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Corrected Trapezoidal Rule For The Riemann-Stieltjes Integral

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Abstract. This study investigates the derivation of a corrected trapezoidal rule for approximating the Riemann-Stieltjes integral. The corrected trapezoidal rule is derived by approximating certain monomial functions to obtain optimal method coefficients. The proposed method has an accuracy of order three. Furthermore, an error analysis is conducted to assess the accuracy of the obtained approximation. In the final section, numerical computations are presented to compare the performance of the proposed method with existing methods. The results demonstrate that the proposed method produces smaller errors compared to previously developed approaches.

Keywords: corrected trapezoidal rule; Riemann-Stieltjes integral; error term.

I. INTRODUCTION

The Riemann-Stieltjes integral was originally introduced by Thomas Stieltjes in 1894 in his seminal work "Recherches sur les fractions continues", which was published in the Annales de la Faculté des Sciences de Toulouse [1]. The Riemann-Stieltjes integral generalizes the standard Riemann integral by introducing a second integrator function, thereby achieving greater flexibility in mathematical modeling and analysis. Specifically, for two real-valued functions $f(x)$ and $g(x)$ defined on $[a, b]$, the Riemann–Stieltjes integral is defined as

$$\int_a^b f(x)\, dg(x) = \lim_{\|\Delta\|\to 0} \sum_{i=0}^{n-1} f(\xi_i)\, [g(x_{i+1}) - g(x_i)],$$

where $\xi_i \in [x_i, x_{i+1}]$ and $\|\Delta\|$ denotes the norm of the partition. When $g(x) = x$, it reduces to the standard Riemann integral $\int_a^b f(x)\, dx$. Hence, the Riemann–Stieltjes integral can be interpreted as a weighted form of integration where the increment of g replaces the uniform measure dx, allowing one to model situations where weights or transformations are non-uniform. This flexibility makes it an essential tool in probability theory, stochastic processes, and functional analysis, particularly in describing probability distributions and expected values. Owing to its broad applicability, the Riemann-Stieltjes integral has also been widely applied in numerical analysis and differential equation studies, and thus remains an active area of research and development.

Various quadrature rules have been developed to approximate the Riemann-Stieltjes integral with the aim of achieving higher accuracy. Mercer [2] proposed a trapezoidal rule for the Riemann-Stieltjes integral, leading to Hadamard's inequality for general integrals. Later, Mercer [3] extended this work by developing the Midpoint and Simpson's $1/3$ rules using the concept of relative convexity. TThe trapezoidal rule has also been modified for Riemann-Stieltjes integrals by Zhao and Zhang [6, 7], by including the value of the derivative at the endpoint and the derivative at the midpoint. Zhao and Zhang [8] also modified Simpsons $1/3$ to approximate this integral. Furthermore, Zhao, Zhang, and Ye [5] introduced a composite Trapezoidal rule for the Riemann-Stieltjes integral. Additionally, Memon et al. [9] proposed efficient derivative-based and derivative-free quadrature schemes, verified through numerical experiments. In another study, Memon et al. [10, 11, 12] modified Simpson's $1/3$ rule by incorporating Heronian, centroidal, and harmonic mean derivative values to approximate the Riemann-Stieltjes integral. Memon [13] also modified the four-point quadrature to approximate the Riemann-Stieltjes integral. Several adjustments have been made in the numerical method over time to get better performance based on our needs. For example, a recent variation of the Double Midpoint Rule for approximating the Riemann-Stieltjes Integral [14].

Despite these advancements, most existing quadrature rules either require higher-order derivatives or involve complex correction terms, which increase computational cost. Therefore, there remains a need for a simple yet accurate method that can effectively approximate the Riemann–Stieltjes integral. Building on the approach presented in [4], this study aims to develop a modified corrected trapezoidal rule to improve the numerical approximation of the Riemann-Stieltjes integral.

The motivation behind the study arises from the classical trapezoidal approximation,

$$I_T = \frac{f(a) + f(b)}{2} [g(b) - g(a)]. \tag{1}$$

This study aims to modify the corrected Trapezoidal rule to improve the numerical approximation of the Riemann-Stieltjes integral. The article provides a comprehensive analysis of the corrected Trapezoidal rule in the context of the Riemann-Stieltjes integral, including the derivation of its error formulation. Furthermore, numerical simulations are conducted to evaluate the accuracy and effectiveness of the proposed method.

II. RESULTS AND DISCUSSION

This section presents the findings of the study, including the development of the Corrected Trapezoidal Rule for the Riemann-Stieltjes integral, its associated error analysis, and numerical examples demonstrating its effectiveness. The proposed method is analyzed in detail to evaluate its accuracy and applicability in numerical integration.

2.1. Corrected Trapezoidal Rule For The Riemann-Stieltjes Integral

In this section, the formulation of the corrected Trapezoidal Rule (CTRS) for the Riemann–Stieltjes integral is presented. The formulation, stated in Theorem 2.1, is derived by approximating monomial functions at selected powers and solving a nonlinear system that determines the coefficients of the corrected trapezoidal rule in (1).

Theorem 1 *Let $f'(s)$ and $g(s)$ are continuous on $[v, w]$ and $g(t)$ is increasing function in the* *interval $[v, w]$. The corrected Trapezoidal Rule (CTR) for the Riemann-Stieltjes integral is*

$$\begin{aligned} CTRS \;=\; & \Big(-g(v) + \frac{6}{(v-w)^2}\int_v^w\int_v^s g(x)dxds + \frac{12}{(v-w)^3}\int_v^w\int_v^s\int_v^y g(x)dxdyds\Big) f(v) \\ & +\Big(g(w) - \frac{6}{(v-w)^2}\int_v^w\int_v^s g(x)dxds - \frac{12}{(v-w)^3}\int_v^w\int_v^s\int_v^y g(x)dxdyds\Big) f(w) \\ & -\Big(\int_v^w g(s)ds + \frac{4}{(v-w)}\int_v^w\int_v^s g(x)dxds \\ & +\frac{6}{(v-w)^2}\int_v^w\int_v^s\int_v^y g(x)dxdyds\Big) f'(w) \\ & +\Big(\frac{-2}{(v-w)}\int_v^w\int_v^s g(x)dxds - \frac{6}{(v-w)^2}\int_v^w\int_v^s\int_v^y g(x)dxdyds\Big) f'(v). \end{aligned} \tag{2}$$

Proof. The general form of corrected trapezoid rule is [15, h. 174]

$$CT = \frac{w-v}{2}\left(f(v)+f(w)\right) - \frac{(w-v)^2}{12}\left(f'(w)-f'(v)\right), \tag{3}$$

Obtaining the corrected trapezoidal rule for the Riemann-Stieltjes integral requires rewriting Equation (3) as follows:

$$CT = a_0 f(v) + b_0 f(w) - \left[c_0 f'(w) - d_0 f'(v)\right]. \tag{4}$$

The values of a_0, b_0, c_0, and d_0 will be determined such that the integral in Equation (4) is exact for $f(s) = 1, s, s^2, s^3$ and obtain the following equations

$$\int_v^w 1dg(s) = a_0 + b_0; \tag{5}$$

$$\int_v^w sdg(s) = a_0 v + b_0 w - c_0 + d_0; \tag{6}$$

$$\int_v^w s^2 dg(s) = a_0 v^2 + b_0 w^2 - 2wc_0 + 2vd_0; \tag{7}$$

$$\int_v^w s^3 dg(s) = a_0 v^3 + b_0 w^3 - 3c_0 w^2 + 3d_0 v^2. \tag{8}$$

The Riemann-Stieltjes integral formula is applied to the left side of equations (5), (6), (7), and (8) to derive the following expression

$$a_0 + b_0 = g(w) - g(v); \tag{9}$$

$$a_0 v + b_0 w - c_0 + d_0 = wg(w) - vg(v) - \int_v^w g(s)ds; \tag{10}$$

$$\begin{aligned} a_0 v^2 + b_0 w^2 - 2wc_0 + 2vd_0 \;=\; & w^2 g(w) - v^2 g(v) - 2\int_v^w g(s)dt \\ & +2\int_v^w\int_v^t g(x)dxds; \end{aligned} \tag{11}$$

$$\begin{aligned} a_0v^3 + b_0w^3 - 3c_0w^2 + 3d_0v^2 &= b^3g(b) - a^3g(a) - 3b^2\int_v^w g(s)ds \\ &\quad +6b\int_v^w\int_v^s g(x)dxds \\ &\quad -6\int_v^w\int_v^s\int_v^y g(x)dxdyds. \end{aligned} \tag{12}$$

The coefficients a_0, a_1, c_0, and c are determined by solving the system of equations (9), (10), (11), and (12), yielding the following results:

$$\begin{aligned} a_0 &= -g(v) + \frac{6}{(v-w)^2}\int_v^w\int_v^s g(x)dxds + \frac{12}{(v-w)^3}\int_v^w\int_v^t\int_v^y g(x)dxdyds \\ b_0 &= g(w) - \frac{6}{(v-w)^2}\int_v^w\int_v^t g(x)dxds - \frac{12}{(v-w)^3}\int_v^w\int_v^t\int_v^y g(x)dxdyds \\ c_0 &= \int_v^w g(s)ds + \frac{4}{(v-w)}\int_v^w\int_v^t g(x)dxds + \frac{6}{(v-w)^2}\int_v^w\int_v^t\int_v^y g(x)dxdyds \\ d_0 &= \frac{-2}{(v-w)}\int_v^w\int_v^t g(x)dxds - \frac{6}{(v-w)^2}\int_v^w\int_v^t\int_v^y g(x)dxdyds. \end{aligned}$$

□

Based on Theorem (1), it can be seen that for $f(s) = s^4$ the quadrature is not equal. Therefore, the accuracy of this method is 3.

2.2. Error Term Of Corrected Trapezoidal Rule For The Riemann-Stieltjes Integral.

In this section, the error form of the corrected Trapezoidal Rule for the Riemann-Stieltjes integral is given. We use the precision notion to calculate the error form associated to the difference between the quadrature formula for the monomial $\frac{s^{r+1}}{(r+1)!}$ and the exact value of the

$$\frac{1}{(r+1)!}\int_v^w s^{r+1}dx = \frac{w^{r+2} - v^{r+2}}{(r+2)!},$$

where r is the precision of the quadrature formula.

Theorem 2 *Suppose that $f'(s)$ and $g'(s)$ are continuous on $[v, w]$ and $g(s)$ is increasing there. The corrected Trapezoidal Rule (CTRS) for the Riemann-Stieltjes integral with the error term is*

$$\begin{aligned} R(f) &= \Bigg[\left(\frac{v^2w - 2vw^2 + w^2}{12}\right)\int_v^w\int_v^s g(x)dxds \\ &\quad + \left(\frac{v-w}{2}\right)\int_v^w\int_v^s\int_v^y g(x)dxdyds \\ &\quad + \int_v^w\int_v^s\int_v^z\int_v^y g(x)dxdydzds\Bigg] f^{(4)}(\xi)\, g'(\eta) \end{aligned} \tag{13}$$

Proof. The error in the equation (13) is obtained by using a monomial of order 4, which

$$f(s) = \frac{s^4}{4!}. \tag{14}$$

The exact solution of equation (14) is

$$\begin{aligned} \frac{1}{4!}\int_v^w t^4 dg &= \frac{1}{24}\left(w^4 g(w) - v^4 g(v)\right) - \frac{w^3}{6}\int_v^w g(s)ds + \frac{w^2}{6}\int_v^w\int_v^s g(x)dxds \\ &\quad -w\int_v^w\int_v^s\int_v^y g(x)dxdyds + \int_v^w\int_v^s\int_v^z\int_v^y g(x)dxdydzds. \end{aligned} \tag{15}$$

Based on Theorem (1), the approximate solution is

$$\begin{aligned} CNTR &= \frac{1}{24}\left(w^4 g(w) - v^4 g(v)\right) - \frac{w^3}{6}\int_v^w g(s)ds \\ &\quad - \frac{-v^2 + 2vw + 5b^2}{12}\int_v^w\int_v^s g(x)dxds \\ &\quad - \frac{-v - w}{2}\int_v^w\int_v^s\int_v^y g(x)dxdyds. \end{aligned} \tag{16}$$

The error term is obtained by subtracting the exact solution (15) from the approximate solution (16), hence

$$\begin{aligned} R(f) &= \Bigg[\left(\frac{v^2 w - 2vw^2 + w^2}{12}\right)\int_v^w\int_v^s g(x)dxds \\ &\quad + \left(\frac{v-w}{2}\right)\int_v^w\int_v^s\int_v^y g(x)dxdyds \\ &\quad + \int_v^w\int_v^s\int_v^z\int_v^y g(x)dxdydzds\Bigg] f^{(4)}(\xi)\, g^{'}(\eta). \end{aligned}$$

□

2.3. Numerical Examples

To validate the theoretical findings, numerical examples are given to illustrate the performance of the Corrected Trapezoidal Rule for Riemann-Stieltjes integrals $(CTRS)$. These examples compare the proposed method with existing numerical integration techniques, namely Trapezoidal Rule for Riemann-Stieltjes integral (AT) [6], Simpson Centroidal Riemann-Stieltjes integral (SC) [11], Simpson Heronian Riemann-Stieltjes integral (SH) [10], and Simpson Harmonic Riemann-Stieltjes integral (SHM) [12], highlighting its efficiency and precision.

Example 1 Integral $\int_{0.0}^{1.0} s^3 d\left(\sin s^2\right)$, $\int_{0.0}^{1.0} e^{-s} d\left(\cos s\right)$, $\int_0^1 ln(s+1)d\left(\cos s\right)$ are approximated by using $CTRS$, AT, SC, SH, and SHM.

Table 1. Comparison of computational results of $CTRS$, AT, SC, SH, and SHM methods

Methods	Integral		
	$\int_{0.0}^{1.0} s^3 d\left(\sin s^2\right)$	$\int_{0.0}^{1.0} e^{-s} d\left(\cos s\right)$	$\int_{0}^{1} ln(s+1) d\left(\cos s\right)$
	Error	*Error*	*Error*
$CTRS$	0.0000000000000	0.0003985494540	0.0008831519270
AT	0.0000000000000	0.0005691286670	0.0011528711856
SC	0.0119205077121	0.0010307315662	0.0011632818783
SH	0.0119205077121	0.0011843421351	0.0020109453758
SHM	0.0119205077121	0.0013987229536	0.0051128623031

The computational results in Table (1.) compare five numerical integration methods: $CTRS$, AT, SC, SH, and SHM, with the $CTRS$ method consistently demonstrating superior accuracy. In the first integral $\int_0^1 s^3 d(\sin s^2)$, all methods achieve zero error, indicating excellent performance for relatively simple integrals. However, the CTRS method stands out in more complex cases.

For the second integral $\int_0^1 e^{-s} d(\cos s)$, $CTRS$ achieves the lowest error of 0.0003985494540, outperforming other methods such as AT and SC. This demonstrates its efficiency in handling exponential and trigonometric components with better numerical stability. Similarly, for the third integral $\int_0^1 \ln(s+1) d(\cos s)$, $CTRS$ again records the smallest error of 0.0008831519270, indicating its robustness with logarithmic-trigonometric integrals.

Overall, the $CTRS$ method consistently delivers minimal error across all tested integrals, especially in complex scenarios. Its superior precision makes it a reliable choice for mathematical and engineering applications requiring high numerical accuracy.

III. CONCLUSIONS AND FUTURE RESEARCH DIRECTION

In conclusion, the Corrected Trapezoidal Rule for the Riemann-Stieltjes integral is derived from the formulation of the Corrected Trapezoidal Rule itself. This method achieves third-order accuracy. The error expression for the Corrected Trapezoidal Rule in Riemann-Stieltjes integration is obtained by calculating the difference between the exact value and the quadrature formula applied to monomials of a certain degree.the Corrected Trapezoidal Rule for the Riemann-Stieltjes integral $(CTRS)$ has proven to be a highly accurate and reliable numerical method. Its minimal error across various types of integrals, including polynomial, exponential, trigonometric, and logarithmic functions, highlights its versatility and robustness. The computational results clearly demonstrate that $CTRS$ outperforms other conventional methods such as AT, SC, SH, and SHM, particularly in complex integration scenarios. Future research could focus on modifying other numerical methods to approximate the Riemann-Stieltjes integral. Exploring variations of classical methods might reveal new techniques that enhance accuracy or computational efficiency.

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