JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 8 NO. 1 (2025)
Available online at www.jfma.math.fsm.undip.ac.id

ON A HIGHLY ROTUND NORM AND UNIFORMLY ROTUND NORM IN EVERY DIRECTION ON A FRECH'ET SPACE

Amos Otieno Wanjara

Kaimosi Friends University, Kenya.
School of Science, Department of Mathematics and Statistics
Email: awanjara@kafu.ac.ke

Abstract. The word rotund comes from Latin word "rotundus" implying wheel-shaped or round (from rota wheel). Rotundity is the roundness of a 3-dimensional object. Some of the properties of rotundity include: UR-Uniformly Rotund, LUR-Locally Uniformly Rotund, MLUR-Midpoint Locally Uniformly Rotund, WUR-Weakly Uniformly Rotund, URED-Uniformly Rotund in Every Direction, HR- Highly Rotund, WLUR-Weakly Locally Uniformly Rotund and URWC-Uniformly Rotund in Weakly Compact sets of directions. Problems on Rotundity properties are still open. Smith gave a summary chart on rotundity of norms in Banach spaces. The chart left an open question whether or not a Highly Rotund norm(HR) implies Uniformly Rotund norm on Every Direction(URED). It is not clear whether if a Banach space has a Highly Rotund(HR) norm it follows that it has and equivalently URED. In this paper, we investigated the relationship between a Highly Rotund norm(HR) and a Uniformly Rotund norm in Every Direction(URED) on a Freche't Space. The result shows that both Highly Rotund norm and Uniformly Rotund norm on Every Direction(URED) exist in Freche't spaces. The implication of this result is that rotundity properties can be extended within spaces. This research work is very important since rotundity properties are strongly applicable in many branches of mathematics.

Keywords: Rotund, Norm, Highly Rotund, Uniformly Rotund in Every Direction, Freche't Space

I. INTRODUCTION

A rotunda is a circular building with a dome over it. The rotundity or roundness of its structure is what gives it its name. The study of rotundity of the unit ball in Banach space has been of interest to many mathematicians. Any 3-dimensional object that has a roundness property to it can be described in terms of its rotundity. Some of the properties of rotundity include: Uniform Rotund(UR), Locally Uniformly Rotund(LUR), Midpoint Locally Uniformly Rotund(MLUR), Weakly Uniformly Rotund(WUR), Uniformly Rotund in Every Direction(URED), Highly Rotund(HR), Weakly Locally Uniformly Rotund(WLUR) and Uniformly Rotund in Weakly Compact sets of directions(URWC) among others. In 1936, Clarckson [1] introduced the stronger notion of uniform rotundity of the norm in a Banach space. He proved that for $p > 1$, the space $\boldsymbol{L^p}$and $\boldsymbol{l^p}$ are uniformly rotund. According to Smith in [2], a Banach space B is uniformly rotund in weakly compact sets of directions(URWC) if whenever Z is in B and $\{\boldsymbol{x_n}\}$ and $\{\boldsymbol{y_n}\}$ are sequences in B such that $\| \boldsymbol{x_n} \| \to 1$, $\| \boldsymbol{y_n} \| \to 1$, $\| \boldsymbol{x_n} + \boldsymbol{y_n} \| \to 2$ and $\boldsymbol{x_n} - \boldsymbol{y_n} \to z$ weakly, then $z = 0$. Garkavi in [3] brought the idea of uniformly rotund in every direction. A Banach space B is uniformly rotund in every direction(URED) if whenever non zero z is in B and $\{\boldsymbol{x_n}\}$ and $\{\boldsymbol{y_n}\}$ are sequences in B such that $\| \boldsymbol{x_n}\| \to 1$, $\| \boldsymbol{y_n} \| \to 1$, $\| \boldsymbol{x_n} + \boldsymbol{y_n}\| \to 2$ and

$\boldsymbol{x_n} - \boldsymbol{y_n} = \boldsymbol{\alpha_n} z$ then $\boldsymbol{\alpha_n} \to 0$. It is well known that among the many kinds of rotundities of Banach spaces, local uniform rotundity is the most important one. One reason is that this kind of rotundity ensures the fixed point property [4]. Jim Ming in [4] gave criteria for local uniform rotundity and weak local uniform rotundity of Musielak-Orlicz sequence spaces equipped with the Luxemburg norm. Criteria for locally uniform rotundity of Orlicz space have been obtained in [5, 6], locally uniform rotundity of Musielak-Orlicz space was discussed also and the result plus proof are similar to those of Orlicz space [7].

In this paper, we investigated the relationship between a Highly Rotund norm(HR) and a Uniformly Rotund norm in Every Direction(URED) on a Freche't Space.

II. PRELIMINARIES

In this section, we give the fundamental concepts underlying this research that are very important in understanding Highly Rotund Norm and Uniformly Rotund Norm in Every Direction on a Freche't space.

Definition 1 *[3] A Banach space B is uniformly rotund in every direction(URED) if whenever non zero z is in B and* $\{x_n\}$ *and* y_n *are sequences in B such that* $\|x_n\| \to 1$, $\|y_n\| \to 1$, $\|x_n + y_n\| \to 2$, *and* $x_n - y_n = \alpha_n z$, *then* $\alpha_n \to 0$.

Definition 2 *[8, Definition 1.1] A Banach space* $(X, \|.\|)$ *is Rotund(R) if given* x , $y \in S_X$ *with* $x \neq y$, *then* $\left\|\frac{x+y}{2}\right\| < 1$.

Definition 3 *[9] A Banach space B has property H(B) if whenever x is in B and* $\{x_n\}$ *is a sequence in B such that* $\|x_n\| \to \|x\|$ *and* $x_n \to x$ *weakly, then* $x_n \to x$. *If* $B \in H$ *and R, write B is Highly Rotund.*

Definition 4. *[13, Definition 1.0] Let F denote either the field of R or C. A norm on a real or complex vector space X is a real-valued function on X whose value at an* $x \in X$ *is denoted by* $\| x \|$ *(read as norm of x) and which has the properties:*

(i) $\| x \| \geq 0 \ \forall x \in X$

(ii) $\| x \| = 0 \Leftrightarrow x = 0$

(iii) $\| \alpha x \| = | \alpha | \| x \|$

(iv) $\| x + y \| \leq \| x \| + \| y \|$ *(Triangular inequality).*

Definition 5. *[14, Definition 1.5] A Fre'chet space ((F) for short) is a locally convex topological vector space. Trivial example of a Fre'chet space is a Banach space. Every Banach space (in particular,* l^p $\forall 1 \leq p \leq \infty$*) is a Fre'chet space.*

As part of the preliminaries, the following theorems, some stated without proof were important:

Theorem 1 *[10, Theorem 7.2] Any uniformly convex Banach space is reflexive.*

Proof. Assume that the norm of X is uniformly convex. Then the dual norm of X* is uniformly Frechet differentiable, therefore X* is reflexive thus X is reflexive. □

Theorem 2 *[10, Theorem 6.2] Any space X with a Frechet differentiable norm which has a Gateaux defferentiable dual norm admits an equivalent LUR norm.*

Theorem 3 *[11, Theorem 4.1] The space $c(\hat{T}_0)^\star$ admits an equivalent LUR norm but $c(\hat{T}_0)$ does not admit an equivalent R norm.*

Proof. Since T_0 is scattered, $\hat{T}_0$ is also scattered. Therefore $c(\hat{T}_0)^\star$ is isometrically isomorphic to $l_1(\Gamma)$ for some Γ [11] and consequently admits an equivalent LUR norm[12]. $c_0(T_0)$ is closed subspace of $c(\hat{T}_0)$. It is proved in [12] that $c_0(T_0)$ does not admit an equivalent R norm. Hence $c(\hat{T}_0)$ does not admit an equivalent R. □

III. RESULT AND DISCUSSION

In this section we presented the results and discussions on the research.

Theorem 2.0. *Let $(E, \|.\|_F)$ be a Fre'chet space. Then the norm $\|.\|_F$ is Highly rotund.*

Proof. From the hypothesis, $\|.\|_F$ is Highly rotund. It suffices to show that $\|.\|_E$ is rotund. For $(x_1^1, x_2^2, \dots)$ in E, let $x^1 = (0, x^2, \dots)$ and define the equivalent norm

$$\| x \|_F = |x^1| + \| x^1 \|_2 \qquad (1)$$

Let $\{\alpha_n\}$ be a sequence of positive real numbers decreasing to zero and define the continuous linear mapping $T: E \to E$ by

$$T(x^1, x^2, \dots) = (\alpha_2 x^2, \alpha_3 x^3, \dots \ \forall \alpha \in \mathbb{R}, n \in \mathbb{N}) \qquad (2)$$

For $x \in E$, define

$$\| x \|_F = (\| x \|_F^2 + \| Tx \|_2^2)^{\frac{1}{2}} \qquad (3)$$

Then to show that $\|.\|_F$ is rotund, let x and $\{x_n\}$ be given such that $\| x_n \|_F \longrightarrow \| x \|_F = 1$ and $x_n \longrightarrow x$ weakly. In this case, we may assume that $\| x_n \|_F = \| x \|_F = 1$; otherwise we normalize x_n and x. Thus

$$|x_n^1| + \| x_n^1 \|_2 = |x^1| + \| x^1 \|_2 \quad \forall n \in \mathbb{N} \qquad (4)$$

Since $x_n \longrightarrow x$ weakly, it follows that $x_n^1 \longrightarrow x$ and hence $\| x_n^1 \|_2 \longrightarrow \| x^1 \|_2$ and $x_n^1 \longrightarrow x^1$ weakly. Since $\|.\|_2$ is rotund, it follows that $x_n^1 \longrightarrow x^1$ and hence $x_n^1 \longrightarrow x$ implying a higly rotund norm in a Fre'chet space F from Eq. (4) □

Theorem 2.1. *Let $(E, \|.\|)$ be any separable Fre'chet space and $(E^*, \|.\|)$be its associated separable dual Fre'chet space. Then E admits an equivalently Highly Rotund and Uniformly Rotund norm in Every Direction.*

Proof. From the hypothesis and supposing that $(e_i)_{i=1}^\infty$ is dense in $S(E)$ and $(f_i)_{i=1}^\infty$ be dense in $S(E^*)$. Given $i \in I \subseteq \mathbb{N}$, put $F_i = span\{f_1, f_2, \dots, f_i\}$. We define a norm $| \|.\| |$ on E^* by

$$| \| f \| |^2 = \| f \|^{*2} + \sum_{i=1}^{\infty} 2^{-i} dist(f, F_i)^2 + \sum_{i=1}^{\infty} 2^{-i} f^2(e_i) \qquad (5)$$

Then $|\ \|.\|\ |$ is a $weak^*$ lower semicontinous function on E^* equivalent with $\|.\|^*$streched from $\|.\|$. hence $|\ \|.\|\ |$ is the dual of a norm $|.|$ equivalent with $\|.\|$ which is highly rotund(HR) and also it is URED in both $(E, \|.\|)$ and $(E^*, \|.\|)$ ☐

IV. CONCLUSIONS

Based on the results and discussion, we conclude that a Fre'chet space admits both Highly Rotund norm and Uniformly Rotund Norm in Every Direction. we investigated whether a Highly Rotund norm implies Uniformly Rotund norm in Every Direction on a Fre'chet space. It is shown that a Fre'chet space admits both Highly Rotund norm and Uniformly Rotund Norm in Every Direction.

REFERENCES

[1] J. A. Clarkson, Uniformly convex spaces. Trans. Amer. Math. Soc. 1936, **40**:396-414

[2] Smith M. A. Banach spaces that are uniformly rotund in weakly compact sets of directions. Canad. J. Math Vol. 29, (1977), 963-970 .

[3] A. L. Garkavi, The Best possible net and best possible cross section of a set in a normed space, Izv. Akad. Nauk SSSR Ser. Mat. (1962-1963), **26**:87-106

[4] Jun Ming et al., Local uniform rotundity in Musielak-Orlicz sequence space equipped with the Luxemburg norm, Annales Societiatis mathematicae polonae Vol.1 (2006): 131-139.

[5] Y. A. Cui and H. Hudzik Maluta, coefficient and opial Property in Musielak- Orlicz sequence spaces equipped with the Luxemburg norm, to appear in Nonlinear Anal. Theory method and Appl.

[6] A. Kaminska, The criteria for local uniform rotundity of Orlics spaces, Studia Math. 1984 **74**: 201-215

[7] A. Kaminska, On some convexity properties of Musielak-Orlicz Spaces, Suppl. Rend. Circ. Math. 1984 **2**:589-594

[8] V. Montesinos and J. R. Torregrosa, Rotund and uniformly rotund Banach spaces. Collect. Math. 1991 **42**(2): 137-146.

[9] M. A. Smith, Some examples concerning rotundity in Banach spaces. Math. Ann. 1978, **233**: 155-161

[10] Amanollah Assadi and Hadi Haghshenas, *Some classical and Recent results concerning renorming theory,* Thai Journal of mathematics Vol.10 No.3 (2012), 481-495.

[11] M. Fabian et al., Functional analysis and infinite-dimensional geometry, CMS Books in Mathematics, Springer-Verlag, New York, 2001

[12] R. Deville, G. Godefroy, V. Zizler, Smoothness and Renorming in Banach spaces. Pitman Monography and surveys in Pure and Applied Math. Longman Scientific and Technical, Longman House, Burnt Mill, Harlow. 1993, **64**,183-187.

[13] Kronheimer P. B., *Banach spaces, definitions and examples*, October, 27 (2010).

[14] Andreas Kriegl, *Frechet Spaces*. http://www.mat.univie.ac.at/ kriegl/Skripten/2016SS, (2016).