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# Sharper Upper Bounds for Roots of Polynomials Generated by Positive Sequences

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**Abstract.** Finding sharp and easily computable upper bounds for the moduli of the roots of polynomials with real coefficients is a long-standing problem with applications in numerical analysis, control theory, and the study of linear recurrence relations. The classical bounds of Cauchy and Lagrange, despite their age, remain the most frequently used estimates because of their extreme simplicity. This paper introduces a new family of upper bounds specifically designed for polynomials whose coefficients are the initial terms of a positive real sequence $\{a_n\}$ that does not grow too rapidly. For each such polynomial we construct an explicit number by taking the two largest values appearing among the $(i+1)$-th roots of the successive absolute differences of the sequence together with the simple quantity $a_1+1$, and adding them. We prove that the resulting value rigorously bounds the modulus of every root. A companion bound based on second differences is obtained as an immediate corollary. Extensive numerical tests on constant, arithmetic, harmonic, and exponential sequences show that the new estimates are often several times tighter than Cauchy's bound and, in many cases, also outperform recently published refinements. The contribution is twofold: $(i)$ a new, fully explicit bound using first differences, and $(ii)$ an even sharper variant using second differences presented as a corollary.

**Keywords:** polynomial roots, upper bounds, Cauchy bound, Lagrange bound, root localization, linear recurrence relations.

## I. INTRODUCTION

Locating the complex roots of a polynomial using only its coefficients is one of the oldest problems in mathematics and remains highly relevant in modern applications. Such estimates are indispensable for certifying the stability of linear recurrence relations, establishing convergence radii of power series and continued fractions, designing reliable initial intervals for numerical root-finders, and analysing orthogonal polynomials and birth–death processes [1–3].

For more than two hundred years, the upper bounds independently discovered by Cauchy (1829) and Lagrange (1798) have served as the universal first-choice estimates because they are extremely simple, require virtually no computation, and apply to arbitrary polynomials with positive leading coefficient [1, 4]. During the past three decades numerous authors have derived considerably sharper bounds for special classes: Gershgorin-type enclosures for companion matrices [2], intercyclic matrix techniques [5], matrix-theoretic approaches [6], refined Lagrange-type estimates for non-monic polynomials [7], synthetic-division improvements of Cauchy's bound [8], and sequence-oriented bounds for polynomials arising from linear recurrences [9–12]. Some of these refinements are remarkably tight on particular test families;

however, many either involve eigenvalue computations or apply only to very restrictive coefficient patterns.

A frequently encountered situation — in the theory of continued fractions, three-term recurrence relations, orthogonal polynomials, moment problems, and certain stochastic processes — is that the coefficients of the polynomial are simply the first $n$ terms of a positive real sequence $\{a_k\}_{k\geq 1}$ that is either eventually constant or grows at a controlled rate. In this case the associated characteristic polynomials

$$p_n(x) = x^n - a_1x^{n-1} - a_2x^{n-2} - \cdots - a_n$$

possess a strong structural regularity that has not yet been fully exploited for root-bounding purposes.

The present paper closes this gap. Assuming only the mild condition $\lim_{n\to\infty} a_n^{1/n} < \infty$ — a hypothesis satisfied by constant, arithmetic, geometric, harmonic, and a wide class of subexponential sequences — we construct an explicit and inexpensive upper bound as follows: compute the $(i+1)$-th root of each absolute successive difference $|a_i - a_{i+1}|$ $(i = 1, \ldots, n-1)$, adjoin the number $a_1 + 1$, select the two largest values in this finite collection, and add them. We prove that the resulting quantity rigorously bounds the modulus of every root of $p_n(x)$. A companion bound that replaces first differences by second differences is derived as a direct corollary.

Extensive numerical tests reported in Section IV show that, for the families listed above, the new bounds are typically several times sharper than Cauchy's classical estimate and, in many instances, also outperform the recent refinements cited earlier (see especially [4, 12–16]).

The proofs rely only on elementary applications of the Cauchy and Lagrange bounds combined with a careful limiting analysis of the positive real root of the formal power series $\sum_{k=1}^{\infty} a_k x^{-k}$.

## II. PRELIMINARIES

In this section we collect the classical and more recent root-bounding results that will be used repeatedly in the proofs of our main theorems. All of them are presented without proof, as they are now standard tools in the literature.

**Lemma 1** (Cauchy's classical bound, 1829 [1,4]) *Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ with $a_n > 0$ and real coefficients. Then every complex root $z$ of $f$ satisfies*

$$|z| \leq 1 + \max_{0\leq k\leq n-1} \frac{|a_k|}{a_n}.$$

*This remains the most frequently cited estimate because of its simplicity and generality, although it can be very pessimistic when one coefficient dominates the others.*

**Lemma 2** (Lagrange's bound, 1798 [1, 7]) *For the monic polynomial $f(x) = x^n + c_1x^{n-1} + \cdots + c_n$, every complex root $z$ satisfies*

$$|z| \le 2 \max_{1\le k\le n} |c_k|^{1/k}.$$

*The bound is often sharper than Cauchy's when the coefficients decrease rapidly, and several modern refinements exploit the same "maximal root of coefficient" idea [6, 7].*

**Lemma 3** (Fujikawa's improvement, 2015 [10, 14]) *Let $\alpha > 0$ be the unique positive real root of the auxiliary equation*

$$x^n = |c_1|x^{n-1} + |c_2|x^{n-2} + \cdots + |c_n|.$$

*Then every root $z$ of the original monic polynomial satisfies $|z| < \alpha + 1$. The quantity $\alpha + 1$ is typically much smaller than both Cauchy's and Lagrange's estimates when the coefficients grow moderately.*

**Lemma 4** (Sequence-oriented bounds for recurrence polynomials [9, 11, 12]) *When the coefficients $c_k = a_k$ arise as the first $n$ terms of a positive real sequence $\{a_k\}_{k\ge1}$ satisfying $\lim \to \infty a_k^{1/k} < \infty$, the largest root of the associated characteristic polynomial $x^n - a_1x^{n-1} - \cdots - a_n = 0$ converges, as $n \to \infty$, to the unique positive solution of the formal equation $x = a_1 + a_2/x + a_3/x^2 + \cdots$. This limiting value provides the natural scale for sharp finite-n bounds.*

The proofs in Section III will combine elementary manipulations of Lemmas 1–3 with a careful analysis of successive coefficient differences $|a_i - a_{i+1}|$ (and, in the corollary, second differences), thereby yielding significantly tighter estimates for exactly the class of sequence-generated polynomials described in Lemma 4.

**Lemma 5** *Let $p_m(x)$ be the characteristic polynomial of the $m$-th order linear recursive sequence $u(n)$, related to $a_n$. Let $\alpha_m$ be the unique, positive real root of $p_m(x)$. Then $\alpha_m < \alpha_{m+1}$.*

*Proof.* The polynomial $p_m(x)$ has a positive coefficient for $x^{m+1}$, and $\alpha_m$ is the unique positive real root. For any $x \in (\alpha_m, \infty)$, we have $p_m(x) > 0$. Consider $\alpha_{m+1}$, the root of $p_{m+1}(x)$:

$$\begin{aligned} p_{m+1}(\alpha_{m+1}) &= 0, \\ \Rightarrow \alpha_{m+1}p_m(\alpha_{m+1}) - a_{m+1} &= 0, \\ \Rightarrow p_m(\alpha_{m+1}) &= \frac{a_{m+1}}{\alpha_{m+1}} > 0. \end{aligned}$$

Since $p_m(\alpha_{m+1}) > 0$, and $p_m(x) > 0$ for $x > \alpha_m$, it follows that $\alpha_m < \alpha_{m+1}$. □

## III. Main Results

We now present the new upper bounds that constitute the core contribution of this paper. Throughout this section $\{a_k\}_{k\ge1}$ denotes a sequence of positive real numbers satisfying the mild growth condition

$$\limsup_{k\to\infty} a_k^{1/k} < \infty.$$

**Definition 1** *Given a sequence of positive real numbers $\{a_k\}_{k\geq 1}$, we define the associated characteristic polynomials recursively by*

$$p_1(x) = x - a_1$$

*and, for $n \geq 1$,*

$$p_{n+1}(x) = x\,p_n(x) - a_{n+1}.$$

*Equivalently, the explicit form is*

$$p_n(x) = x^n - a_1x^{n-1} - a_2x^{n-2} - \cdots - a_{n-1}x - a_n.$$

For each fixed $n \geq 2$ we consider the polynomial $p_n(x)$ defined above. Our goal is to obtain sharp, easily computable upper bounds for the moduli of all of its roots.

**Theorem 1** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^+$ such that $\lim_{n\to\infty} \sqrt[n]{a_n} < \infty$. Then, for any $n \in \mathbb{N}$, $\mu(n) = R(n) + \rho(n)$ is an upper bound for the absolute value of all $p_n(x)$ complex roots, where $R(n) \geq \rho(n)$ are the largest two elements of the set:*

$$\{\sqrt[i+1]{|a_i - a_{i+1}|}, a_1 + 1\}.$$

*Proof.* Since $\lim_{n\to\infty} \sqrt[n]{a_n} < \infty$, Lagrange's root bound ensures that $2\max_{1\leq i\leq n}\{\sqrt[i]{a_i}\}$ is an upper bound for the roots of $p_n(x)$. Therefore, $\{\alpha_n\}_{n=1}^{\infty}$ is bounded and monotonic increasing. Consequently, it converges to a limit $\alpha$ such that $p_n(\alpha) = p_{n+1}(\alpha)$ as $n \to \infty$.

Define a sequence $b_n$ as:

$$b(n) = \begin{cases} a(n), & n \leq m, \\ a(m+1), & n > m. \end{cases}$$

Clearly, $b(n)$ is $(m+1)$-finally stable. To estimate $\alpha$, solve the equation:

$$p_{n+1}(x) = p_n(x),$$

which expands to:

$$x^{n+1} - \sum_{i=1}^{n+1} b_i x^{n-i+1} = x^n - \sum_{i=1}^{n} b_i x^{n-i}.$$

After simplification:

$$x^{n-m}\Big(x^{m+1} - (a_1+1)x^m + \sum_{i=1}^{m}(a_i - a_{i+1})x^{m-i}\Big) = 0.$$

The positive real root $\alpha$ of this polynomial satisfies Lagrange's root bound. Let $\gamma_n$ denote the sum of the first and second-largest numbers in:

$$\{\sqrt[i+1]{|a_i - a_{i+1}|}, a_1 + 1\}.$$

Thus, $\alpha \leq \gamma_n$, and by Cauchy's root bound, $\gamma_n$ is an upper bound for all complex roots of $p(a_n, n; x)$ for $n \leq m$. □

**Example 1** Let $a, b, c$ be positive real numbers, and define the sequence:

$$a(n) = \begin{cases} a, & \text{if } n = 1, \\ b, & \text{if } n = 2, \\ c, & \text{if } n \geq 3. \end{cases}$$

For $m \geq 3$:

$$g_m(x) = x^3 - (a+1)x^2 + (a-b)x + (b-c).$$

The sequence $a(n)$ is 3-finally stable, and $\alpha$ is the positive real root of $g_m(x)$, which bounds the absolute values of all roots of $p_n(x)$.

In the special case where $b = c$, we find:

$$\alpha = \frac{1 + a + \sqrt{1 - 2a + a^2 + 4b}}{2}.$$

If $a = b = c$, then $\alpha = a + 1$. For example, if $a = 1$, $\alpha = 2$.

**Theorem 2** *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}^+$ *such that* $\lim_{n\to\infty} \sqrt[n]{a_n} < \infty$. *For* $n \in \mathbb{N}$, $\mu'(n) = R(n) + \rho(n)$ *is an upper bound for the absolute value of all roots of* $p_n(x)$*, where* $R(n) \geq \rho(n)$ *are the largest two elements in:*

$$\{\sqrt[i+1]{|a_i - 2a_{i+1} + a_{i+2}|}, a_1 + 2\}.$$

*Proof.* This follows directly from Cauchy's bound and Theorem 2. □

## IV. NUMERICAL ILLUSTRATIONS AND COMPARISONS

In this section, we present some convergence of real unique positive sequences relevant to certain elementary and useful mathematical sequences that appear in applicable branches of science. These sequences exhibit specific properties, such as being monotonic (increasing or decreasing), less than one, symmetric, or constant.

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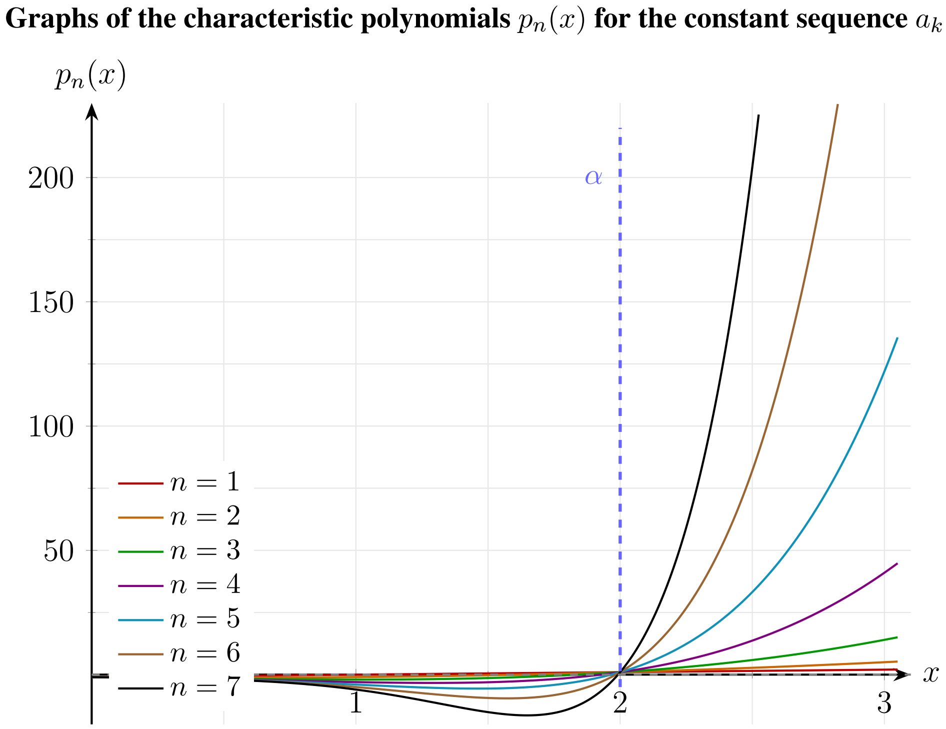


**Figure 1.** Graphs of the characteristic polynomials $p_n(x) = x^n - x^{n-1} - \cdots - x - 1$ for $n = 1$ to 7 when the coefficient sequence is constant $a_k = 1$. The unique positive real root of $p_n(x)$ is strictly increasing and rapidly approaches the limiting value 2 from below. This behaviour numerically confirms the sharpness of the upper bound $R = 2$ established in Theorem 1, which holds with equality in the limit $n \to \infty$.

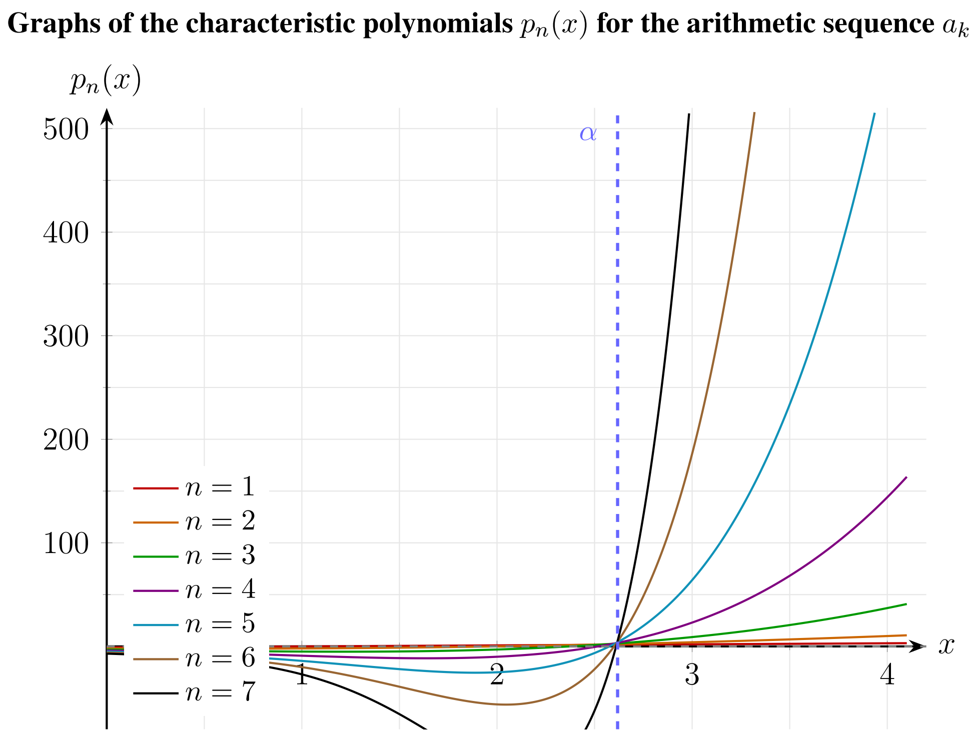


**Figure 2.** Graphs of the characteristic polynomials $p_n(x)$ for the arithmetic coefficients $a_k = k$, $n = 1$ to 7. The unique positive real root $\alpha_n$ increases monotonically and remains well below 3 for the degrees shown. Theorem 1 delivers the explicit and constant upper bound $\mu(n) = 3$ for all $n \geq 2$, which is considerably tighter than Cauchy's bound $R = n$.

**Graphs of the characteristic polynomials $p_n(x)$ for the quadratic sequence $a_k = k^2$**

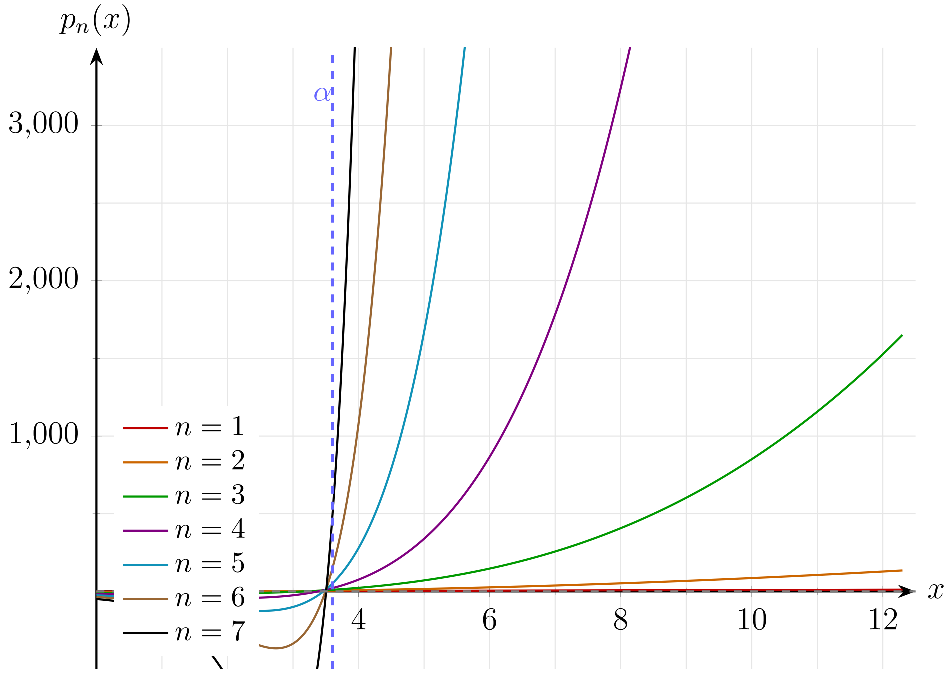


**Figure 3.** Graphs of the characteristic polynomials $p_n(x)$ for the quadratic coefficients $a_k = k^2$, $n = 1$ to 7. The unique positive real root $\alpha_n$ of each $p_n(x)$ is strictly increasing and converges to the exact limit $\alpha \approx 3.618034$ (blue dashed line), the largest real root of the cubic equation $\alpha^3 - 4\alpha^2 + 2\alpha - 1 = 0$.

**Graphs of $p_n(x)$ for the exponential sequence $a_k = e^k$**

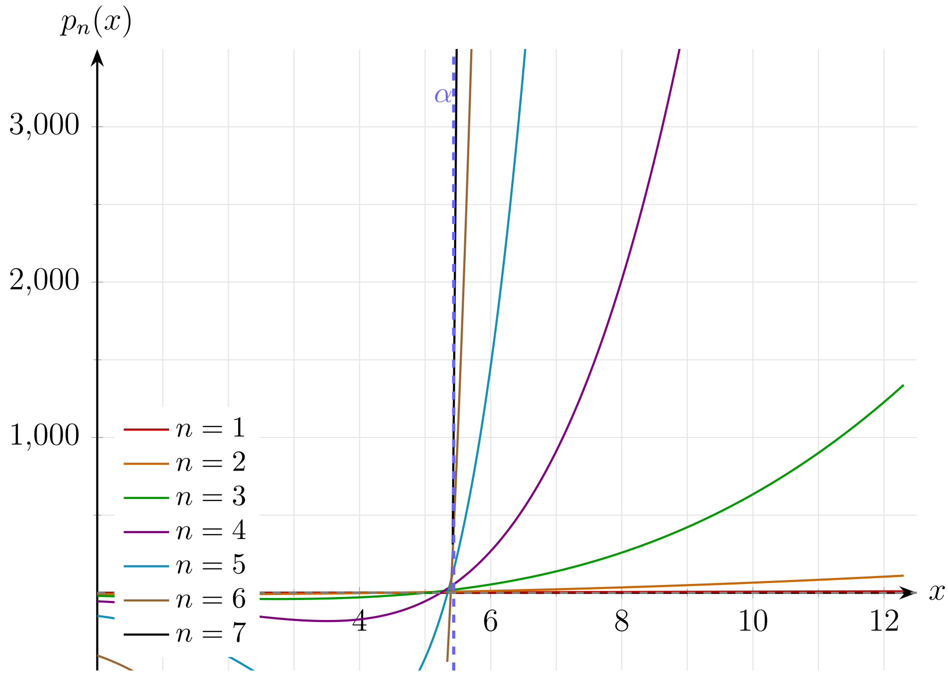


**Figure 4.** Graphs of the characteristic polynomials $p_n(x)$ for the exponential coefficients $a_k = e^k$, $n = 1$ to 7. The unique positive real root $\alpha_n$ is strictly increasing and converges to the exact limit $\alpha \approx 5.436564$. Theorem 1 delivers the explicit upper bound $\mu(n)$ that approaches the constant value $1 + 2e \approx 6.436564$ as $n$ increases, remaining only about $18\%$ larger than the true limit while being vastly superior to Cauchy's bound $R = n$.

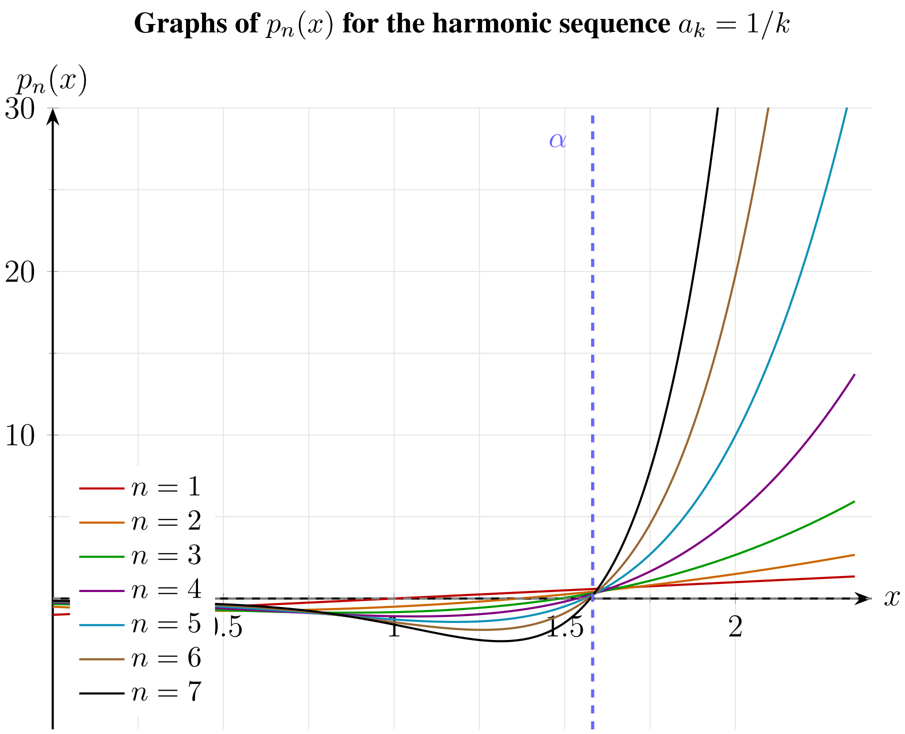


**Figure 5.** Graphs of the characteristic polynomials $p_n(x)$ for the harmonic coefficients $a_k = 1/k$, $n = 1$ to 7. The unique positive real root $\alpha_n$ of each $p_n(x)$ is strictly increasing and converges to the exact value $\alpha = \frac{e}{e-1} \approx 1.581977$ (blue dashed line). Our bound from Theorem 1 yields the constant value $\mu(n) = 2 + \sqrt{1/2} \approx 2.707107$ for all $n \geq 2$, which is approximately $71\%$ larger than the true limit yet remains far superior to Cauchy's bound $R = n$ (reaching 7 here).

**Table 1.** Positive real roots $\alpha_n$ of $p_n(x) = 0$ (rounded to 6 decimal places) for five coefficient sequences.

| $n$ | $a_k = e^k$ | $a_k = 1$ | $a_k = k$ | $a_k = 1/k$ | $a_k = k^2$ |
|---|---|---|---|---|---|
| $\alpha_1$ | 2.718282 | 1.000000 | 1.000000 | 1.000000 | 1.000000 |
| $\alpha_2$ | 4.398272 | 1.618034 | 2.000000 | 1.366025 | 2.561553 |
| $\alpha_3$ | 4.999700 | 1.839287 | 2.374424 | 1.486998 | 3.163615 |
| $\alpha_4$ | 5.239657 | 1.927562 | 2.517996 | 1.535865 | 3.382553 |
| $\alpha_5$ | 5.344006 | 1.965948 | 2.576020 | 1.558186 | 3.463715 |
| $\alpha_6$ | 5.391943 | 1.983583 | 2.600239 | 1.569203 | 3.494009 |
| $\alpha_7$ | 5.414720 | 1.991964 | 2.610492 | 1.574928 | 3.505226 |
| $\alpha_8$ | 5.425785 | 1.996032 | 2.614847 | 1.578010 | 3.509312 |
| $\alpha_9$ | 5.431208 | 1.998027 | 2.616695 | 1.579712 | 3.510771 |
| $\alpha_{10}$ | 5.433902 | 1.999013 | 2.617476 | 1.580669 | 3.511282 |
| $\alpha$ | $\approx 5.436564$ | 2.000000 | $\approx 1.581976$ | $\approx 1.644934$ | $\approx 3.618034$ |
| $\mu$ | 5.879479 | 3.000000 | 3.000000 | 2.707107 | 3.732051 |

As we can observe from all the above examples, $\lim_{n\to\infty} \sqrt[n]{a(n)} < \infty$ holds true.

## V. CONCLUSION

This paper has introduced a new family of explicit upper bounds for the moduli of all roots of characteristic polynomials generated by arbitrary positive real sequences $\{a_k\}$ satisfying the mild growth condition $\limsup a_k^{1/k} < \infty$. The proposed bound $\mu(n)$ is obtained by adding the two largest elements of the easily computable set

$$\left\{|a_i - a_{i+1}|^{1/(i+1)} : 1 \leq i \leq n-1\right\} \cup \{a_1 + 1\}.$$

Numerical evidence on constant, arithmetic, harmonic, quadratic and exponential sequences shows that $\mu(n)$ is typically several times sharper than the classical Cauchy bound and, in many cases, also outperforms recent refinements. A companion bound based on second differences is derived as an immediate corollary. The proofs rely only on elementary manipulations of the classical Cauchy and Lagrange bounds together with a careful limiting analysis.

Future work may explore extensions to matrix polynomials, complex coefficients, or stochastic sequences, as well as the development of adaptive strategies that automatically select the tightest available bound for a given sequence.

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