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On the necessary and sufficient condition of a k-Euler pair

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Abstract. In this paper, we discuss George Andrews' definition of an Euler pair and Subbarao's generalization of the Euler pair to a k-Euler pair. Let N and M be non-empty sets of natural numbers. A pair (N, M) is called a k-Euler pair if, for any natural number n, the number of partitions of n into parts from N is equal to the number of partitions of n into parts from M, with the condition that each part appears fewer than k times. We further explore several theorems concerning Euler pairs that were established by Andrews and Subbarao, and we present proofs using a method distinct from those previously utilized.

Keywords: Partitions, Number Theory, Euler pair.

1 INTRODUCTION

In the theory of partitions, a well-known identity, known as Euler's partition theorem, states that the number of partitions of an integer n into odd parts is equal to the number of partitions of n into distinct parts. Inspired by this identity, Andrews [2] introduced the concept of an Euler pair, defined as follows:

Definition 1.1. *Let N and M be any subsets of all natural number set $\mathbb{N}$. A pair (N, M) is called an Euler pair if for any $n \in \mathbb{N}$ it satisfies*

$$p\left(n \mid \textit{parts in } N\right) = p\left(n \mid \textit{distinct parts in } M\right).$$

For example, by Euler's partition theorem, we know that for any $n \in \mathbb{N}$, the following equality holds:

$$p\left(n \mid \text{parts in } \{1, 3, 5, 7, 9, \dots\}\right) = p\left(n \mid \text{distinct parts in } \mathbb{N}\right),$$

which implies that the pair $(\{1, 3, 5, 7, 9, \dots\}, \mathbb{N})$ is an Euler pair.

Several related studies have been conducted on Euler's pair. For instance , Subbarao [10] presents two additional examples of non-trivial Euler pairs:

$$\left(\{n \in \mathbb{N} \mid n \equiv 1, 5 \pmod 6\}, \{n \in \mathbb{N} \mid n \not\equiv 0 \pmod 3\}\right)$$

and

$$\left(\{n \in \mathbb{N} \mid n \equiv 2, 5, 11 \pmod{12}\}, \{n \in \mathbb{N} \mid n \equiv 2, 4, 5 \pmod 6\}\right).$$

Additionally, Subbarao's paper [10] was the first to introduce the concept of Euler pair of order r, which is equivalent to the definition of k-Euler pair that will be defined later in this paper.

For another example, Merca [9] provides a combinatorial interpretation on the Euler pair (N, M), where $N = \{n \in \mathbb{N} \mid n \equiv \pm 1 \pmod 6\}$ and $M = \{n \in \mathbb{N} \mid n \not\equiv 0 \pmod 3\}$.

There are also a lot of studies that doesn't directly dwell on the Euler pair itself but more on some topics that are developed from and/or inspired by Euler pair. For example, on Euler's partition identities (see [1] and [3]) and on Beck-type identities (see [4], [5], [6], and [8]).

Furthermore, Andrews [2] discovered the following theorem:

Theorem 1.1. *[2] If N is any set of integers such that no element of N is a power of two times an element of N, and M is the set containing all elements of N together with all their multiples of powers of two, then (N, M) is an Euler pair.*

Theorem 1.1 was proven by Andrews [2] through a technique known as merging. This technique involves, for any partition of n, the repeated combination of two identical summands until all summands are distinct. Andrews then demonstrated the existence of a bijection between the set of all partitions of n with parts in N and the set of all partitions of n with distinct parts in M.

We can now strengthen Theorem 1.1 into the following result.

Theorem 1.2. *The pair of two sets (N, M) is an Euler pair, if and only if N is a set of integers such that no element of N is a power of two times another element of N and M is the set containing all elements of N together with all their multiples of powers of two.*

Alternatively, the pair of sets (N, M) is called an Euler pair if and only if

$$2M \subset M \quad \textit{and} \quad N = M \setminus 2M.$$

Theorem 1.2 is derived from two exercises presented in Andrews' book [2], although no proof is provided there. Consequently, one of the objectives of this paper is to present a detailed proof of the theorem.

In another book, Berndt [7] discusses a generalization of Euler's partition theorem as follows:

Theorem 1.3. *Let k be an integer where $k \geq 2$, then for any positive integer n we have that*

$$p\left(n \mid \textit{parts not divisible by } k\right) = p\left(n \mid \textit{no parts appears} \geq k \textit{ times}\right).$$

The identity mentioned above is presented as an exercise in Berndt's book [7]. Therefore, the proof of this theorem will also be included in this paper.

Motivated by the construction of Euler pairs from Euler's partition theorem, we propose a generalization of Euler pairs based on Theorem 1.3, which extends Euler's partition theorem. Indeed, Subbarao [10] provides the following definition.

Definition 1.2. *Let N and M be any subsets of $\mathbb{N}$, a pair (N, M) is called an Euler pair of order k if for any $n \in \mathbb{N}$ it satisfies*

$$p\left(n \mid \textit{parts in } N\right) = p\left(n \mid \textit{parts in } M, \textit{ appears} < k \textit{ times}\right).$$

From this definition, we observe that when $k = 2$, the result is the standard Euler pair. For simplicity, we will refer to an Euler pair of order k as a k-Euler pair.

In his paper, Subbarao [10] presents the necessary and sufficient conditions for a k-Euler pair, which generalize Theorem 1.2, as follows.

Theorem 1.4. *[10] A pair of sets (N, M) is a k-Euler pair if and only if*

$$kM \subset M \quad \text{and} \quad N = M - kM.$$

Alternatively, (N, M) forms a k-Euler pair if and only if N is a set of integers such that no element of N is a multiple of k times another element of N, and

$$M = \left\{k^x \cdot n \mid n \in N \text{ and } x \in \mathbb{Z}^{\geq 0}\right\},$$

where $\mathbb{Z}^{\geq 0}$ denotes the set of all non-negative integers.

In this paper, we aim to prove the above theorem using a method different from that employed by Subbarao.

In the next section, we will prove several lemmas that are essential for establishing the main results, provide the proofs of Theorems 1.2 and 1.3, and provide alternative proof of 1.4.

2 Results and Discussions

2.1. Some premilinary lemmas

To prove Theorem 1.2, we first present some preliminary lemmas.

Lemma 2.1. *For any given set N, there can be at most one set M such that (N, M) is an Euler pair.*

Proof. For any given set N, suppose that there exist two different sets M_a and M_b such that (N, M_a) and (N, M_b) are both Euler pairs. Then, necesarilly

$$p\left(n \mid \text{parts in } N\right) = p\left(n \mid \text{distinct parts in } M_a\right) = p\left(n \mid \text{distinct parts in } M_b\right).$$

More specifically,

$$p\left(n \mid \text{distinct parts in } M_a\right) = p\left(n \mid \text{distinct parts in } M_b\right). \tag{1}$$

Let x be the smallest number that belongs to one set but not the other. In other words, x is the smallest number that satisfies either $x \in M_a \cap M_b^c$ or $x \in M_a^c \cap M_b$. Without loss of generality, suppose that $x \in M_a \cap M_b^c$.

Now, we define the set

$$M_0 = \{m \mid m \in M_a \cap M_b \text{ and } m < x\}.$$

It is straightforward to verify that $M_0 \subseteq M_a$ and $M_0 \subseteq M_b$. Moreover, we have $M_0 \cup \{x\} \subseteq M_a$, but $M_0 \cup \{x\} \nsubseteq M_b$.

We observe that

$$\begin{aligned} & p\left(x \mid \text{distinct parts in } M_a\right) \\ & \quad = p\left(x \mid \text{parts in } M_a, \text{ distinct and at most } x\right) \\ & \quad = p\left(x \mid \text{distinct parts in } M_0 \cup \{x\}\right) \\ & \quad = p\left(x \mid \text{distinct parts in } M_0\right) + \underbrace{1}_{\text{partitioning } x \text{ as itself}} . \end{aligned}$$

On the other hand

$$\begin{aligned} & p\left(x \mid \text{distinct parts in } M_b\right) \\ & \quad = p\left(x \mid \text{parts in } M_b, \text{ distinct and at most } x\right) \\ & \quad = p\left(x \mid \text{distinct parts in } M_0\right) . \end{aligned}$$

Thus we have

$$p\left(n \mid \text{distinct parts in } M_a\right) = p\left(n \mid \text{distinct parts in } M_b\right) + 1 \tag{2}$$

Since Equations (1) and (2) contradict each other, it follows that our initial assumption was incorrect. Therefore, for any given set N, there can be at most one set M such that (N, M) forms an Euler pair. □

Next, we have the following lemma.

Lemma 2.2. *For any given set N which contains an element of the form $2^k \cdot a$ such that $a \in N$ and $k > 0$, there exists no set M such that (N, M) is an Euler pair.*

Proof. For any given set N that contains elements in which one is a power of two times another, let $2^k \cdot a$ be the smallest element in N that is a power of two times some other element in N (in this case, a). Suppose that there exists a set M such that the pair (N, M) forms an Euler pair.

Now, we construct a set

$$N_0 = \left\{n \,|n \in N \text{ and } n < 2^k \cdot a\right\} .$$

Since $2^k \cdot a$ is the smallest element in N that is a power of two multiplied by some other element in N, no element of N_0 is a power of two times an element of N_0. By applying Theorem 1.1, we can construct an Euler pair (N_0, M_0) where

$$M_0 = \left\{2^x \cdot n \mid n \in N_0 \text{ and } x \in \mathbb{Z}^{\geq 0}\right\} .$$

From Lemma 2.1, we know that there is at most one way to construct a set M for any given set N such that (N, M) forms an Euler pair. Since $N_0 \subseteq N$, it must follow that $M_0 \subseteq M$. Additionally, from the definition of M_0, it is clear that $2^k \cdot a \in M_0$, which implies that $2^k \cdot a \in M$.

Now, let m be any natural number such that $m < 2^k \cdot a$ and $m \notin M_0$. Suppose that $m \in M$. Since $M_0 \subseteq M$ and $m \notin M_0$ but $m \in M$, it follows that the following condition must hold:

$$p\left(m \mid \text{distinct parts in } M\right) \geq p\left(m \mid \text{distinct parts in } M_0\right) + 1.$$

However, we know that

$$\begin{aligned} & p\left(m \mid \text{distinct parts in } M\right) \\ & \quad = p\left(m \mid \text{parts in } N\right) \\ & \quad = p\left(m \mid \text{parts in } N \text{ at most } m\right) \\ & \quad = p\left(m \mid \text{parts in } N \text{ less than } 2^k \cdot a\right) \\ & \quad = p\left(m \mid \text{parts in } N_0\right) \\ & \quad = p\left(m \mid \text{distinct parts in } M_0\right). \end{aligned}$$

We arrive at a contradiction, which means that our assumption was incorrect. Therefore, for any natural number $m < 2^k \cdot a$, if $m \notin M_0$, then $m \notin M$. More specifically, for any natural number $m < 2^k \cdot a$, we have $m \in M$ if and only if $m \in M_0$.

Because $2^k \cdot a \in M_0$ and $2^k \cdot a \in M$, and for any natural number $m < 2^k \cdot a$, $m \in M$ if and only if $m \in M_0$, we can construct the set

$$M_1 = \left\{m \mid m \in M \text{ and } m \leq 2^k \cdot a\right\} = \left\{m \mid m \in M_0 \text{ and } m \leq 2^k \cdot a\right\}.$$

Since (N, M) and (N_0, M_0) are Euler pairs, then we have

$$\begin{aligned} & p\left(2^k \cdot a \mid \text{parts in } N\right) \\ & \quad = p\left(2^k \cdot a \mid \text{distinct parts in } M\right) \\ & \quad = p\left(2^k \cdot a \mid \text{parts in } M \text{ distinct and at most } 2^k \cdot a\right) \\ & \quad = p\left(2^k \cdot a \mid \text{distinct parts in } M_1\right) \\ & \quad = p\left(2^k \cdot a \mid \text{parts in } M_0 \text{ distinct and at most } 2^k \cdot a\right) \\ & \quad = p\left(2^k \cdot a \mid \text{distinct parts in } M_0\right) \\ & \quad = p\left(2^k \cdot a \mid \text{parts in } N_0\right). \end{aligned}$$

However, we can see that

$$\begin{aligned} & p\left(2^k \cdot a \mid \text{parts in } N\right) \\ & \quad = p\left(2^k \cdot a \mid \text{parts in } N \text{ that is at most } 2^k \cdot a\right) \\ & \quad = p\left(2^k \cdot a \mid \text{parts in } N_0 \cup \left\{2^k \cdot a\right\}\right) \\ & \quad = p\left(2^k \cdot a \mid \text{parts in } N_0\right) + \underbrace{1}_{\text{partitioning } 2^k \cdot a \text{ as } 2^k \cdot a \text{ itself}}. \end{aligned}$$

We reach a contradiction, which means that our initial assumption was incorrect. Thus, for any given set N, where one element is a power of two times another element, no set M can exist such that (N, M) forms an Euler pair.

□

2.2. Proof of Theorem 1.2

Using the results of the previous subsection, we present the proof of Theorem 1.2.

($\Leftarrow$) Let (N, M) be a pair where N is a set of integers such that no element in N is a power of two times another element in N. The set M consists of all elements of N along with their multiples by powers of two. By Theorem 1.1, it is evident that (N, M) forms an Euler pair.

($\Rightarrow$) Since (N, M) is an Euler pair, by applying Lemma 2.2, we conclude that N must be a set of integers such that no element of N is a power of two times another element in N.

Next, we construct a set M' such that M' contains all elements of N along with all of their multiples by powers of two. According to Theorem 1.1, the pair (N, M') is an Euler pair.

On the other hand, from Lemma 2.1, we know that there can be at most one set M such that (N, M) is an Euler pair. Therefore, since both (N, M) and (N, M') are Euler pairs, it follows that $M = M'$. Hence, M is the set containing all elements of N along with all their multiples of powers of two.

Thus, we define (N, M) as an Euler pair if and only if N is a set such that for any $n \in N$ and for any $k \in \mathbb{N}$, we have $2^k \cdot n \notin N$. The set M is defined as

$$M = \left\{ 2^x \cdot n \,\middle|\, n \in N \text{ and } x \in \mathbb{Z}^{\geq 0} \right\}.$$

Therefore,

$$2M = \left\{ 2^x \cdot n \,\middle|\, n \in N \text{ and } x \in \mathbb{N} \right\}.$$

For any element $2^x \cdot n \in 2M$, we have $n \in N$ and $x \in \mathbb{N}$. Since $\mathbb{N} \subset \mathbb{Z}^{\geq 0}$, it follows that $x \in \mathbb{Z}^{\geq 0}$. Therefore, $2^x \cdot n \in M$, or equivalently, $2M \subseteq M$.

For any element $n \in N$, since $n = 2^0 \cdot n$, it follows that $n \in M$ but $n \notin 2M$. Hence, $n \in M - 2M$, which implies that $N \subseteq M - 2M$.

Next, we pick any element $2^x \cdot n \in M \setminus 2M$. In other words, $2^x \cdot n \in M$ but $2^x \cdot n \notin 2M$. Since $2^x \cdot n \in M$, it follows that $n \in N$ and $x \in \mathbb{Z}^{\geq 0}$. Additionally, because $2^x \cdot n \notin 2M$, we have either $n \notin N$ or $x \notin \mathbb{N}$. From this, we can conclude that $n \in N$ and $x \in \mathbb{Z}^{\geq 0} \setminus \mathbb{N}$. Therefore, $x = 0$, which implies that $2^x \cdot n = n \in N$, or simply $M \setminus 2M \subset N$.

We can conclude that $N = 2M - M$, thus the proof of Theorem 1.2 is complete. □

2.3. Proof of Theorem 1.3

Now we present the proof of Theorem 1.3.

Let $A(n) = p(n \mid \text{parts not divisible by } k)$, the generating function of $A(n)$ is,

$$\sum_{n=0}^{\infty} A(n) \cdot q^n = \prod_{j \not\equiv 0 \pmod{k}}^{j \geq 1} \frac{1}{(1-q^j)}.$$

Let $B(n) = p(n \mid \text{no parts appears} \geq k \text{ times})$, the generating function of $B(n)$ is,

$$\sum_{n=0}^{\infty} B(n) \cdot q^n = \prod_{j=1}^{\infty} \left(1 + q^j + q^{2j} + \cdots + q^{(k-1)j}\right).$$

Since

$$\begin{aligned}
\sum_{n=0}^{\infty} A(n) \cdot q^n &= \prod_{j \not\equiv 0 \pmod{k}}^{j \geq 1} \frac{1}{(1-q^j)} \\
&= \prod_{j=1}^{\infty} \frac{(1-q^{kj})}{(1-q^j)} \\
&= \prod_{j=1}^{\infty} \left(1 + q^j + q^{2j} + \cdots + q^{(k-1)j}\right) \\
&= \sum_{n=0}^{\infty} B(n) \cdot q^n,
\end{aligned}$$

then $A(n) = B(n)$, which proves Theorem 1.3. □

2.4. Some properties for k-Euler pair

Since a k-Euler pair is constructed as a generalization of the Euler pair, one may naturally ask whether Theorem 1.1 and Theorem 1.2 can also be generalized. As it turns out, Theorem 1.1 can indeed be generalized, as shown in the following theorem.

Theorem 2.1. *If N is any set of integers such that no element of N is a power of k times an element of N, and M is the set containing all elements of N together with all their multiples of powers of k, then (N, M) is a k-Euler pair.*

Proof. Suppose $N = \{n_1, n_2, n_3, \cdots\}$, where $n_i \neq (k)^h \cdot n_j$ for any natural number h, i, and j, while $M = \{k^h \cdot n_j \,|n_j \in N \text{ and } h \in \mathbb{Z}^{\geq 0}\}$.

For any partition with parts in N, we do the following procedure:

1. If in that partition, any number appears k times, then add the k of them together into one.
2. After step 1, see if there are still any numbers that appears k times. If there are, we repeat the process.
3. Do this until there are no longer any number that appears at least k times.

Since this procedure will not terminate until the resulting partition has no number that appears at least k times, the final result must contain no parts that appear more than k times. Additionally, since we start with a partition that only uses elements of N as its parts, the resulting partition will consist solely of parts that are elements of M.

Since $N = \{n_1, n_2, n_3, \cdots\}$, where $n_i \neq k^h \cdot n_j$ for any natural numbers h, i, and j, any two distinct partitions with parts in N will result in two different outcomes after applying the aforementioned procedure.

Thus, this merging procedure defines an injection from the set of all partitions of n with parts in N into the set of all partitions of n with parts in M, where no part appears at least k times. Additionally, we can reverse the procedure: for any partition with parts in M where no part appears more than k times, the following steps can be applied:

1. If in that partition, there are numbers that is not an element of N, for example in the form of $k^h \cdot n_j$ where $n_j \in N$, then divide them into k^h equal parts.
2. Do this until all the numbers that appears is an element of N.

It is evident that the final result of this procedure is a partition with parts in $\mathbb{N}$.

Using the fact that any natural number can be uniquely represented as the sum of powers of k, where no number appears at least k times, it follows that any two different partitions with parts in M, where no part appears at least k times, will yield two distinct results after applying the procedure described above.

Thus, there exists an injection from the set of all partitions of n with parts in M, where no part appears at least k times, into the set of all partitions of n with parts in N.

We can conclude that a bijection exists, which means that the number of partitions of n with parts in M and no part appearing at least k times is the same as the number of partitions of n with parts in N.

Thus, $p\left(n \mid \text{parts in } N\right) = p\left(n \mid \text{parts in } M, \text{ appears less than } k \text{ times}\right)$. In other words, (N, M) is a k-Euler pair.

□

Further investigation reveals that Lemmas 2.1 and 2.2 can also be generalized for k-Euler pairs. We begin by presenting a generalization of Lemma 2.1 as follows.

Lemma 2.3. *For any given set N, there can be at most one set M such that (N, M) is a k-Euler pair.*

Proof. For any given set N, suppose there exist two distinct sets M_a and M_b such that both pairs (N, M_a) and (N, M_b) are Euler pairs. Necessarily,

$$\begin{aligned} &p\left(x \mid \text{parts in } M_a, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(x \mid \text{parts in } M_b, \text{ appears less than } k \text{ times}\right). \end{aligned} \tag{3}$$

Let x be the smallest number that belongs to one set but not the other. Specifically, x is the smallest number such that $x \in M_a \cap M_b^c$ or $x \in M_a^c \cap M_b$. Without loss of generality, assume that $x \in M_a \cap M_b^c$.

Now, we construct a set

$$M_0 = \{m \,|m \in M_a \cap M_b \text{ and } m < x\}.$$

Then $M_0 \subseteq M_a$ and $M_0 \subseteq M_b$, while $M_0 \cup \{x\} \subseteq M_a$ but $M_0 \cup \{x\} \nsubseteq M_b$.

We observe that

$$\begin{aligned} &p\left(x \mid \text{parts in } M_a, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(x \mid \text{parts in } M_a \text{ that is at most } x, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(x \mid \text{parts in } M_0 \cup \{x\}, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(x \mid \text{parts in } M_0, \text{ appears less than } k \text{ times}\right) + \underbrace{1}_{\text{partitioning } x \text{ as itself}}. \end{aligned}$$

On the other hand

$$\begin{aligned} &p\left(x \mid \text{parts in } M_b, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(x \mid \text{parts in } M_b \text{ that is at most } x, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(x \mid \text{parts in } M_0, \text{ appears less than } k \text{ times}\right). \end{aligned}$$

Thus we have

$$\begin{aligned} &p\left(x \mid \text{parts in } M_a, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(x \mid \text{parts in } M_b, \text{ appears less than } k \text{ times}\right) + 1. \end{aligned} \tag{4}$$

Since Equations (3) and (4) contradict each other, this implies that our initial assumption was incorrect. Therefore, for any given set N, there can be only one set M such that (N, M) forms a k-Euler pair.

□

Next, we have a generalization of Lemma 2.2 as follows.

Lemma 2.4. *For any given set N, if there exists a natural number p such that $k^p \cdot a \in N$ for some $a \in N$, then there is no set M such that (N, M) is a k-Euler pair.*

Proof. Suppose that $k^p \cdot a$ is the smallest element in N that is a power of k multiplied by some other element in N (in this case, a). Furthermore, suppose that there exists a set M such that (N, M) forms a k-Euler pair.

Now, we construct a set

$$N_0 = \{n \,|n \in N \text{ and } n < k^p \cdot a\}.$$

Since $k^p \cdot a$ is the smallest element in N that is a power of k multiplied by some element of N, then there is no element of N_0 that can be written as a power of k times an element of N_0. By applying Theorem 2.1, we can construct a k-Euler pair (N_0, M_0), where

$$M_0 = \left\{k^x \cdot n \,|n \in N_0 \text{ and } x \in \mathbb{Z}^{\geq 0}\right\}.$$

From Lemma 2.3, we know that there is at most one way to construct a set M for any given set N such that (N, M) forms a k-Euler pair. Since $N_0 \subseteq N$, it must follow that $M_0 \subseteq M$. Furthermore, from the definition of M_0, it is clear that $k^p \cdot a \in M_0$, which implies $k^p \cdot a \in M$.

Now, let m be any natural number such that $m < k^p \cdot a$ and $m \notin M_0$. Suppose that $m \in M$. Since $M_0 \subseteq M$ and $m \notin M_0$, but $m \in M$, the following condition must be satisfied:

$$\begin{aligned} &p\left(m \mid \text{parts in } M, \text{ appears less than } k \text{ times}\right) \\ &\quad \geq p\left(m \mid \text{parts in } M_0, \text{ appears least than } k \text{ times}\right) + 1. \end{aligned} \tag{5}$$

However, we know that

$$\begin{aligned} &p\left(m \mid \text{parts in } M, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(m \mid \text{parts in } N\right) \\ &\quad = p\left(m \mid \text{parts in } N \text{ that is at most } m\right) \\ &\quad = p\left(m \mid \text{parts in } N \text{ that is at most } k^p \cdot a\right) \\ &\quad = p\left(m \mid \text{parts in } N_0\right) \\ &\quad = p\left(m \mid \text{parts in } M_0, \text{ appears less than } k \text{ times}\right). \end{aligned} \tag{6}$$

From Equation (5) and (6), we have a contradiction, which means that our assumption was incorrect. Therefore, for any natural number $m < k^p \cdot a$, if $m \notin M_0$, then $m \notin M$. More specifically, for any natural number $m < k^p \cdot a$, we have $m \in M$ if and only if $m \in M_0$.

As $k^p \cdot a \in M_0$ and $k^p \cdot a \in M$, and for any natural number $m < k^p \cdot a$, $m \in M$ if and only if $m \in M_0$, we construct a set

$$M_1 = \{m \,|m \in M \text{ and } m \leq k^p \cdot a\} = \{m \,|m \in M_0 \text{ and } m \leq k^p \cdot a\}.$$

Since (N, M) and (N_0, M_0) are k-Euler pairs, then we have

$$\begin{aligned} &p\left(k^p \cdot a \mid \text{parts in } N\right) \\ &\quad = p\left(k^p \cdot a \mid \text{parts in } M, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(k^p \cdot a \mid \text{parts in } M \text{ that is at most } k^p \cdot a, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(k^p \cdot a \mid \text{parts in } M_1, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(k^p \cdot a \mid \text{parts in } M_0 \text{ that is at most } k^p \cdot a, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(k^p \cdot a \mid \text{parts in } M_0, \text{ appears less than } k \text{ times}\right) \\ &\quad = p\left(k^p \cdot a \mid \text{parts in } N_0\right). \end{aligned} \tag{7}$$

However,

$$\begin{aligned} & p\left(k^p \cdot a \mid \text{parts in } N\right) \\ & \quad = p\left(k^p \cdot a \mid \text{parts in } N \text{ that is at most } k^p \cdot a\right) \\ & \quad = p\left(k^p \cdot a \mid \text{parts in } N_0 \cup \{k^p \cdot a\}\right) \\ & \quad = p\left(k^p \cdot a \mid \text{parts in } N_0\right) + \underbrace{1}_{\text{partitioning } k^p \cdot a \text{ as itself}} . \end{aligned} \tag{8}$$

From Equation (7) and (8), we have a contradiction, which means that our initial assumption was incorrect. Thus, for any given set N, where some element is a power of k times another element, there cannot exist a set M such that (N, M) forms a k-Euler pair.

□

2.5. Alternative proof of Theorem 1.4

Using Lemma 2.3 and 2.4, we can now prove Theorem 1.4.

($\Leftarrow$) Let (N, M) be a pair where N is a set of integers such that no element of N is a multiple of any other element of N by a power of k, and M is the set consisting of all elements of N along with all their multiples by powers of k. By applying Theorem 2.1, it follows that (N, M) forms a k-Euler pair.

($\Rightarrow$) Since (N, M) is a k-Euler pair, by using Lemma 2.4, we can conclude that N must be a set of integers such that no element of N is a power of k times another element of N.

Next, we construct a set M', where

$$M = \left\{k^x \cdot n \,|n \in N \text{ and } x \in \mathbb{Z}^{\geq 0}\right\}.$$

According to Theorem 2.1, (N, M') is a k-Euler pair.

On the other hand, from Lemma 2.3 we know that there can be at most one set M such that (N, M) is a k-Euler pair. Therefore, since (N, M) and (N, M') are both k-Euler pair, then $M = M'$. Thus, we have that (N, M) is a k-Euler pair if and only if $k^x \cdot n \notin N$ for any $n \in N$ and $x \in \mathbb{N}$ and $M = \{k^x \cdot n \,|n \in N \text{ and } x \in \mathbb{Z}^{\geq 0}\}$.

Therefore, if (N, M) is a k-Euler pair, then $kM = \{k^x \cdot n \,|n \in N \text{ and } x \in \mathbb{N}\}$. For any element $k^x \cdot n \in kM$, it satisfies $n \in N$ and $x \in \mathbb{N} \subset \mathbb{Z}^{\geq 0}$. Thus, $k^x \cdot n \in M$ or in other word $kM \subset M$.

For any element $n \in N$, since $n = k^0 \cdot n$ then $n \in M$ but $n \notin kM$. Thus, $n \in M - kM$, that is to say $N \subset M - kM$. Next, we pick any element $k^x \cdot n \in M - kM$, or in other word, $k^x \cdot n \in M$ but $k^x \cdot n \notin M$. Since $k^x \cdot n \in M$, then $n \in N$ and $x \in \mathbb{Z}^{\geq 0}$. And since

$k^x \cdot n \notin kM$, then $n \notin N$ or $x \notin \mathbb{N}$. Therefore, we can conclude that $n \in N$ and $x \in \mathbb{Z}^{\geq 0} \setminus \mathbb{N}$, thus $x = 0$. It means that $k^x \cdot n = n \in N$, and hence $M - kM \subset N$. We conclude that $N = M - kM$, which completes the proof of Theorem 1.4. □

2.6. Differences with Subbarao's proof of Theorem 1.4

Before we compare our new proofs to the existing proof, we will summarize how Subbarao proved Theorem 1.4.

First, Subbarao defined $p(N; n)$ as the number of partitions of n into parts taken from N, while $q_k(M; n)$ as the numbers of partitions of n into parts taken from M with no part repeated more than $(k-1)$ times.

Next, Subbarao calculate the generating function of $p(N; n)$ and $q_k(M; n)$, which is given by

$$\sum_{n=0}^{\infty} q_k(M; n) \cdot x^n = \prod_{a \in M} \frac{1 - x^{ra}}{1 - x^a},$$

$$\sum_{n=0}^{\infty} p(N; n) \cdot x^n = \prod_{b \in N} \frac{1}{1 - x^b}.$$

Thus by Definition 1.2, a pair (N, M) is called an Euler pair of order k if it satisfies

$$\prod_{a \in M} \frac{1 - x^{ra}}{1 - x^a} = \prod_{b \in N} \frac{1}{1 - x^b}$$

Then, Subbarao proved that $\prod_{a \in M} \frac{1-x^{ra}}{1-x^a} = \prod_{b \in N} \frac{1}{1-x^b}$ holds if and only if $kM \subset M$ and $N = M - kM$.

From this we can see that Subbarao's proof mainly used algebraic manipulation with the help of generating functions of the partitions.

On the other hand our proof analyze the relation of the two partitions directly using the help of set theory.

3 CONCLUSIONS AND FUTURE RESEARCH DIRECTION

In this paper, we have established alternative proof for necessary and sufficient conditions for a k-Euler pair, generalizing the concept of Euler pairs introduced by Andrews and expanded upon by Subbarao. Future research could further investigate generalized Euler pairs with additional constraints or explore applications of k-Euler pairs in related combinatorial problems. Additionally, exploring the implications of these results in modular forms or partition identities could be a promising avenue for advancing this field.

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