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SOME PROPERTIES OF MODULAR TOPOLOGY IN THE ORLICZ SEQUENCE SPACE

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Abstract. In this article, we examined some properties of modular topology on the Orlicz sequence space. Discussions were conducted by constructing the topology on the sequence space using a modular neighborhood of zero. The neighborhood forms a local base that is balanced, absorbing, and symmetrical. Furthermore, for the Orlicz function that grows not too rapidly, the modular neighborhood induces a topological vector space. We also characterize the modular boundedness, modular convergence, and modular closed set on the sequence space.
Keywords: modular, Orliz, sequence, topology.

I. INTRODUCTION

Let X be a sequence space. For $1 < p < \infty$, the collection of all the set $\{(x_k) : \sum_{k=1}^{\infty} |x_k|^p < r, r > 0\}$ is a local base for a topology on the sequence space ℓ_p. If the function $|\cdot|^p$ is replaced by the Orlicz function ϕ, then the topological properties of the sequence space can be discussed through modular space.

Researchers have worked on several approaches to discussing the properties of topology in modular spaces. The topology of some sequence spaces is examined in [1]. In [2], Kolk defines the topology in the Orlicz sequence space using the modulus function. The construction of modular spaces into modular metric spaces recently has been widely used ([3], [4], [5]). In [6], the modular metric is used to induce a topology. Meanwhile, [7] examines the weak* topology in modular space. Some concepts of boundedness were introduced in [8]. The ball on modular space was used by [9] to examine its topological properties. In [10] Hajji constructs a topological vector space on a modular space using a local base and study its properties. The topological structures induced on vector spaces by convex modulars was studied in [17]. The w-open set was introduced by [14] to study topology of w-mutiplicative modular metric space. A new concept of modularity and its convergence properties is discussed in [20]. Further applications of modular to fixed-point theory have been carried out by [11] and [12]. Another development was the introduction of modulated topology vector space by Kozlowski [13].

Boundedness is closely related to convergence. The study of strong convergence in topological space was carried out by [15]. In the discussion of bounded sets in normed spaces, bounded sets in the sense of norms are often used. In a modular space, in addition, there is also a type of modular bounded. Modular convergence is weaker than norm convergence, that is, norm convergent implies modular convergent, but on the other hand, modular convergent is not necessarily norm convergent [16].

One of the most recognizable modular spaces is the Orlicz sequence space. The modular in this sequence space has the property of convex, which is not necessarily possessed by modular spaces in general. The linear structure and convexity of the sequence space therefore can be used to construct a topological vector space where its local base is defined through modular.

In this paper, we examine topological some properties of the Orlicz sequence space. To discuss these properties, we topology in the space using the neighborhood modular of zero. Furthermore, the properties of modular boundedness and modular convergence in the space are examined.

II. PRELIMINARIES

In this section, we present some basic concepts that are used for further discussion in this paper.

Definition 1 *[18] Let X be linear space over a real field. Pseumodular on $X$is a function $\rho : X \to [0, \infty]$ such that for each $x, y \in X$*

(1) $\rho(0) = 0$,

(2) $\rho(-x) = \rho(x)$,

(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *with* $\alpha, \beta > 0,\ \alpha + \beta = 1$.

If condition (1) is added with $\rho(x) = 0$ implies $x = 0$, then ρ is called modular. If condition (3) is relpaced by: $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ with $\alpha, \beta > 0,\ \alpha + \beta = 1$, then ρ is called modular convex.

Let ρ be pseumodular on X. Then

$$X_\rho = \left\{ x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0 \right\}$$

is called modular space. If ρ convex modular, then on X_ρ can be defined Luxemburg norm $\|\cdot\|_\rho$ as follows:

$$\|x\|_\rho = \inf \left\{ \epsilon > 0 : \rho\left(\frac{x}{\epsilon}\right) \leq 1 \right\}.$$

Definition 2 *A sequence (x_n) in modular space X_ρ is called ρ-convergent (or modular convergent) to x, written as $x_n \xrightarrow{\rho} x$, if there exists $\lambda > 0$ such that $\rho(\lambda(x_n - x)) \to 0$ if $n \to \infty$. A sequence (x_n) in X_ρ is called ρ-Cauchy (or Cauchy modular) if there exists $\lambda > 0$ such that $\rho(\lambda(x_n - x_m)) \to 0$ if $m, n \to \infty$.*

Definition 3 *Let E be a subset of modular space X_ρ.*

(1) E is called ρ-bounded (or modular bounded), if for every sequence $(x_n) \subset E$ and arbitrary $\varepsilon_n \to 0$ then $\varepsilon_n x_n \xrightarrow{\rho} 0$ as $n \to \infty$.

(2) E is called ρ-closed, if $(x_n) \subset E$ and $x_n \xrightarrow{\rho} x$ then $x \in E$. A smallest ρ-closed containing the $E \subset X_\rho$ is called ρ-closure of E and is denoted by $\overline{E}^\rho$.

We recall one of the important results in modular theory (see [18]) which states the equivalence of norm convergence and modular convergence.

Given a sequence (x_n) in X_ρ. The following statements are equivalent:

(1) $\lim_{n\to\infty} \|x_n - x\|_\rho = 0$

(2) $\lim_{n\to\infty} \rho(\lambda(x_n - x)) = 0$ for each $\lambda > 0$.

Recall that, a topological vector space $(X, \mathcal{T})$ is a topology on linear space X, where the addition and scalar multiplication is continuous. To construct a topology vector space, the following (see e.g. [19]) can be used.

Let X be linear space and $\mathcal{B}$ a nonempty family of a subset of X which satisfy:

(i) for each $U, V \in \mathcal{B}$ there exists $W \in \mathcal{B}$ such that $W \subset U \cap V$;

(ii) for each $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $V + V \subset U$;

(iii) for each $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $aV \subset U$ for each scalar a with $|a| \leq 1$;

(iv) for each $x \in X$ and $U \in \mathcal{B}$ there exist a scalar a such that $x \in aU$, i.e. every member of $\mathcal{B}$ is absorbing.

If $\mathcal{T}$ is the family all set G such that for each $x \in G$, there exists $U \in \mathcal{B}$ with $x + U \subset G$, then $\mathcal{T}$ is a linear topological space for X.

Definition 4 *[19] Let $(X, \mathcal{T})$ be a linear space and $U \subset X$.*

(1) U is called balanced if $\lambda U \subset U$ for each λ with $|\lambda| \leq 1$

(2) U is called convex if $\lambda, \beta \geq 0$ with $\lambda + \beta \leq 1$ implies $\lambda x + \beta y \in U$ for each $x, y \in U$.

(3) U is called symmetric if $-U_r = U_r$. Here $-U_r = \{x : x = -y,\ y \in U_r\}$.

The following description of Orlicz functions can be found in [21]. The function $\phi : \mathbb{R} \to [0, \infty)$ is called Orlicz function if the following conditions are satisfied: even, monotonically increasing, continuous, convex, $\phi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \phi(y) = \infty$. Orlicz function ϕ is said to satisfy the Δ_2-condition if for each $t > 0$ it is true that $\phi(2t) \leq K\phi(t)$ for a positive constant K.

Since the space being studied is a sequence space, it is necessary to re-emphasize some of the notation used. We use ω to denote the space of all sequences of real numbers with addition and multiplication defined as follows: for each $x = (x_k), y = (y_k) \in \omega$ and scalar α, $x+y = (x_k+y_k)$ and $\alpha x = (\alpha x_k)$. The symbol θ denotes the zero sequence, i.e. $\theta = (0, 0, \cdots)$. A sequence with term in ω is written as $(x^{(n)})$, where $x^{(n)} = (x_k^{(n)}) = (x_1^{(n)}, x_2^{(n)}, \cdots)$ for each integer n. Let $U, V \subset \omega$ and α be any scalar. We use the following notations: (i) $x + U$ the set of all $x + y$ where $y \in U$, (ii) αU the set of all αx where $x \in U$ and (iii) $U + V$ the set of all $x + y$ where $x \in U, y \in V$.

III. MODULAR BOUNDED AND CONVERGENCE

Given the Orlicz function ϕ. The function $\rho_\phi : \omega \to \mathbb{R}$ where

$$\rho_\phi(x) = \sum_{k=1}^{\infty} \phi(x_k), \quad x = (x_1, x_2, \cdots) \tag{1}$$

is a modular. Furthermore, since ϕ is convex then ρ_ϕ is a convex modular [10]. For further discussion in this article, the notation ρ_ϕ denotes the modular as in definition (1). Sinceρ_ϕ is modular, we can define modular space ℓ_ϕ as follows:

$$\ell_\phi = \left\{ x \in \omega : \rho_\phi(\lambda x) = \sum_{k=1}^{\infty} \phi(\lambda x_k) \to 0, \text{ if } \lambda \to 0 \right\}$$

with Luxemburg norm $\|x\|_{\rho_\phi} = \inf\{\epsilon > 0 : \sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\epsilon}\right) \leq 1\}$. This modular space is also called the Orlicz sequence space.

We'll also use notation $\ell_\phi^0 = \{x : \sum_{k=1}^{\infty} \phi(x_k) < \infty\}$ and $E^\phi = \{x : \sum_{k=1}^{\infty} \phi(\lambda x_k) < \infty\}$ for each $\lambda > 0$. Obviously $E^\phi \subset \ell_\phi^0 \subset \ell_\phi$.

Definition 1 *Given Orlicz function ϕ. For $r > 0$, the ρ_ϕ-neighborhood of zero U_r in ℓ_ϕ is defined as $U_r = \{x \in \ell_\phi : \sum_{k=1}^{\infty} \phi(x_k) < r\}$.*

For further discussion, the collection of all ρ_ϕ-neighborhood of zero U_r is written as $\mathcal{B}_\phi$. Some basic properties of the ρ_ϕ-neighborhood of zero that are useful for constructing a topological space are stated in the following theorem.

Theorem 1 *Every element of $\mathcal{B}_\phi$ is convex, symmetric, and balanced.*

Proof. The convex properties are a result of the convex properties of the function ϕ. While the symmetric property is a direct result of the even property of the function ϕ. Let $U_r \in \mathcal{B}_\phi$. For each $x \in U_r$ and λ with $|\lambda| \leq 1$, the convexity of ϕ give the result $\sum_{k=1}^{\infty} \phi(\lambda x_k) \leq |\lambda| \sum_{k=1}^{\infty} \phi(x_k) < r$, which mean $\lambda x \in U_r$; thus U_r is balanced. □

The following two lemmas are used to construct a topological vector space in the Orlicz sequence space as stated in the Theorem 2.

Lemma 1 *The family $\mathcal{B}_\phi$ satisfies the following conditions :*

(i) for each $U, V \in \mathcal{B}_\phi$ there exists $W \in \mathcal{B}_\phi$ such that $W \subset U \cap V$;

(ii) for each $U \in \mathcal{B}_\phi$ there exists $V \in \mathcal{B}_\phi$ such that $V + V \subset U$;

(iii) for each $x \in X$ and $U \in \mathcal{B}_\phi$ there exist a scalar a such that $x \in aU$.

Proof. For condition (i), let $U_s, U_t \in \mathcal{B}_\phi$ and $r = \min\{s, t\}$. Then for each $x \in U_r$, $\sum_{k=1}^{\infty} \phi(x_k) < r \leq s$ and $\sum_{k=1}^{\infty} \phi(x_k) < r \leq t$, i.e. $x \in U_s \cap U_t$.

Condition (ii) can be found as follows: for $U_r \in \mathcal{B}_\phi$, then U_s with $0 < s < r$ is a member of $\mathcal{B}_\phi$. Since ϕ is convex and even function, then for each scalar a with $|a| \leq 1$, we have

$$\sum_{k=1}^{\infty} \phi(ax_k) \leq |a| \sum_{k=1}^{\infty} \phi(x_k) \leq r$$

for each $x = (x_k) \in U_s$. Thus $ax \in U_r$.
Finnaly given $U_r \in \mathcal{B}_\phi$. Let $x \in \ell_\phi$, i.e. $\sum_{k=1}^{\infty} \phi(\lambda x) \to 0$ if $\lambda \to 0$. There exists $0 < \alpha < 1$ such that $\sum_{k=1}^{\infty} \phi(\alpha x) < r$. Let $y = (y_k)$ where $y_k = \alpha x_k,\ k = 1, 2, \cdots$. Then $y \in U_r$. Hence $x = \alpha^{-1} y \in \alpha^{-1} U$. Hence, the condition (iii) is satisfied. □

Based on that Lemma 1, the collection of set $\{x + U_r : x \in \ell_\phi, U_r \in \mathcal{B}_\phi\}$ forms a local basis for the topology in ℓ_ϕ. However, this topology is not necessarily a topological vector space, because the requirement of continuous addition has not been met. For Orlicz functions under certain conditions, the following lemma ensures that the vectors addition is continuous.

Lemma 2 *If the Orlicz function ϕ satisfies the Δ_2-condition, then for each $U_r \in \mathcal{B}_\phi$ there exists $U_s \in \mathcal{B}_\phi$ such that $U_s + U_s \subset U_r$.*

Proof. Let $U_r \in \mathcal{B}_\phi$ and let $s = \frac{r}{K}$ where K is the constant in the Δ_2-condition. For each $x, y \in U_s$, the Δ_2-condition and the convexity of ϕ implies

$$\begin{aligned} \sum_{k=1}^{\infty} \phi(x_k + y_k) &= \sum_{k=1}^{\infty} \phi\left(2\left(\frac{1}{2}\cdot x_k + \frac{1}{2}\cdot y_k\right)\right) \\ &\leq \frac{K}{2}\left(\sum_{k=1}^{\infty} \phi(x_k) + \sum_{k=1}^{\infty} \phi(y_k)\right) < \frac{K}{2}\left(\frac{r}{K} + \frac{r}{K}\right) = r. \end{aligned}$$

□

By Lemma 1 and Lemma 2, we have the following result.

Theorem 2 *Let the Orlicz function ϕ satisfy the Δ_2-condition. The collection $\mathcal{T}$ of all $G \subset \ell_\phi$ such that for each sequence $x \in G$ there exists $U_r \in \mathcal{B}_\phi$ such that $x + U_r \subset G$ is a topological vector space on ℓ_ϕ with local base $\mathcal{B}_\phi$.*

Thus, we have formed a topological vector space on the Orlicz sequence space with local base $\mathcal{B}$.

Example 1 The Orlicz function $\phi(t) = |t|^p,\ 1 < p < \infty$ satisfies the Δ_2-condition. Hence, according to Theorem 2, the collection of all $U_r = \{(x_k) : \sum_{k=1}^{\infty} |x_k|^p < r\}$ is a local base for topological vector space in ℓ_p, $1 \leq p < \infty$.

Theorem 3 *If the Orlicz function ϕ satisfies Δ_2-condition, then the topological space $\mathcal{T}$ is Hausdorff space.*

Proof. Let $x, y \in \ell_\phi$ with $x \neq y$. There exists $r > 0$ such that $\rho_\phi(x - y) = r$. Let $U_{r/2} \in \mathcal{B}_\phi$. By Lemma 2 there exists $U_s \in \mathcal{B}_\phi$ such that $U_s + U_s \subset U_{r/2}$. It will be shown that $(x + U_s) \cap$

$(y + U_s) = \emptyset$. Suppose that $z \in (x + U_s) \cap (y + U_s)$ for some $z \in \ell_\phi$; i.e. $z = x + u = y + v$ for some $u, v \in U_s$. By Theorem 1, U_s is symmetric and implies $U_s - U_s = U_s + U_s$. So,

$$x - y = v - u \in U_s - U_s = U_s + U_s \subset U_{r/2}.$$

Therefore $\rho_\phi(x - y) = \sum_{k=1}^{\infty} \phi(x_k - y_k) < r/2$, contradicting to $\rho_\phi(x - y) = r$. □

Example 2 Let $x = (1, 0, 0, \cdots)$ and $y = (0, 1, 0, 0, \cdots)$. For $r > 0$, the set $U_r = \{(z_k) \in \ell_2 : \sum_{k=1}^{\infty} |z_k|^2 < r\}$ is open ball in ℓ_2. Then we have $(x + U_{1/2}) \cap (y + U_{1/2}) = \emptyset$, since every element of $x + U_{1/2}$ is of the form $(1 + z_1, z_2, z_3, z_4, \cdots)$ and every element of $y + U_{1/2}$ is of the form $(z_1, 1 + z_2, z_3, z_4, \cdots)$ where $|x_k| < \frac{1}{4}$ for each $k \in \mathbb{N}$.

Furthermore, with the topology that has been formed, several properties related to modular convergence can be analyzed. Modular convergence in ℓ_ϕ can be expressed using a local base. A sequence $(x^{(n)})$ is ρ_ϕ-convergent to x there exists $\lambda > 0$ such that for each $U_r \in \mathcal{B}_\phi$ there exists natural number n_0 such that $\lambda(x^{(n)} - x) \in U_r$ if $n \geq n_0$ or equivalently $x^{(n)} \in x + \alpha U_r$ for some $\alpha > 0$; and it is ρ_ϕ-Cauchy $x^{(m)} - x^{(n)} \in \alpha U_r$ for some $\alpha > 0$ and $m, n \geq n_0$. For the Orlicz function that satisfies Δ_2-condition, the ρ_ϕ-convergent can be described as follows.

Theorem 4 *If the Orlicz function ϕ satisfies the Δ_2-condition and $(x^{(n)})$ sequence in ℓ_ϕ such that $x^{(n)} \xrightarrow{\rho_\phi} x$, then for each $\lambda > 0$ and $U_r \in \mathcal{B}_\phi$ there exits natural number n_0 such that $\lambda(x^{(n)} - x) \in U_r$ for each $n \geq n_0$.*

Proof. Let $x^{(n)} \xrightarrow{\rho_\phi} x$, i.e. there exists $\alpha > 0$ such that $\rho_\phi(\alpha(x^{(n)} - x)) \to 0$ if $n \to \infty$. Given any $\lambda > 0$ and $U_r \in \mathcal{B}_\phi$. There exists K^{s_0} such that $\phi(\frac{\lambda}{\alpha} t) \leq K^{s_0} \phi(t)$ for each $t > 0$. Here K is constant in the Δ_2-condition. Given any natural number n_0 such that $\rho_\phi(\alpha(x^{(n)} - x)) = \sum_{k=1}^{\infty} \phi(\alpha(x_k^{(n)} - x_k)) < \frac{r}{K^{s_0}}$ for each $n \geq n_0$. Then

$$\sum_{k=1}^{\infty} \phi(\lambda(x_k^{(n)} - x_k) = \sum_{k=1}^{\infty} \phi\left(\frac{\lambda}{\alpha} \alpha(x_k^{(n)} - x_k)\right) \leq K^{s_0} \sum_{k=1}^{\infty} \phi(\alpha(x_k^{(n)} - x_k)) < r$$

for each $n \geq n_0$, which mean $\lambda(x^{(n)} - x) \in U_r$ for each $n \geq n_0$. □

By the equivalence of norm convergence and modular convergence stated in Section II. and by Theorem 4 Section III., we have the following corollary.

Corollary 1 *Let the Orlicz function ϕ satisfy the Δ_2-condition. The sequence $(x^{(n)})$ is ρ_ϕ-convergent if only if the sequence is convergent (in the sense of norm $\|\cdot\|_{\rho_\phi}$).*

The following result illustrates the nature of a Cauchy sequence within the topology $\mathcal{T}$.

Theorem 5 *Let the Orlicz function ϕ satisfy the Δ_2-condition. If $(x^{(n)})$ is a ρ_ϕ-Cauchy sequence in ℓ_ϕ and $U_r \in \mathcal{B}_\phi$, then there exists $\lambda > 0$ such that $x^{(n)} \in \lambda U_r$ for all n.*

Proof. Let $U_r \in \mathcal{B}_\phi$. By Lemma 2 there exists $U_s \in \mathcal{B}_\phi$ such that $U_s + U_s \subset U_r$. Since $(x^{(n)})$ is a ρ_ϕ-Cauchy, there exists n_0 such that $x^{(n)} \in x^{(n_0)} + U_s$ for all $n \geq n_0$. Since U_s absorbing, then there exist $\alpha > 0$ such that $x^{(n_0)} \in \alpha U_s$ and then implies

$$x^{(n)} \in \alpha U_s + U_s \subset \alpha U_r, \quad n \geq n_0.$$

For $n = 1, 2, \cdots, n_0-1$, there exist α_n such that $x^{(n)} \in \alpha_n U_r$. Let $\lambda = \max\{\alpha, \alpha_1, \cdots, \alpha_{n_0-1}\}$. We get $x^{(n)} \in \alpha U_r$ for all n. □

Example 3 Let $(x^{(n)})$ be sequence such that $x_k^{(n)} = \frac{1}{n}$ if $k = n$ and 0 otherwise. Then $(x^{(n)})$ is a ρ_ϕ-Cauchy sequence in ℓ_2, since

$$\rho_\phi(x^{(n)} - x^{(m)}) = \sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^2 = \frac{1}{n^2} + \frac{1}{m^2} \to 0$$

as $n, m \to \infty$. For any $U_r = \{(x_k) : \sum_{k=1}^{\infty} |x_k|^2 < r\} \in \mathcal{B}_\theta$, let $\lambda = r$. Since $\rho_\phi(\lambda x^{(n)}) = |\frac{r}{n^2}| < r$, then $x^{(n)} \in \lambda U_r$.

Before examining the nature of ρ_ϕ-bounded, first, the following example was presented.

Example 4 (i) Every member of $\mathcal{B}_\phi$ is ρ_ϕ-bounded. This can be shown as follows. Given sequence $(x^{(n)})$ with $x^{(n)} \in U_r \in \mathcal{B}_\phi$ and (ε_n) such that $\varepsilon_n \to 0$ if $n \to \infty$. Let n_0 be an integer such that $\varepsilon_n < 1$ for each $n \geq n_0$. Since ϕ is convex, then we have

$$\sum_{k=1}^{\infty} \phi(\varepsilon_n x_k^n) \leq \varepsilon_n \sum_{k=1}^{\infty} \phi(x_k^n) \leq \varepsilon_n \cdot r,$$

which implies $\sum_{k=1}^{\infty} \phi(\varepsilon_n x_k^n) \to 0$ if $n \to \infty$.
(ii) The linear space ℓ_ϕ is not ρ_ϕ-bounded as we show as follows. Let $(x^{(n)})$ be a sequence in ℓ_ϕ with $x_k^{(n)} = 1$ for $k \leq n$ and $x_k^{(n)} = 0$, for $k > n$, and given $\lambda > 0$. Let (ε_n) be a sequence of real number such that $\lambda \varepsilon_n = \phi^{-1}(1/n)$. We get

$$\rho_\phi(\lambda \varepsilon_n x^{(n)}) = \sum_{k=1}^{\infty} \phi(\lambda \varepsilon_n x_k^{(n)}) = \sum_{k=1}^{n} \phi(\lambda \varepsilon_n) = n \cdot \frac{1}{n} \to 1,$$

i.e. $\varepsilon_n \to 0$ but $(\varepsilon_n x^{(n)})$ is not ρ_ϕ-convergent to 0.

The necessary and sufficient condition for the subset of ℓ_ϕ to be bounded is stated in the following theorem.

Theorem 6 *Let $E \subset \ell_\phi$. Then E is ρ_ϕ-bounded if and only if there exists $\lambda > 0$ and $M > 0$ such that $\lambda x \in U_M$ for each $x \in E$.*

Proof. Suppose that E is ρ_ϕ-bounded, but no such number λ and M exists. We can find a sequence $(x^{(n)})$ in E such that $\frac{1}{n^2} x^{(n)} \notin U_1$, $n = 1, 2, \cdots$. Since for each n, $x^{(n)} \in E$, then there exist $\alpha > 0$ such that $\rho_\phi(\frac{\alpha}{n} x^{(n)}) \to 0$ as $n \to \infty$, and we can find n_0 such that $\alpha n > 1$ and $\rho_\phi(\frac{\alpha}{n} x^{(n)}) < 1$ if $n \geq n_0$. Hence, for $n \geq n_0$ we get

$$\rho_\phi\left(\frac{1}{n^2} x^{(n)}\right) = \rho_\phi\left(\frac{1}{\alpha n} \frac{\alpha}{n} x^{(n)}\right) \leq \frac{1}{\alpha n} \rho_\phi\left(\frac{\alpha}{n} x^{(n)}\right) < 1$$

as $n \to \infty$, which mean $\frac{1}{n^2} x^{(n)} \in U_1$, a contradiction.
Conversely, let $\lambda x \in U_M$ for each $x \in E$. Given a sequence $(x^{(n)})$ with $x^{(n)} \in E$ and (ε_n) such

that $\varepsilon_n \to 0$; hence we can take $|\lambda\varepsilon_n| \leq 1$ for each n. By the convexity of ρ_ϕ we have

$$\rho_\phi(\lambda\varepsilon_n x^n) \leq |\lambda\varepsilon_n|\rho_\phi(x^n) \leq |\varepsilon_n|\lambda M \to 0$$

if $n \to \infty$. □

Example 5 The set of all $x \in \ell_\phi$ such that $\rho_\phi(x) < M$ is obviously ρ_ϕ-bounded subset of ℓ_ϕ, since $x \in U_M$.

Corollary 2 *If $E \subset \ell_\phi$ is ρ-bounded, then E is bounded with respect to $\|\cdot\|_{\rho_\phi}$.*

Proof. Let $x \in E$ and let λ and M as in Theorem 6. Since $\lambda x \in U_M$, then

$$\rho_\phi\left(\frac{\lambda x}{M}\right) \leq \frac{1}{M}\rho_\phi(\lambda x) \leq 1,$$

and therefore $\|x\|_{\rho_\phi} = \inf\left\{\epsilon > 0 : \rho_\phi\left(\frac{x}{\epsilon}\right) \leq 1\right\} \leq M/\lambda$. □

Theorem 7 *Let the Orlicz function ϕ satisfy the Δ_2-condition and $E \subset \ell_\phi$. Then E is ρ_ϕ-bounded if and only if $\rho_\phi(\lambda\varepsilon_n x^{(n)}) \to 0$ for each $\lambda > 0$.*

Proof. Let $x^{(n)} \in E$. Given any $\lambda > 0$ and a sequence of real number (ε_n) such that $\varepsilon_n \to 0$. Since E is ρ_ϕ-bounded, then there exists $\alpha > 0$ suc that $\sum_{k=1}^{\infty} \phi(\alpha\varepsilon_n x_k^{(n)}) \to 0$ if $n \to \infty$. Hence,

$$\sum_{k=1}^{\infty} \phi(\lambda\varepsilon_n x_k^n) = \sum_{k=1}^{\infty} \phi(\frac{\lambda}{\alpha}\alpha\varepsilon_n x_k^n) \leq K^{s_0} \sum_{k=1}^{\infty} \phi(\alpha\varepsilon_n x_k^n) \to 0$$

if $n \to \infty$, where K is constant in the Δ_2-condition. For sufficiency is obvious from the definition 3. □

The following theorem explains the relationship between modular convergence and modular boundedness.

Theorem 8 *The set of all ρ_ϕ-convergent sequence in ℓ_ϕ is ρ_ϕ-bounded.*

Proof. Suppose that $(x^{(n)})$ is ρ_ϕ-convergent to $x \in \ell_\phi$. Here, there exists $\lambda > 0$ such that $\rho_\phi(\lambda(x^{(n)} - x)) \to 0$ if $n \to \infty$. Let (ε_n) be a sequence of real numbers such that $\varepsilon_n \to 0$. There exists n_0 such that $2\varepsilon_n < \lambda$ for each $n \geq n_0$. Therefore for each $n \geq n_0$ we have

$$\rho_\phi(\varepsilon_n x^{(n)}) \leq \rho_\phi(2\varepsilon_n(x^{(n)} - x)) + \rho_\phi(2\varepsilon_n x) \leq \rho_\phi(\lambda(x^{(n)} - x)) + \rho_\phi(2\varepsilon_n x).$$

Since $x \in \ell_\phi$ then $\rho_\phi(2\varepsilon_n x) \to 0$, and hence the last term of the inequality tends to 0 as $n \to \infty$. □

Example 6 The sequence $(x^{(n)})$ where $x_k^{(n)} = \frac{1}{2^{n/2}}$ if $n = k$ and 0 otherwise is ρ_ϕ-convergent in ℓ_2, since $\rho_\phi(x^{(n)}) = \sum_{k=1}^{\infty} |x_k^{(n)}|^2 = \frac{1}{2^n} \to 0$ as $n \to \infty$, and hence $\rho_\phi(\varepsilon_n x^{(n)}) \to 0$ if $\varepsilon \to 0$ as $n \to \infty$.

By Theorem 6 and 8 the following result is obtained.

Corollary 3 *If a sequence $(x^{(n)})$ is ρ_ϕ-convergent, then there exist $\lambda > 0$ and $M > 0$ such that $\lambda x^{(n)} \in U_M$, for $n = 1, 2, \cdots$.*

The following results describe some of the properties related to modular closed sets in the sequence space.

Theorem 9 *If $E \subset \ell_\phi$ is ρ_ϕ-closed then for each $x \notin E$ there exists $U_r \in \mathcal{B}_\phi$ such that $E \cap (x + U_r) = \emptyset$.*

Proof. Assume that there exists $x \notin E$ such that for each $U_r \in \mathcal{B}$, $E \cap (x + U_r) \neq \emptyset$. Then for each n there exists $x^{(n)} \in E \cap (x + U_{1/n})$. Hence, the sequence $(x^{(n)}) \subset E$ and $\rho_\phi(x^{(n)} - x) < 1/n \to 0$ if $n \to \infty$. Since E is ρ_ϕ-closed, then $x \in E$, contradicts the assumption that $x \notin E$. □

Finally, we obtain the property of the density of the set E^ϕ within ℓ_ϕ^0 in this topology, which is stated in the following theorem.

Theorem 10 *Then E^ϕ is dense in ℓ_ϕ^0 (in the sense of ρ_ϕ-convergent).*

Proof. Let $x = (x_k) \in \ell_\phi^0$. For each n, let $x_k^{(n)} = x_k$ if $k \leq n$ and $x_k^{(n)} = 0$ if $k > n$. Then $x^{(n)} \in E^\phi$, since $\sum_{k=1}^\infty \phi(\lambda x_k^{(n)}) = \sum_{k=1}^n \phi(\lambda x_k^{(n)}) < \infty$ for each $\lambda > 0$. Let $\varepsilon > 0$ be arbitary. Since $x \in \ell_\phi^0$, then there exists n_0 such that $\sum_{k=n+1}^\infty \phi(x_k) < \varepsilon$ for each $n \geq n_0$. Therefore we get

$$\rho_\phi(x^{(n)} - x) = \sum_{k=1}^\infty \phi(x_k^{(n)} - x_k) = \sum_{k=n+1}^\infty \phi(x_k) < \varepsilon$$

for each $n \geq n_0$, i.e. $(x^{(n)})$ is ρ-convergent to x. □

IV. CONCLUSIONS AND FUTURE RESEARCH DIRECTION

In the Orlicz sequence space, a topological space can be constructed using a local base whose members are ρ_ϕ-neighborhood of zero. Topological properties such as convergence, bounded sets, and closed sets can be expressed in local bases. Furthermore, some results in this paper are based on the Δ_2-condition of the Orlicz function. Based on the results that have been obtained, the next possible research to be done is to examine the topological properties when the Δ_2-condition is omitted.

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