-coprime graph associated with the dihedral group D_{2n} , where $n = p^k$, p is a prime number, and k is a positive integer. The results distinguish two structurally different cases, namely the complete graph case occurring when n is a power of two and the case in which the graph decomposes into two disjoint complete subgraphs when n is a power of an odd prime, thereby demonstrating how the underlying algebraic structure influences the spectral properties of the associated non-coprime graph.

For future research, the approach developed in this work may be extended to non-coprime graphs arising from other classes of finite groups, such as quaternion, dicyclic, or symmetric groups. In addition, further studies may investigate other spectral-based graph invariants, including the signless Laplacian Estrada index, the Sombor index, or graph energy, which could deepen the connections between algebraic graph theory, combinatorics, and potential applications in network analysis and chemical graph theory.

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