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PMC-Labeling of Certain Classes of Graphs

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Abstract. Let $G = (V, E)$ be a graph with p vertices and q edges. Define $\rho = \frac{p}{2}$ if p is even or $\frac{p-1}{2}$ if p is odd and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p - 1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling thus defined is called a pair mean cordial labeling (PMC-labeling) if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there is a pair mean cordial labeling is called a pair mean cordial graph (PMC-graph). In this paper, we investigate the PMC-labeling behavior of some new graphs such as the double fan graph, triple fan graph, m–enriched fan graph, C_n–snake, stripe blade graph, G_n, $Sf_n + K_1$, armed helm graph, alternate armed helm graph and spectrum graph.

Keywords: double fan graph, C_n–snake, stripe blade graph, armed helm graph, spectrum graph.

I. INTRODUCTION

Mathematics owes a lot to the genius of Leonhard Euler (1707-1783) whose pioneering works led to the birth of new branches of the subject like Algebraic number theory, Analytic number theory and Graph theory and laid the foundation for the branches such as Geometry, Calculus, Topology, Complex analysis, Mechanics, etc. Born at Basel in Switzerland, Euler had his education in his motherland and was taught by Johann Bernoulli. In 1727, Euler joined the St. Petersburg Academy of Sciences in Russia. There he came across an outstanding problem of the time known as the Konisberg bridge problem. The river Pragel was flowing across the city of Konisberg, presently known as Kaliningrad, and caused the division of the city into four land areas which were linked by seven bridges. People were interested to start from any one land area, walk through all the seven bridges and cover all the four land areas with the restriction that any bridge or any land area shall be traversed exactly once. Several persons ventured to accomplish the task but none could succeed. The problem became very famous and the significant question in everybody's mind was whether the solution could be achieved and how to achieve the solution. On hearing the problem, Euler evinced keen interest to systematically analyse it. He did not consider the physical travel through the bridges and the land areas to find a solution. Rather he conceived the problem from geometrical perspective. He considered each land area as a point (vertex) and each bridge as a line (edge) in a plane. With his masterstroke of novel representation, he proved that it was impossible to cover all the four land areas by crossing each bridge exactly once. His method of solution led to the origin of the branch of Mathematics which came to be known later as Graph theory. Euler's procedure to analyse the problem has become a classical example of Mathematical modeling. It is an elegant illustration for the transformation of an interesting puzzle into a new discipline of research. With the growth of the subject of Graph theory, several of its applications have come into play in various diversified areas such as Operations Research, Computer Science, Communication Engineering, Network Design, Structural Chemistry, Biological Systems, Cryptography, etc.

During the early stages of the development of Graph theory, the vertices and edges of a graph G were assigned the labels $v_1, v_2, \dots$ and $e_1, e_2, \dots$, respectively. As the subject registered phenomenal growth emanating from different methodologies of various researchers, different ingenious techniques emerged for labeling the vertices and edges of a graph, reflecting the undercurrent of the relevant mathematical properties. Nowadays, graph labeling has become an active area of research.

In this paper, we consider only finite, simple and undirected graphs. A labeling of a graph is a mapping that translates graph elements (vertices, edges, or both) to integers or colors subject to certain conditions. Graph labeling has received significant attention from various graph theory researchers recently. It has a wide range of applications including coding theory, radar, X-ray, circuit design, communication networks, astronomy, network addressing and graph decomposition problems. For the purpose of having a broad outlook of the present status of research in the area of graph labeling, it is worthwhile to list down some glimpses of the works carried out by various researchers in this area and their significant contributions. With this objective, a brief review of literature is presented here.

The first paper on graph labeling was presented by Rosa in [26]. The idea of cordial labeling was introduced by Cahit [4]. For a dynamic survey on graph labeling, one may follow Gallian [6]. Difference cordial labeling of some special graphs and related to fan graphs was studied in [28]. Ghodasara et al. [8] have proved that the path union of finite copies of helm, closed helm and gear graph is product cordial. The k-enriched fan graphs and their characterisations of graceful labeling were examined in [11]. Harmonic mean cordial labeling of some cycle related graphs was discussed in [10]. Prajapati and Patel [24] have studied the edge product cordial labeling behavior of some cycle related graphs such as closed web graph, lotus inside circle, sunflower graph and super subdivision of flower graph. Catherine Lee [5] has determined the minimum coprime number for a few well-studied classes of graphs including the coronas of complete graphs with empty graphs and the joins of two paths.
The prism graphs, generalized Petersen graphs with $k = 2$, and stacked prism graphs are investigated for minimum coprime labelings in [14]. Berliner et al. [3] have investigated the coprime labelings for complete bipartite graphs. The rainbow connection number of spectrum graphs was discussed by Melina and Salman in [13]. Cordial labeling of the graph obtained by joining two copies of fan graph and star of fan graph by a path of arbitrary length was explored in [7]. Cordiality in the context of duplication in web and armed helm has been investigated by Prajapti and Gajjar [23]. Topological cordial labeling of some graphs were discussed in [12].

Power cordial labeling was studied in [1]. Several sunflower extended graphs were defined and the upper bounds of the radio number and radial radio number of these graphs were investigated in [18]. Basher [2] has proven that the super subdivision of path, comb, cycle, ladder, crown, circular ladder, planar grid and prism are k-Zumkeller graphs. For all terms and notations of graph theory, we refer to Harary [9]. Legendre cordial labeling of some graphs were discussed in [13]. E_A-cordial labeling of graphs and its implications for A-antimagic labeling of trees was studied in [27]. Exploring mean cordial labeling possible combination position of vertices in graphs: A computational approach was explored in [17].

A new labeling method called PMC-labeling has been introduced in [19]. The PMC-labeling behavior of path, cycle, complete graph, star, fan, friendship, gear, some snake related graphs and some special graphs have been explored in [20, 21, 22]. In this paper, we investigate the PMC-labeling behavior of some new graphs such as the double fan graph, triple fan graph, m–enriched fan graph, C_n–snake, stripe blade graph, G_n, $Sf_n + K_1$, armed helm graph, alternate armed helm graph and spectrum graph.

Gallian [6] is a standard source of reference for a detailed account of the different concepts related to the research on graph labeling. The following is one of the crucial concepts required for the present study.

Definition 1 *[6] Let G_1 and G_2 be two graphs with no vertex in common. The join of G_1 and G_2, denoted by $G_1 + G_2$ is defined to be the graph with vertex set and edge set given as follows: $V(G_1 + G_2) = V(G_1) \cup V(G_2)$, $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in G_1, v \in G_2\}$.*

In their interesting study on graceful labeling of graphs, Haviar and Kurtulik [11] considered enriched fan graphs. From their work, we have the following concepts.

Definition 2 *[11] A fan graph f_n is a join of a path P_n and a single vertex K_1. That is $f_n = P_n + K_1$.*

Definition 3 *[11] The k-enriched fan graph $EF_{n,k}$, $n \geq 2$, is the graph of size $(k+1)n - 1$ obtained by connecting n copies of the star graph $K_{1,k}$ of order k to the fan graph f_n such that one vertex of each copy of the star $K_{1,k}$ is identified with one vertex of the main path P_n of f_n.*

Definition 4 *[6] The C_n–snake $C_{n,m}$ is the graph obtained from a path P_m by replacing each edge of the path P_m by a cycle C_n.*

Definition 5 *[6] The graph obtained by replacing every edge of a star graph $K_{1,n}$ by the tripartite graph $K_{1,2,1}$ is a stripe blade graph SB_n.*

Definition 6 *[6] The graph G_n is a connected graph with the vertex set $V(G_n) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and the edge set $E(G_n) = \{v_iw_i, u_iv_i \mid 1 \leq i \leq n\} \cup \{v_iv_{i+1}, v_nv_1, u_iu_{i+1}, u_nu_1, v_iu_{i+1}, v_nu_1 \mid 1 \leq i \leq n-1\}$.*

Prajapati and Gajjar [23] examined the cordiality of certain graphs in the context of duplication in web and armed helm. Their study has led to the following concepts.

Definition 7 *[23] The wheel graph W_n is a join of the graphs C_n and K_1. That is $W_n = C_n + K_1$. The vertices corresponding to C_n are called the rim vertices of W_n and the vertex corresponding to K_1 is called the apex vertex.*

Definition 8 *[23] The helm graph H_n is the graph obtained from the wheel W_n by adding a pendent edge at each rim vertex.*

Definition 9 *[23] The armed helm graph AH_n is a graph in which path P_2 is attached with each vertex of wheel graph W_n by an edge.*

Definition 10 *[6] The alternate armed helm graph AAH_n, is a graph with the vertex set $V(AAH_n) = \{u_0, u_i, v_i, w_i, x_i \mid 1 \leq i \leq n\}$ and the edge set $E(AAH_n) = \{u_0u_i, u_iv_i, v_iw_i, w_ix_i \mid 1 \leq i \leq n\} \cup \{u_iv_{i+1}, u_nv_1 \mid 1 \leq i \leq n-1\}$.*

Melina and Salman [16] focused attention on the rainbow connection number of spectrum graphs. Their study has provided the following concept.

Definition 11 *[16] The spectrum graph ST_n, $n \geq 1$ is a connected graph with the vertex set and the edge set, respectively, as follows: $V(ST_n) = \{u, v, w, u_i, v_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(ST_n) = \{uv, uw, uv_i, uu_i, ux_i, uy_i, vu_i, vx_i, wv_i, wy_i, u_iv_i, x_iy_i \mid 1 \leq i \leq n\}$.*

Definition 12 *Let $G = (V, E)$ be a graph with p vertices and q edges. Define*

$$\rho = \begin{cases} \frac{p}{2} & \text{if } p \text{ is even} \\ \frac{p-1}{2} & \text{if } p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is called a pair mean cordial labeling (PMC-labeling) if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there is a pair mean cordial labeling is called a pair mean cordial graph (PMC-graph). Figure (1) presents a simple example of PMC-Graph.

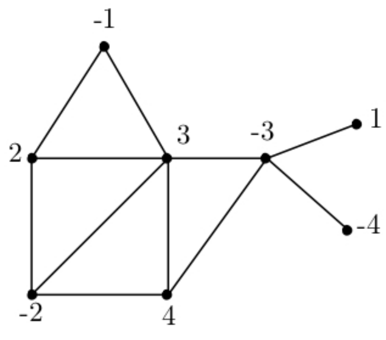

Figure 1. An Example of PMC-graph

II. MAIN RESULTS

In this section, we investigate the PMC-labeling behavior of the double fan graph, triple fan graph, m–enriched fan graph, C_n–snake, stripe blade graph, G_n, $Sf_n + K_1$, armed helm graph, alternate armed helm graph and spectrum graph.

Theorem 1 *The double fan graph DF_n is a PMC-graph for all $n \geq 2$.*

Proof. Consider the double fan graph DF_n, $n \geq 2$. Let $V(DF_n) = \{v_0, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(DF_n) = \{v_0v_i, v_iw_i \mid 1 \leq i \leq n\} \cup \{v_iv_{i+1} \mid 1 \leq i \leq n-1\}$ denote, respectively, the vertex set and edge set of the double fan graph DF_n. Hence it is seen that DF_n has $3n-1$ edges and $2n+1$ vertices. We have to consider the following two cases.
Case (i) : n is odd
Let $\lambda(v_0) = 2$ and $\lambda(w_n) = 1$. Assign the labels $-1, -3, \ldots, -n$ and $3, 5, \ldots, n$ to the corresponding vertices $v_1, v_3, \ldots, v_n$ and $v_2, v_4, \ldots, v_{n-1}$. Next assign the labels $2, 4, \ldots, n-1$ and $-2, -4, \ldots, -n+1$ to the corresponding vertices $w_1, w_3, \ldots, w_{n-1}$ and $w_2, w_4, \ldots, w_{n-1}$.
Case (ii) : n is even
Let $\lambda(v_0) = 1$ and $\lambda(v_n) = 1$. We assign the labels $-1, -3, \ldots, -n+1$ and $3, 5, \ldots, n-1$ to the corresponding vertices $v_1, v_3, \ldots, v_{n-1}$ and $v_2, v_4, \ldots, v_{n-2}$. Thereafter assign the labels $2, 4, \ldots, n$ and $-2, -4, \ldots, -n$ to the corresponding vertices $w_1, w_3, \ldots, w_{n-1}$ and $w_2, w_4, \ldots,$ w_n. □

Table 1. The following table establishes the PMC-labeling of the double fan graph DF_n for all $n \geq 2$.

Values of n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
n is odd	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$
n is even	$\frac{3n-2}{2}$	$\frac{3n}{2}$

Example 1 Figure (2) presents a PMC-labeling of a double fan graph DF_3.

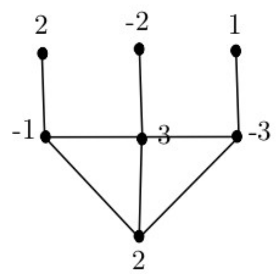

Figure 2. A PMC-labeling of a double fan graph DF_3

Theorem 2 *The triple fan graph TF_n is a PMC-graph for all $n \geq 2$.*

Proof. Consider the triple fan graph TF_n, $n \geq 2$. Let $V(TF_n) = \{v_0, u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(TF_n) = \{v_0u_i, u_iv_i, v_iw_i \mid 1 \leq i \leq n\} \cup \{u_iu_{i+1} \mid 1 \leq i \leq n-1\}$ denote, respectively, the vertex set and edge set of the triple fan graph TF_n. Then, TF_n has $4n-1$ edges and $3n+1$ vertices. Define $\lambda(v_0) = 1$. We have to consider the following two cases.
Case (i) : n is odd
Let us assign the labels $3, 6, \ldots, \frac{3n-3}{2}$ and $-2, -5, \ldots, \frac{-3n+5}{2}$ to the corresponding vertices

$u_1, u_3, \ldots, u_{n-2}$ and $u_2, u_4, \ldots, u_{n-1}$. Fix the vertices u_n by the label $\frac{-3n-1}{2}$. Thereafter assign the labels $-1, -4, \ldots, \frac{-3n+1}{2}$ and $-3, -6, \ldots, \frac{-3n+3}{2}$ to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $v_2, v_4, \ldots, v_{n-1}$ respectively. Next we assign the labels $2, 5, \ldots, \frac{3n+1}{2}$ and $4, 7, \ldots, \frac{3n-1}{2}$ respectively to the vertices $w_1, w_3, \ldots, w_n$ and $w_2, w_4, \ldots, w_{n-1}$.

Case (ii) : n is even

In this case, assign the labels $3, 6, \ldots, \frac{3n}{2}$ and $-2, -5, \ldots, \frac{-3n+2}{2}$ to the corresponding vertices $u_1, u_3, \ldots, u_{n-1}$ and $u_2, u_4, \ldots, u_n$. Further, assign the labels $-1, -4, \ldots, \frac{-3n+4}{2}$ and $-3, -6, \ldots, \frac{-3n}{2}$ to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $v_2, v_4, \ldots, v_n$ respectively. Next we assign the labels $2, 5, \ldots, \frac{3n-2}{2}$ and $4, 7, \ldots, \frac{3n-4}{2}$ respectively to the vertices $w_1, w_3, \ldots, w_{n-1}$ and $w_2, w_4, \ldots, w_{n-2}$. Fix the vertices w_n by the label $\frac{-3n}{2}$. In both cases, $\bar{\mathbb{S}}_{\lambda_1} = 2n - 1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2n$. □

Example 2 Figure (3) presents a PMC-labeling of a triple fan graph TF_3.

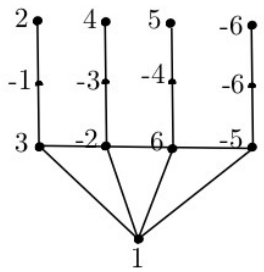

Figure 3. A PMC-labeling of a triple fan graph TF_3

Theorem 3 *The m–enriched fan graph $EF_{n,m}$, $n \geq 2$ is a PMC-graph for all $m \leq 5$.*

Proof. Consider the m–enriched fan graph $EF_{n,m}$, $n \geq 2$. Let $V(EF_{n,m}) = \{v_0, v_{i,j} \mid 1 \leq i \leq n \,\text{and}\, 1 \leq j \leq m\}$ and $E(EF_{n,m}) = \{v_0 v_{i,1}, v_{i,1}v_{i,2}, v_{i,j}v_{i,j+1} \mid 1 \leq i \leq n \,\text{and}\, 2 \leq j \leq m\} \cup \{v_{i,1}v_{i+1,1} \mid 1 \leq i \leq n-1\}$ denote, respectively, the vertex set and edge set of the m–enriched fan graph $EF_{n,m}$. Then, $EF_{n,m}$ has $n(m+1)-1$ edges and $nm+1$ vertices. When $m = 1$, the m–enriched fan graph $EF_{n,m} \simeq F_n$. Note that the fan graph F_n is a PMC-graph. When $m = 2$, the m–enriched fan graph $EF_{n,m} \simeq DF_n$. The double fan graph DF_n is a PMC-graph in Theorem 1. When $m = 3$, the m–enriched fan graph $EF_{n,m} \simeq TF_n$. The triple fan graph TF_n is a PMC-graph in theorem (4.2). Define $\lambda(v_0) = 1$. We have to consider the following two cases.

Case (i) : $m = 4$

Subcase (i) : n is even

We assign the labels $-1, -4, \ldots, \frac{-3n+4}{2}$ and $4, 7, \ldots, \frac{3n+2}{2}$ to the corresponding vertices $v_{1,2}, v_{2,2}, \ldots, v_{n-1,2}$ and $v_{2,2}, u_{4,2}, \ldots, v_{n,2}$. Further, assign the labels $2, 5, \ldots, \frac{3n-2}{2}$ and $-2, -5, \ldots, \frac{-3n+2}{2}$ to the vertices $v_{1,3}, v_{3,3}, \ldots, v_{n-1,3}$ and $v_{2,3}, v_{4,3}, \ldots, v_{n,3}$ respectively. Next we assign the labels $3, 6, \ldots, \frac{3n}{2}$ and $-3, -6, \ldots, \frac{-3n}{2}$ to the vertices $v_{1,4}, v_{3,4}, \ldots, v_{n-1,4}$ and $v_{2,4}, v_{4,4}, \ldots, v_{n,4}$ respectively. Continue the labeling process by assigning the labels $\frac{-3n-2}{2}, \frac{-3n-4}{2}, \ldots, -2n$ and $\frac{3n+4}{2}, \frac{3n+6}{2}, \ldots, 2n$ respectively to the vertices $v_{1,1}, v_{3,1}, \ldots, v_{n-1,1}$ and $v_{2,1}, v_{4,1}, \ldots, v_{n-2,1}$. Fix the vertices $v_{n,1}$ by the label $-2n$. With these assignments, we have $\bar{\mathbb{S}}_{\lambda_1} = \frac{5n-2}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{5n}{2}$.

Subcase (ii) : n is odd
Assign the labels $-1, -4, \ldots, \frac{-3n+1}{2}$ and $4, 7, \ldots, \frac{3n-1}{2}$ to the corresponding vertices $v_{1,2}, v_{2,2}, \ldots, v_{n,2}$ and $v_{2,2}, u_{4,2}, \ldots, v_{n-1,2}$. Next assign the labels $2, 5, \ldots, \frac{3n+1}{2}$ and $-2, -5, \ldots, \frac{-3n+5}{2}$ to the vertices $v_{1,3}, v_{3,3}, \ldots, v_{n,3}$ and $v_{2,3}, v_{4,3}, \ldots, v_{n-1,3}$ respectively. Also assign the labels $3, 6, \ldots, \frac{3n+3}{2}$ and $-3, -6, \ldots, \frac{-3n+3}{2}$ to the vertices $v_{1,4}, v_{3,4}, \ldots, v_{n,4}$ and $v_{2,4}, v_{4,4}, \ldots, v_{n-1,4}$ respectively. Finally assign the labels $\frac{-3n-1}{2}, \frac{-3n-3}{2}, \ldots, -2n$ and $\frac{3n+3}{2}, \frac{3n+5}{2}, \ldots, 2n$ respectively to the vertices $v_{1,1}, v_{3,1}, \ldots, v_{n,1}$ and $v_{2,1}, v_{4,1}, \ldots, v_{n-1,1}$. Then $\bar{\mathbb{S}}_{\lambda_1} = \frac{5n-1}{2} = \bar{\mathbb{S}}_{\lambda_1^c}$.
Case (ii) : $m = 5$
Let us now label the vertices $v_{i,j}$, $1 \leq i \leq n$, $2 \leq j \leq 4$ in the same manner as in Case (i).
Subcase (i) : n is even
Continue the labeling process by assigning the labels $\frac{3n+4}{2}, \frac{3n+6}{2}, \ldots, 2n+1$ and $\frac{-3n-2}{2}, \frac{-3n-4}{2}, \ldots, -2n$ respectively to the vertices $v_{1,1}, v_{3,1}, \ldots, v_{n-1,1}$ and $v_{2,1}, v_{4,1}, \ldots, v_{n,1}$. Also assign the labels $2n+2, 2n+3, \ldots, 3n-3$ and $-2n-1, -2n-2, \ldots, -3n+3$ to the vertices $v_{1,5}, v_{3,5}, \ldots, v_{n-1,5}$ and $v_{2,5}, v_{4,5}, \ldots, v_{n,5}$ respectively.
Subcase (ii) : n is odd
Continue the labeling process by assigning the labels $\frac{-3n-1}{2}, \frac{-3n-3}{2}, \ldots, -2n$ and $\frac{3n+5}{2}, \frac{3n+7}{2}, \ldots, 2n+1$ respectively to the vertices $v_{1,1}, v_{3,1}, \ldots, v_{n,1}$ and $v_{2,1}, v_{4,1}, \ldots, v_{n-1,1}$. Finally assign the labels $-2n-1, -2n-2, \ldots, -3n+2$ and $2n+2, 2n+3, \ldots, 3n-2$ to the vertices $v_{1,5}, v_{3,5}, \ldots, v_{n,5}$ and $v_{2,5}, v_{4,5}, \ldots, v_{n-1,5}$ respectively. In both cases $\bar{\mathbb{S}}_{\lambda_1} = 3n-1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 3n$.

□

Example 3 Figure (4) presents a PMC-labeling of a 4-enriched fan graph $EF_{5,5}$.

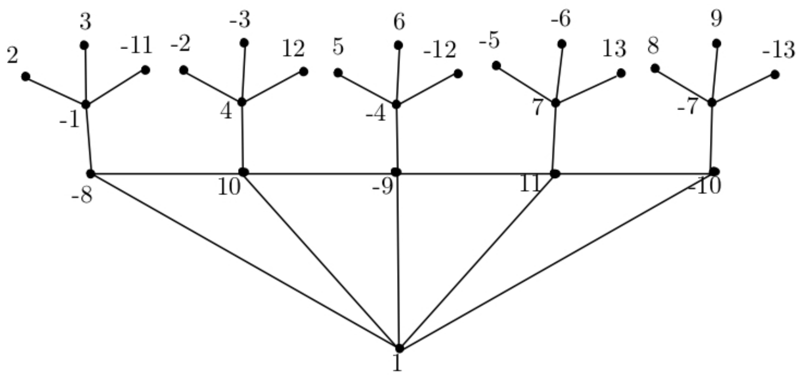

Figure 4. A PMC-labeling of a 4-enriched fan graph $EF_{5,5}$

Theorem 4 *The* 6*–enriched fan graph* $EF_{n,6}$ *is a PMC-graph if and only if* n *is even.*

Proof. Consider the 6–enriched fan graph $EF_{n,6}$. We have to consider the following two cases.
Case (i) : n is even
Let us now label the vertices $v_{i,j}$, $1 \leq i \leq n$, $2 \leq j \leq 4$ in the same manner as in Case (i) of above theorem. Define $\lambda(v_0) = 2n+1$. Thereafter assign the labels $\frac{3n+4}{2}, \frac{3n+6}{2}, \ldots, 2n+1$ and $\frac{-3n-2}{2}, \frac{-3n-4}{2}, \ldots, -2n$ respectively to the vertices $v_{1,1}, v_{3,1}, \ldots, v_{n-1,1}$ and $v_{2,1}, v_{4,1}, \ldots, v_{n-2,1}$.

Thereafter assign the remaining labels to the remaining vertices in any order. We obtain $\bar{\mathbb{S}}_{\lambda_1} = \frac{7n-2}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{7n}{2}$.

Case (ii) : n is odd

Suppose the 6–enriched fan graph $EF_{n,6}$ is a PMC-graph. To get the edge uv with label 1, the condition to be fulfilled is that the sum of the labels of the vertices i. e., $\lambda(u)+\lambda(v) = 1$ or 2. The maximum possible number of edges labelled with 1 is $3n$ for these vertices. That is, $\bar{\mathbb{S}}_{\lambda_1} \leq 3n$. Subsequently, $\bar{\mathbb{S}}_{\lambda_1^c} \leq q-(3n) = nm-2n-1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq nm-2n-1-(3n) = nm-5n-1 \geq 2 > 1$, and thus we come across a contradiction. □

Theorem 5 *The m–enriched fan graph $EF_{n,m}$, $n \geq 2$ is not a PMC-graph for all $m \geq 7$.*

Proof. Suppose the m–enriched fan graph $EF_{n,m}$, $m \geq 7$ is a PMC-graph. To get the edge uv with label 1, the requirement is that the sum $\lambda(u)+\lambda(v) = 1$ or 2.

Case (i) : n is even

The vertices have $3n+1$ as the maximum possible number of edges labeled with 1. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 3n+1$. Subsequently, $\bar{\mathbb{S}}_{\lambda_1^c} \leq q-(3n+1) = nm-2n-2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq nm-2n-2-(3n+1) = nm-5n-3 \geq 3 > 1$, and thus we obtain a contradiction.

Case (ii) : n is odd

The maximum possible number of edges labeled with 1 is $3n$ for the vertices. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 3n$. Subsequently, $\bar{\mathbb{S}}_{\lambda_1^c} \leq q-(3n) = nm-2n-1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq nm-2n-1-(3n) = nm-5n-1 \geq 2 > 1$, and therefore we come across a contradiction. □

Theorem 6 *The C_n-snake $C_{n,m}$ is a PMC graph for all $n \geq 3$ and $m \geq 2$.*

Proof. Consider the C_n-snake $C_{n,m}$, $n \geq 3$ and $m \geq 2$. Let $V(C_{n,m}) = \{u_{i,j}, u_0 \mid 1 \leq i \leq n-1 \text{ and} 1 \leq j \leq m\}$ and $E(C_{n,m}) = \{u_{i,j}u_{i+1,j}, u_{n-1,m}u_0 \mid 1 \leq i \leq n-2 \text{ and} 1 \leq j \leq m\} \cup \{u_{1,j}u_{1,j+1}, u_{n-1,j+1}u_{1,j+1}, u_{1,m}u_0 \mid 1 \leq j \leq m-1\}$ denote, respectively the vertex set and edge set of the C_n-snake $C_{n,m}$. Then it has $m(n-1)+1$ vertices and mn edges. Let $\lambda(u_0) = 1$. We have to consider the following two cases.

Case (i) : m is even

The mapping $\lambda : V(C_{n,m}) \to \{\pm1, \pm2, \ldots, \pm\frac{m(n-1)}{2}\}$ is defined as follows:

Subcase (i) : n is even

Define

$$\lambda(u_{i,j}) = \begin{cases} \frac{(j-1)(n-1)+i+3}{2} & 1 \leq i \leq n-1, \quad 1 \leq j \leq m, \ i \,\&\, j \text{ are odd} \\ \frac{(-j+1)(n-1)-i}{2} & 1 \leq i \leq n-1, \quad 1 \leq j \leq m, \ i \text{ is even } \& \ j \text{ is odd,} \end{cases}$$

$$\lambda(u_{i,j}) = \begin{cases} \frac{-j(n-1)-2i+1}{2} & i = 1, 2, \ldots, \frac{n}{2}, \quad 1 \leq j \leq m, \ j \text{ is even} \\ \frac{(j+2)(n-1)-2i-4}{2} & i = \frac{n+2}{2}, \frac{n+4}{2}, \ldots, n-1, \quad 1 \leq j \leq m-1, \ j \text{ is even,} \end{cases}$$

$$\lambda(u_{i,m}) = \begin{cases} \frac{(m+2)(n-1)-2i-2}{2} & i = \frac{n+2}{2}, \frac{n+4}{2}, \ldots, n-2 \\ 1 & i = n-1 \end{cases}$$

Subcase (ii) : n is odd

Label the vertices $u_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq m-1 \,\&\, j$ is odd in the same manner as in Subcase

(i). Then

$$\lambda(u_{i,j}) = \begin{cases} \frac{(j-1)(n-1)+2i+1}{2} & i = 1, 2, \ldots, \frac{n-1}{2}, \quad 1 \le j \le m-1, \ j \text{ is even} \\ \frac{(-j-1)(n-1)+2i-2}{2} & i = \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-1, \quad 1 \le j \le m-1, \ j \text{ is even,} \end{cases}$$

$$\lambda(u_{i,m}) = \begin{cases} \frac{(m-1)(n-1)+2i+1}{2} & i = 1, 2, \ldots, \frac{n-3}{2} \\ \frac{(-m-1)(n-1)-2i}{2} & i = \frac{n-1}{2}, \frac{n+1}{2}, \ldots, n-2 \\ 1 & i = n-1 \end{cases}$$

Hence, $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{mn}{2} = \bar{\mathbb{S}}_{\lambda_1}$.
Case (ii) : m is odd
The mapping $\lambda : V(C_{n,m}) \to \{\pm 1, \pm 2, \ldots, \pm \frac{m(n-1)+1}{2}\}$ is defined as follows. Label the vertices $u_{i,j}$, $1 \le i \le n$, $1 \le j \le m-1$ in the same manner as in Case (i). Then
Subcase (i) : n is even

$$\lambda(u_{i,m}) = \begin{cases} \frac{(m-1)(n-1)+2i+2}{2} & i = 1, 3 \\ \frac{(-m+1)(n-1)-i}{2} & i = 2, 4 \\ \frac{(-m+1)(n-1)-i-1}{2} & i = 5, 7, \ldots, n-1 \\ \frac{(m-1)(n-1)+i+2}{2} & i = 6, 8, \ldots, n-2 \end{cases}$$

Hence, $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{mn}{2} = \bar{\mathbb{S}}_{\lambda_1}$.
Subcase (ii) : n is odd

$$\lambda(u_{i,m}) = \begin{cases} \frac{(-m+1)(n-1)-i-1}{2} & i = 1, 3, \ldots, n-2 \\ \frac{(m-1)(n-1)+i+2}{2} & i = 2, 4, \ldots, n-3 \\ 1 & i = n-1 \end{cases}$$

Thus, $\bar{\mathbb{S}}_{\lambda_1} = \frac{mn-1}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{mn+1}{2}$.

□

Example 4 Figure (5) presents a PMC-labeling of a C_n-snake graph $C_{6,4}$.

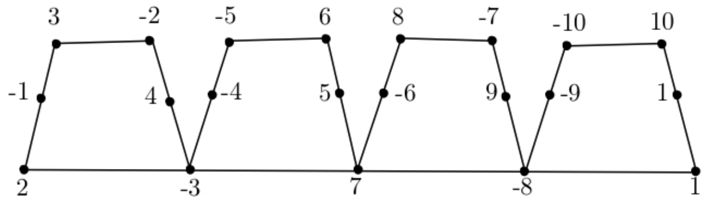

Figure 5. A PMC-labeling of a C_n-snake graph $C_{6,4}$

Theorem 7 *The stripe blade graph SB_n is not a PMC-graph for all $n \ge 1$.*

Proof. Consider the stripe blade graph SB_n, $n \geq 1$. Denote by $V(SB_n) = \{u_0, u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(SB_n) = \{u_0u_i, u_0v_i, u_0w_i, u_iv_i, u_iw_i \mid 1 \leq i \leq n\}$, respectively the vertex set and edge set of the stripe blade graph SB_n. Then SB_n has $3n+1$ vertices and $5n$ edges. We have to consider the following two cases.

Case $(i): n = 1$

Suppose the stripe blade graph SB_1 is a PMC-graph. Clearly we get $\bar{\mathbb{S}}_{\lambda_1} \leq 1$. Subsequently, $\bar{\mathbb{S}}_{\lambda_1^c} \leq 4$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3 > 1$. This is a contradiction.

Case $(ii): n \geq 2$

Suppose the stripe blade graph SB_n is a PMC-graph. To get the edge uv with label 1, the constraint is that the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. This implies that $2n$ is the maximum possible number of edges labeled with 1. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2n$. Subsequently, $\bar{\mathbb{S}}_{\lambda_1^c} \leq q - (2n) = 3n$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3n - (2n) = n \geq 2 > 1$, and this leads to a contradiction. □

Theorem 8 *The graph G_n is a PMC-graph if and only if n is odd and $n \geq 3$.*

Proof. Denote by $V(G_n) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(G_n) = \{v_iw_i, u_iv_i \mid 1 \leq i \leq n\} \cup \{v_iv_{i+1}, v_nv_1, u_iu_{i+1}, u_nu_1, v_iu_{i+1}, v_nu_1 \mid 1 \leq i \leq n-1\}$ respectively, the vertex and edge sets of the graph G_n. Then, G_n has $3n$ vertices and $5n$ edges. We now consider two cases.

Case $(i): n$ is odd

Then, assign the labels $3, 6, \ldots, \frac{3n-3}{2}$ and $-2, -5, \ldots, \frac{-3n+5}{2}$ to the corresponding vertices $u_1, u_3, \ldots, u_{n-1}$ and $u_2, u_4, \ldots, u_{n-1}$. Fix the label $\frac{3n-1}{2}$ with u_n. Assign the labels $4, 7, \ldots, \frac{3n-1}{2}$ and $-4, -7, \ldots, \frac{-3n+7}{2}$ to the corresponding vertices $v_1, v_3, \ldots, v_{n-2}$ and $v_2, v_4, \ldots, v_{n-1}$. Fix the label $\frac{-3n+3}{2}$ with v_n. Also, assign the labels $2, 5, \ldots, \frac{3n-5}{2}$ and $-3, -6, \ldots, \frac{-3n+3}{2}$ to the corresponding vertices $w_1, w_3, \ldots, w_{n-2}$ and $w_2, w_4, \ldots, w_{n-3}$. Finally, assign the labels $\frac{-3n+1}{2}, 1$ to the vertices w_{n-1}, w_n.

Case $(ii): n$ is even

Suppose that the graph G_n is a PMC-graph. To get the edge uv with label 1, the required condition is that the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. This implies that $\frac{5n+2}{2}$ is the maximum possible number of edges labeled with 1. That is $\bar{\mathbb{S}}_{\lambda_1} \leq \frac{5n+2}{2}$. Subsequently, $\bar{\mathbb{S}}_{\lambda_1^c} \leq q - (\frac{5n-2}{2}) = \frac{5n+2}{2}$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq \frac{5n+2}{2} - (\frac{5n-2}{2}) = 2 > 1$, and thus we get a contradiction. □

Example 5 Figure (6) presents a PMC-labeling of a graph G_5.

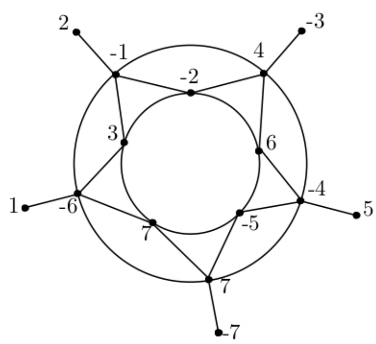

Figure 6. A PMC-labeling of a graph G_5

Theorem 9 *The graph $Sf_n + K_1$ is not PMC-graph for all $n \geq 3$.*

Proof. Consider the graph $Sf_n + K_1$, $n \geq 3$. Let $V(Sf_n + K_1) = \{u_0, u_i, v_i \mid 1 \leq i \leq n\}$ and $E(Sf_n + K_1) = \{u_0u_i, u_0v_i, u_iv_i \mid 1 \leq i \leq n\} \cup \{v_iu_{i+1}, v_nu_1, u_iu_{i+1}, u_nu_1 \mid 1 \leq i \leq n-1\}$ denote, respectively, the vertex set and edge set of the graph $Sf_n + K_1$. It is seen that $Sf_n + K_1$ has $5n$ edges and $2n+1$ vertices. Suppose that the graph $Sf_n + K_1$ is a PMC-graph. To get the edge uv with label 1, the required condition is that the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. This implies that $2n-1$ is the maximum possible number of edges labeled with 1. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2n-1$. Subsequently, $\bar{\mathbb{S}}_{\lambda_1^c} \leq q - (2n-1) = 3n+1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3n+1-(2n-1) = n+2 \geq 5 > 1$, and consequently we get a contradiction.

□

Theorem 10 *The armed helm graph AH_n is a PMC-graph for all $n \geq 3$.*

Proof. Consider the armed helm graph AH_n. Let $V(AH_n) = \{u_0, u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(AH_n) = \{u_0u_i, u_iv_i, v_iw_i \mid 1 \leq i \leq n\} \cup \{u_iu_{i+1}, u_nu_1 \mid 1 \leq i \leq n-1\}$ denote, respectively, the vertex set and edge set of the armed helm graph AH_n. Then AH_n has $3n+1$ vertices and $4n$ edges. Define $\lambda(u_0) = 1$, $\lambda(u_1) = 3$ and $\lambda(w_1) = 2$. We have to consider the following two cases.
Case (i) : n is odd
Let us assign the labels $-1, -4, \ldots, \frac{-3n+1}{2}$ and $4, 7, \ldots, \frac{3n-1}{2}$ to the corresponding vertices $v_1, v_3, \ldots, v_n$ and $v_2, v_4, \ldots, v_{n-1}$. Next, assign the labels $-2, -5, \ldots, \frac{-3n+5}{2}$ and $5, 8, \ldots, \frac{3n+1}{2}$ to the vertices $u_2, u_4, \ldots, u_{n-1}$ and $u_3, u_5, \ldots, u_n$ respectively. Thereafter, we assign the labels $-3, -6, \ldots, \frac{-3n+3}{2}$ and $6, 9, \ldots, \frac{3n-3}{2}$ respectively to the vertices $w_2, w_4, \ldots, w_{n-1}$ and $w_3, w_5, \ldots, w_{n-2}$. Fix the vertices w_n by the label $\frac{-3n-1}{2}$.
Case (ii) : n is even
In this case, assign the labels $-1, -4, \ldots, \frac{-3n+4}{2}$ and $4, 7, \ldots, \frac{3n-4}{2}$ to the corresponding vertices $v_1, v_3, \ldots, v_{n-1}$ and $v_2, v_4, \ldots, v_{n-2}$. Fix the vertices v_n by the label $\frac{3n}{2}$. Then assign the labels $-2, -5, \ldots, \frac{-3n+2}{2}$ and $5, 8, \ldots, \frac{3n-2}{2}$ to the vertices $u_2, u_4, \ldots, u_n$ and $u_3, u_5, \ldots, u_{n-1}$ respectively. Further, assign the labels $-3, -6, \ldots, \frac{-3n}{2}$ and $6, 9, \ldots, \frac{3n}{2}$ respectively to the vertices $w_2, w_4, \ldots, w_n$ and $w_3, w_5, \ldots, w_{n-1}$. In both cases, $\bar{\mathbb{S}}_{\lambda_1} = 2n = \bar{\mathbb{S}}_{\lambda_1^c}$. □

Theorem 11 *The alternate armed helm graph AAH_n is a PMC-graph for all $n \geq 3$.*

Proof. Consider the alternate armed helm graph AAH_n. Let $V(AAH_n) = \{u_0, u_i, v_i, w_i, x_i \mid 1 \leq i \leq n\}$ and $E(AAH_n) = \{u_0u_i, u_iv_i, v_iw_i, w_ix_i \mid 1 \leq i \leq n\} \cup \{u_iv_{i+1}, u_nv_1 \mid 1 \leq i \leq n-1\}$ denote, respectively, the vertex set and edge set of the alternate armed helm graph AAH_n. Then AAH_n has $4n+1$ vertices and $5n$ edges. Define $\lambda(u_0) = -1$ and $\lambda(x_n) = 1$. Assign the labels $-1, -2, \ldots, -n$ and $2, 3, \ldots, n+1$ to the corresponding vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$. We have to consider the following four cases.
Case (i) : $n \equiv 0 \pmod 4$
In this case, assign the labels $-n-1, -n-4, \ldots, -2n+1$; $-n-2, -n-5, \ldots, -2n$, and $n+3, n+5, \ldots, 2n-2$ to the corresponding vertices $w_1, w_4, \ldots, w_{n-1}$; $w_2, w_5, \ldots, w_n$ and $w_3, w_6, \ldots, w_{n-2}$. Further, assign the labels $n+2, n+5, \ldots, 2n$; $-n-3, -n-6, \ldots, -2n+2$, and $n+4, n+7, \ldots, 2n-1$ to the vertices $x_1, x_4, \ldots, x_{n-1}$; $x_2, x_5, \ldots, x_{n-3}$ and $x_3, x_6, \ldots, x_{n-2}$ respectively. Thus $\bar{\mathbb{S}}_{\lambda_1} = \frac{5n}{2} = \bar{\mathbb{S}}_{\lambda_1^c}$.
Case (ii) : $n \equiv 1 \pmod 4$

We assign the labels $-n-1, -n-4, \ldots, -2n+2$; $-n-2, -n-5, \ldots, -2n+1$, and $n+3, n+5, \ldots, 2n$ to the corresponding vertices $w_1, w_4, \ldots, w_{n-2}$; $w_2, w_5, \ldots, w_{n-1}$ and $w_3, w_6, \ldots, w_n$. Thereafter assign the labels $n+2, n+5, \ldots, 2n-1$; $-n-3, -n-6, \ldots, -2n$, and $n+4, n+7, \ldots, 2n-2$ to the vertices $x_1, x_4, \ldots, x_{n-2}$; $x_2, x_5, \ldots, x_{n-1}$ and $x_3, x_6, \ldots, x_{n-3}$ respectively. Therefore $\bar{\mathbb{S}}_{\lambda_1} = \frac{5n-1}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{5n+1}{2}$.

Case $(iii) : n \equiv 2 \pmod 4$

Let us now assign the labels $-n-1, -n-4, \ldots, -2n+2$; $-n-2, -n-5, \ldots, -2n+1$, and $n+3, n+5, \ldots, 2n$ to the corresponding vertices $w_1, w_4, \ldots, w_{n-2}$; $w_2, w_5, \ldots, w_{n-2}$ and $w_3, w_6, \ldots, w_n$. Also, assign the labels $n+2, n+5, \ldots, 2n-1$; $-n-3, -n-6, \ldots, -2n$, and $n+4, n+7, \ldots, 2n-2$ to the vertices $x_1, x_4, \ldots, x_{n-2}$; $x_2, x_5, \ldots, x_{n-1}$ and $x_3, x_6, \ldots, x_{n-3}$ respectively. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{5n}{2} = \bar{\mathbb{S}}_{\lambda_1^c}$.

Case $(iv) : n \equiv 3 \pmod 4$

In this case assign the labels $-n-1, -n-4, \ldots, -2n$; $-n-2, -n-5, \ldots, -2n+2$, and $n+3, n+5, \ldots, 2n-1$ to the corresponding vertices $w_1, w_4, \ldots, w_n$; $w_2, w_5, \ldots, w_{n-2}$ and $w_3, w_6, \ldots, w_{n-1}$. Next, assign the labels $n+2, n+5, \ldots, 2n-2$; $-n-3, -n-6, \ldots, -2n+1$, and $n+4, n+7, \ldots, 2n$ to the vertices $x_1, x_4, \ldots, x_{n-3}$; $x_2, x_5, \ldots, x_{n-2}$ and $x_3, x_6, \ldots, x_{n-1}$ respectively. Then $\bar{\mathbb{S}}_{\lambda_1} = \frac{5n-1}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{5n+1}{2}$.

□

Example 6 Figure (7) presents a PMC-labeling of an alternate armed helm graph AAH_8.

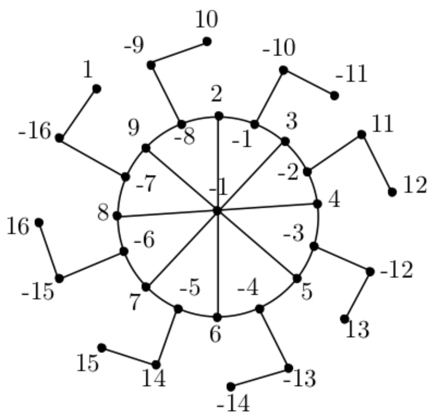

Figure 7. A PMC-labeling of an alternate armed helm graph AAH_8

Theorem 12 *The spectrum graph SP_n is not a PMC-graph for all $n \geq 1$.*

Proof. Consider the spectrum graph $SP_n, n \geq 1$. Denoting by $V(SP_n) = \{u, v, w, u_i, v_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(SP_n) = \{uv, uw, uv_i, uu_i, ux_i, uy_i, vu_i, vx_i, wv_i, wy_i, u_iv_i, x_iy_i \mid 1 \leq i \leq n\}$, respectively the vertex set and edge set of the spectrum graph SP_n. Then it has $4n+3$ vertices and $10n+2$ edges. We have to consider two cases.

Case $(i) : n = 1$

Suppose the spectrum graph SP_1 is a PMC-graph. Clearly we get $\bar{\mathbb{S}}_{\lambda_1} \leq 5$. Subsequently, $\bar{\mathbb{S}}_{\lambda_1^c} \leq 7$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2 > 1$. This is a contradiction.

Case $(ii) : n \geq 2$

Suppose the spectrum graph SP_n is a PMC-graph. To get the edge uv with label 1, the required condition is that the sum of the vertex label $\lambda(u)+\lambda(v)=1$ or 2. This implies that $2n+4$ is the maximum possible number of edges labeled with 1. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2n+4$. Subsequently, $\bar{\mathbb{S}}_{\lambda_1^c} \leq q-(2n+4)=8n-2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c}-\bar{\mathbb{S}}_{\lambda_1} \geq 8n-2-(2n+4)=6n-6 \geq 6 > 1$, and consequently we get a contradiction. □

Example 7 The following theorem says that the Figure (8) is not a PMC-graph.

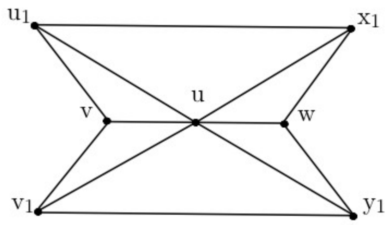

Figure 8. A Spectrum graph SP_1

III. CONCLUSION

In this paper, we have investigated the PMC-labeling behavior of some new graphs such as the double fan graph, triple fan graph, m–enriched fan graph, C_n–snake, stripe blade graph, G_n, Sf_n+K_1, armed helm graph, alternate armed helm graph and spectrum graph. Future research will examine whether PMC-labeling exists for certain standard graphs and graph operations. It is expected that the succeeding research work on PMC- labeling of graphs would bring out various mathematical properties of interest.

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