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CONSTRUCTION OF THE ROUGH QUOTIENT MODULES OVER THE ROUGH RING BY USING COSET CONCEPTS

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Abstract. Consider an ordered pair (U, θ), where U is a universal set and θ is an equivalence relation on U. The equivalence relation θ is a relation that is reflexive, symmetric, and transitive. If the set $X \subseteq U$, then we can determine the upper approximation of the set X, denoted by $\overline{Apr}(X)$, and the lower approximation of the set X, denoted by $\underline{Apr}(X)$. The set X is said to be a rough set on (U, θ) if and only if $\overline{Apr}(X) - \underline{Apr}(X) \neq \emptyset$. A rough set X is a rough module if it satisfies certain axioms. This paper discusses the construction of a rough quotient module over a rough ring using the coset concept to determine its equivalence classes and discusses the properties of a rough quotient module over a rough ring related to a rough torsion module.

Keywords: Approximation space, rough module, rough quotient module over rough ring, rough torsion module.

I. INTRODUCTION

The rough set theory was first introduced by Zdzislaw Pawlak in 1982 [1]. Since then, it has been widely applied in various fields [2, 3, 4, 5, 6]. In addition to these applications, several researchers have explored rough set theory in the context of algebraic structures. Miao et al. [7] studied rough groups, rough subgroups, rough cosets, and their properties. Davvaz and Mahdavipour [8] extended this research to rough modules and rough submodules, while Zhang et al. [9] examined rough modules, rough quotient modules, and their fundamental properties. Later, Jesmalar [10] analyzed the homomorphism properties of rough groups. Alharbi et al. [11] further developed this area by investigating rough quotient groups and rough isomorphism theorems.

More recent studies have continued to expand the framework of rough algebraic structures. Setyaningsih et al. [12] introduced the subexact sequence of rough groups. In 2022, Nugraha et al. [13] explored the application of rough set theory to group structures, while Hafifulloh et al. [14] refined the properties of rough V-exact sequences in rough groups. Agusfrianto et al. [15] also examined rough rings, rough subrings, rough ideals. More recently, Ayuni et al. [16] studied rough U-exact sequences in rough groups, and Yanti et al. [17] applied rough set theory to projective module structures.

Building upon these works, this research focuses on constructing a rough quotient module over a rough ring within an approximation space, using the concept of cosets, where equivalence classes are cosets of a finite set. Additionally, we explore the fundamental properties of rough quotient modules over rough rings, particularly with regard to rough torsion modules. As a foundation, we first establish the key concepts of rough sets necessary to construct rough quotient modules over rough rings.

II. PRELIMINARIES

In this section, we give the fundamental concepts underlying this research. Before delving into rough set theory, we first review the definition of an approximation space.

Definition 2.1 [18] *The pair (U, θ) where U is a set universe with $U \neq \emptyset$ and θ is an equivalence relation on U is called an approximation space.*

Next, we introduce the definitions of lower and upper approximations for nonempty subsets of the universal set.

Definition 2.2 [18] *Given an approximation space $K = (U, \theta)$ and $X \subseteq U$. The set $\overline{Apr}(X) = \{x \in U \mid [x]_\theta \cap X \neq \emptyset\}$ is called the upper approximation of X, and the set $\underline{Apr}(X) = \{x \in U \mid [x]_\theta \subseteq X\}$ is called the lower approximation of X.*

The following definition gives the condition for a subset of a universal set to be a rough set.

Definition 2.3 [19] *Given an approximation space (U, θ) and $X \subseteq U$. The set X is said to be a rough set on (U, θ) if and only if $\overline{Apr}(X) - \underline{Apr}(X) \neq \emptyset$.*

A binary operation can be defined on the universal set. Moreover, a nonempty subset of the universal set qualifies as a rough group if it satisfies the conditions outlined in the following definition. The resulting rough set serves as a generalization of the conventional group structure.

Definition 2.4 [15] *Given an approximation space (U, θ) and a binary operation $*$ in U. The set $G \subseteq U$ is called a rough group if it satisfies the following axioms:*

1. *for every $a, b \in G$, holds $a * b \in \overline{Apr}(G)$;*
2. *for every $a, b, c \in G$, $a * (b * c) = (a * b) * c$, (the associative property in $\overline{Apr}(G)$ holds);*
3. *there exists $e \in \overline{Apr}(G)$, such that for every $a \in G, a * e = e * a = a$, ($e$ is the rough identity element of the rough group G);*
4. *for every $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$, (b is the rough inverse element of a in G).*

As in the case of groups, the term commutative group is given in the following definition given the definition of a commutative rough group.

Definition 2.5 [20] *A rough group G is said to be a commutative rough group if for every $a, b \in G, a * b = b * a$, (the commutative property in $\overline{Apr}(G)$ holds).*

The following definition introduces the concepts of left and right cosets in rough groups, which generalize the conventional notions of left and right cosets in groups.

Definition 2.6 [7] *Given an approximation space (U, θ), a rough group $G \subseteq U$ and a rough subgroup H of G. The set $\overline{Apr}(H) * a = \{h * a \mid h \in \overline{Apr}(H), a \in G, h * a \in G\} \cup a$, $\overline{Apr}(H)$ is said to be the right rough coset of H that contains the element a and the set $a * \overline{Apr}(H) = \{a * h \mid a \in G, h \in \overline{Apr}(H), a * h \in G\} \cup a$ is said to be the left rough coset of H that contains the element a.*

The following definition introduces the concept of a rough ring as a generalization of a ring.

Definition 2.7 [7] *Given an approximation space (U, θ) and two binary operations $(+, *)$ in U. The set The set $R \subseteq U$ is called a rough ring if it satisfies the following axioms:*

1. *$\langle R, + \rangle$ is a rough group commutative to the $+$ operation;*
2. *$\langle R, * \rangle$ is a rough semigroup concerning operation $*$ or R is associative;*
3. *for every $a, b, c \in R$, holds $(a+b)*c = a*c+b*c$ (left distributive law) and $a*(b+c) = a*b+a*c$ (right distributive law).*

The following definition introduces the concept of a rough module over a rough ring.

Definition 2.8 [9] *Given a rough ring $\langle R, +, * \rangle$ with unit element and a commutative rough group $\langle M, +' \rangle$. The set M is said to be a rough right module over the rough ring R if there exists a mapping $\cdot : \overline{Apr}(R) \times \overline{Apr}(M) \to \overline{Apr}(M), (a, x) \mapsto ax$, holds:*

1. *$a(x +' y) = ax +' ay$;*
2. *$(a + b)x = ax +' bx$;*
3. *$(a * b)x = a(bx)$;*
4. *$1x = x$, where 1 is the unit element of R,*

for every $a, b \in R, x, y \in M$

In rough set theory, the rough quotient module is defined as follows.

Definition 2.9 [9] *Given a rough module M over a rough ring R and a rough submodule S of the rough R-module M. There can be formed a rough quotient group $\langle {}^{M}/_{S}, + \rangle$ which is a commutative rough group, with ${}^{M}/_{S} = \{\overline{m} = m + \overline{Apr}(S) \mid m \in M\}$. Define scalar multiplication in the commutative rough group ${}^{M}/_{S}$ as follows:*

$$r\overline{m} = r(m + \overline{Apr}(S)) = rm + \overline{Apr}(S),$$

for every $r \in R$, $\overline{m} \in {}^{M}/_{S}$. Since S is a rough submodule of M, scalar multiplication in the commutative rough group ${}^{M}/_{S}$ is well defined, ${}^{M}/_{S}$ forms a rough module over the rough ring R called the rough quotient module over the rough ring.

III. RESULT AND DISCUSSION

In this part, we will construct the rough quotient module over the rough ring, rough torsion element, and rough torsion module. The following is the construction of a rough quotient module over a rough ring based on Definition 2.9.

Example 3.1 Given an approximation space (U, θ), with $U = \mathbb{Z}_{36}$ there are subgroups on the set U namely $G = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}, \overline{30}, \overline{33}\}$. We know that U is a group under addition modulo 36 $(+_{36})$. Then, we define the relation θ on the set U, for every $a, b \in U$ holds $a\theta b$ if and only if $a -_{36} b \in G$. The equivalence classes on the set U are as follows. Suppose $r \in U$, so the equivalence class containing r is:

$$\begin{aligned}
[r]_\theta &= \{a \in U \mid a\theta r\} \\
&= \{a \in U \mid a -_{36} r \in G\} \\
&= \{a \in U \mid a -_{36} r = g, \text{for } g \in G\} \\
&= \{a \in U \mid -r +_{36} a = g, \text{for } g \in G\} \\
&= \{a \in U \mid a = r +_{36} g, \text{for } g \in G\} \\
&= \{r +_{36} g \mid g \in G\}.
\end{aligned}$$

As per the definition of the left coset of G, for every $r \in U$ the left coset of G in U is $r +_{36} G = \{r +_{36} g \mid g \in G\}$. Thus, the left coset or equivalence class of G in U, is as follows:
$E_1 = \overline{0} +_{36} G = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}, \overline{30}, \overline{33}\}$;
$E_2 = \overline{1} +_{36} G = \{\overline{1}, \overline{4}, \overline{7}, \overline{10}, \overline{13}, \overline{16}, \overline{19}, \overline{22}, \overline{25}, \overline{28}, \overline{31}, \overline{34}\}$;
$E_3 = \overline{2} +_{36} G = \{\overline{2}, \overline{5}, \overline{8}, \overline{11}, \overline{14}, \overline{17}, \overline{20}, \overline{23}, \overline{26}, \overline{29}, \overline{32}, \overline{35}\}$.

Given a nonempty subset $R \subseteq U$ with $R = \{\overline{3}, \overline{6}, \overline{10}, \overline{14}, \overline{22}, \overline{26}, \overline{30}, \overline{33}\}$. The upper approximation and lower approximation of the set $R \subseteq U$ are obtained, namely $\overline{Apr}(R) = E_1 \cup E_2 \cup E_3 = U$, and $\underline{Apr}(R) = \emptyset$. It will be proved R is a rough ring with binary operations $(+_{36}, \cdot_{36})$.

Table 1. Table *Cayley* $+_{36}$ on R

$+_{36}$	$\overline{3}$	$\overline{6}$	$\overline{10}$	$\overline{14}$	$\overline{22}$	$\overline{26}$	$\overline{30}$	$\overline{33}$
$\overline{3}$	$\overline{6}$	$\overline{9}$	$\overline{13}$	$\overline{17}$	$\overline{25}$	$\overline{29}$	$\overline{33}$	$\overline{0}$
$\overline{6}$	$\overline{9}$	$\overline{12}$	$\overline{16}$	$\overline{20}$	$\overline{28}$	$\overline{32}$	$\overline{0}$	$\overline{3}$
$\overline{10}$	$\overline{13}$	$\overline{16}$	$\overline{20}$	$\overline{24}$	$\overline{32}$	$\overline{0}$	$\overline{4}$	$\overline{7}$
$\overline{14}$	$\overline{17}$	$\overline{20}$	$\overline{24}$	$\overline{28}$	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{11}$
$\overline{22}$	$\overline{25}$	$\overline{28}$	$\overline{32}$	$\overline{0}$	$\overline{8}$	$\overline{12}$	$\overline{16}$	$\overline{19}$
$\overline{26}$	$\overline{29}$	$\overline{32}$	$\overline{0}$	$\overline{4}$	$\overline{12}$	$\overline{16}$	$\overline{20}$	$\overline{23}$
$\overline{30}$	$\overline{33}$	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{16}$	$\overline{20}$	$\overline{24}$	$\overline{27}$
$\overline{33}$	$\overline{0}$	$\overline{3}$	$\overline{7}$	$\overline{11}$	$\overline{19}$	$\overline{23}$	$\overline{27}$	$\overline{30}$

Based on Table 1., the following are obtained.

1. For every $a, b \in R$, then $a +_{36} b \in \overline{Apr}(R)$.
2. For every $a, b, c \in R$, $(a +_{36} b) +_{36} c = a +_{36} (b +_{36} c)$ in $\overline{Apr}(R)$ (associative property in $\overline{Apr}(R)$).
3. There is an $\overline{0} \in \overline{Apr}(R)$, for every $a \in R$, then $a +_{36} \overline{0} = \overline{0} +_{36} a = a$.
4. For every $a \in R$, there is $a^{-1} \in R$, then $a +_{36} a^{-1} = a^{-1} +_{36} a = \overline{0}$. We can see this condition in the following table.

Table 2. Inverse table on R

a	$\overline{3}$	$\overline{6}$	$\overline{10}$	$\overline{14}$	$\overline{22}$	$\overline{26}$	$\overline{30}$	$\overline{33}$
$-a$	$\overline{33}$	$\overline{30}$	$\overline{26}$	$\overline{22}$	$\overline{14}$	$\overline{10}$	$\overline{6}$	$\overline{3}$

5. For every $a, b \in R$, then $a +_{36} b = b +_{36} a$.
6. For every $a, b \in R$, then $a \cdot_{36} b \in \overline{Apr}(R)$. We can see this condition in the following table.

Table 3. Tabel *Cayley* $\cdot_{36}$ on R

$\cdot_{36}$	$\overline{3}$	$\overline{6}$	$\overline{10}$	$\overline{14}$	$\overline{22}$	$\overline{26}$	$\overline{30}$	$\overline{33}$
$\overline{3}$	$\overline{9}$	$\overline{18}$	$\overline{30}$	$\overline{6}$	$\overline{30}$	$\overline{6}$	$\overline{18}$	$\overline{27}$
$\overline{6}$	$\overline{18}$	$\overline{0}$	$\overline{24}$	$\overline{12}$	$\overline{24}$	$\overline{12}$	$\overline{0}$	$\overline{18}$
$\overline{10}$	$\overline{30}$	$\overline{24}$	$\overline{28}$	$\overline{32}$	$\overline{4}$	$\overline{8}$	$\overline{12}$	$\overline{6}$
$\overline{14}$	$\overline{6}$	$\overline{12}$	$\overline{32}$	$\overline{16}$	$\overline{20}$	$\overline{4}$	$\overline{24}$	$\overline{30}$
$\overline{22}$	$\overline{30}$	$\overline{24}$	$\overline{4}$	$\overline{20}$	$\overline{16}$	$\overline{32}$	$\overline{12}$	$\overline{6}$
$\overline{26}$	$\overline{6}$	$\overline{12}$	$\overline{8}$	$\overline{4}$	$\overline{32}$	$\overline{28}$	$\overline{24}$	$\overline{30}$
$\overline{30}$	$\overline{18}$	$\overline{0}$	$\overline{12}$	$\overline{24}$	$\overline{12}$	$\overline{24}$	$\overline{0}$	$\overline{18}$
$\overline{33}$	$\overline{27}$	$\overline{18}$	$\overline{6}$	$\overline{30}$	$\overline{6}$	$\overline{30}$	$\overline{18}$	$\overline{9}$

7. For every $a, b, c \in R$, then $(a \cdot_{36} b) \cdot_{36} c = a \cdot_{36} (b \cdot_{36} c)$ in $\overline{Apr}(R)$.
8. For every $a, b, c \in R$, then $(a +_{36} b) \cdot_{36} c = (a \cdot_{36} c) +_{36} (b \cdot_{36} c)$ in $\overline{Apr}(R)$ and $a \cdot_{36} (b +_{36} c) = (a \cdot_{36} b) +_{36} (a \cdot_{36} c)$ in $\overline{Apr}(R)$.

Therefore, it is proved that R is a rough ring of the approximation space (U, θ).

Example 3.2 Given the rough ring R based on Example 3.1, and nonempty set $M \subseteq U$, with $M = \{\overline{0}, \overline{6}, \overline{12}, \overline{18}, \overline{24}, \overline{30}\}$, $\overline{Apr}(M) = E_1 = \overline{0} +_{36} G = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}, \overline{30}, \overline{33}\}$ and $\underline{Apr}(M) = \emptyset$. We will prove that M is a rough commutative group with binary operation $+_{36}$.

1. For every $a, b \in M$, then $a +_{36} b \in \overline{Apr}(M)$.
2. For every $a, b, c \in M$, $(a +_{36} b) +_{36} c = a +_{36} (b +_{36} c)$ in $\overline{Apr}(M)$.
3. There is an $e \in \overline{Apr}(M)$, for every $a \in M$, then $a +_{36} e = e +_{36} a = a$.
4. For every $a \in M$, there is $a^{-1} \in M$, then $a +_{36} a^{-1} = a^{-1} +_{36} a = e$.
5. For every $a, b \in M$, then $a +_{36} b = b +_{36} a$.

It is proved that $\langle M, +_{36} \rangle$ is a commutative rough group. Next, we will prove that the set M is a rough module over the rough ring R by Definition 3.1.

1. Given any $a \in R$ and $x, y \in M$, then $a \cdot_{36} (x +_{36} y) = (a \cdot_{36} x) +_{36} (a \cdot_{36} y)$.
2. Given any $a, b \in R$ and $x \in M$, then $(a +_{36} b) \cdot_{36} x = (a \cdot_{36} x) +_{36} (b \cdot_{36} x)$.
3. Given any $a, b \in R$ and $x \in M$, then $(a \cdot_{36} b) \cdot_{36} x = (a \cdot_{36} x) \cdot_{36} (b \cdot_{36} x)$.
4. There exists $\overline{1} \in \overline{Apr}(R)$, such that for every $x \in M$ holds $\overline{1} \cdot x = x$.

Thus proved, we can conclude that M is a rough module over the rough ring R.

Example 3.3 Given a rough module over the rough ring M described in Example 3.1 and a nonempty set $S \subseteq M$, with $S = \{\overline{0}, \overline{12}, \overline{24}\}$, $\overline{Apr}(S) = E_1 = \overline{0} +_{36} G = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}, \overline{30}, \overline{33}\}$ dan $\underline{Apr}(S) = \emptyset$. It will be shown that S is a rough submodule of M over the rough ring R.

1. For every $x, y \in S$, then $x +_{36} y \in \overline{Apr}(S)$.
2. For every $x \in S$, then $x^{-1} \in S$.
3. For every $a \in R$, $x \in S$, holds $a \cdot_{36} x \in \overline{Apr}(S)$.

So, it is proved that the set S is a rough submodule of M over the rough ring R.

Proposition 3.1 *If given an approximation space (U, θ), with $\langle U, * \rangle$ a finite group and $a\theta b$ if and only if $a - b \in S$, for $a, b \in U$ and S a subgroup of U, there exists an approximation space (U', θ'), with $U' = \{a + E | a \in U\}$, for a subgroup E of U.*

Proof. It will be shown that there exists an approximation space (U', θ'), with $U' = \{a + E | a \in U\}$, for a subgroup E of U. Given an approximation space (U, θ), with $a\theta b$ if and only if $a - b \in S$, for $a, b \in U$ such that there exist equivalence classes $0 + S, a_1 + S, a_2 + S, ..., a_n + S$. Choose $E = 0 + S$ subgroup of U, we have $U' = \{a + E | a \in U\}$. Since $0 + E = E \in U'$ so $U' \neq \emptyset$. Next, define a relation θ' on U', for every $a + E, b + E \in U'$ holds $(a + E)\theta'(b + E)$ if and only if $a - b \in U$. It will be proved that the relation θ' is an equivalence relation on the set U'.

1. Given any $a+E \in U'$, then $(a+E)\theta'(a+E)$. So, θ' is reflexive.
2. Given any $a+E, b+E \in U'$ with $(a+E)\theta'(b+E)$, then $(b+E)\theta'(a+E)$. So, θ' is symmetric.
3. Given any $a+E, b+E, c+E \in U'$ with $(a+E)\theta'(b+E)$ and $(b+E)\theta'(c+E)$, then $(a+E)\theta'(c+E)$. So, θ' is transitive.

So, θ' is an equivalence relation in U'. Therefore, (U', θ') is an approximation space. □

Proposition 3.2 *If given an approximation space (U, θ), with $\langle U, *\rangle$ a finite module and $a\theta b$ if and only if $a-b \in S$, for $a, b \in U$ and S a submodule of U, there exists an approximation space (U', θ'), with $U' = \{a+E | a \in U\}$, for a submodule E of U.*

The proof of Proposition 3.2 follows the similar steps as the proof of Proposition 3.1.

Based on Definition 2.9, since S is a rough submodule in M, $\overline{Apr}(S)$ is a submodule. Thus, the set S is an arbitrary nonempty subset of E and hence related to Proposition 3.2 which results in $\overline{Apr}(S) = E$, with E a submodule of U. The following is the construction of rough quotient modules over rough rings.

Example 3.4 Based on Example 3.1, using Proposition 3.2 we obtain the approximation space (U', θ'), with $U' = \{a +_{36} E_1 | a \in U\} = \{\overline{0} +_{36} E_1, \overline{1} +_{36} E_1, \overline{2} +_{36} E_1\}$, with
$\overline{0} +_{36} E_1 = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}, \overline{30}, \overline{33}\}$;
$\overline{1} +_{36} E_1 = \{\overline{1}, \overline{4}, \overline{7}, \overline{10}, \overline{13}, \overline{16}, \overline{19}, \overline{22}, \overline{25}, \overline{28}, \overline{31}, \overline{34}\}$;
$\overline{2} +_{36} E_1 = \{\overline{2}, \overline{5}, \overline{8}, \overline{11}, \overline{14}, \overline{17}, \overline{20}, \overline{23}, \overline{26}, \overline{29}, \overline{32}, \overline{35}\}$.

We define an equivalence relation θ' on the set U', for every $a +_{36} E_1, b +_{36} E_1 \in U'$ holds $(a +_{36} E_1)\theta'(b +_{36} E_1)$ if and only $a -_{36} b \in U$. Then, the equivalence class on the set $U^{'}$ with respect to the equivalence relation $\theta^{'}$ is $E_1^{'} = \{\overline{0} +_{36} E_1, \overline{1} +_{36} E_1, \overline{2} +_{36} E_1\}$.

Given a nonempty subset $M' \subseteq U'$ with $M^{'} = \{\overline{1} +_{36} E_1, \overline{2} +_{36} E_1\}$. Therefore, the upper approximation and lower approximation of $M' \subseteq U'$ are obtained, namely $\overline{Apr}(M^{'}) = E_1^{'} = \{\overline{0} +_{36} E_1,\ \overline{1} +_{36} E_1,\ \overline{2} +_{36} E_1\}$, and $\underline{Apr}(M^{'}) = \emptyset$.

The set M' is a rough module over the rough ring R concerning the scalar multiplication operation $r \cdot_{36} (m' +_{36} E_1) = (r \cdot_{36} m') +_{36} E_1$, for any $r \in R$ and $m' +_{60} E_1 \in M'$.

Table 4. Tabel *Cayley* $\cdot_{36}$ on M'

$\cdot_{36}$	$\overline{1} +_{36} E_1$	$\overline{2} +_{36} E_1$
$\overline{3}$	$\overline{0} +_{36} E_1$	$\overline{0} +_{36} E_1$
$\overline{6}$	$\overline{0} +_{36} E_1$	$\overline{0} +_{36} E_1$
$\overline{10}$	$\overline{1} +_{36} E_1$	$\overline{2} +_{36} E_1$
$\overline{14}$	$\overline{2} +_{36} E_1$	$\overline{1} +_{36} E_1$
$\overline{22}$	$\overline{1} +_{36} E_1$	$\overline{2} +_{36} E_1$
$\overline{26}$	$\overline{2} +_{36} E_1$	$\overline{1} +_{36} E_1$
$\overline{30}$	$\overline{0} +_{36} E_1$	$\overline{0} +_{36} E_1$
$\overline{33}$	$\overline{0} +_{36} E_1$	$\overline{0} +_{36} E_1$

Thus, the set M' is called the rough quotient module M' over the rough ring R.

After constructing the rough quotient module on the rough ring, the next step is to construct the rough torsion element, rough torsion module. Before that, we will first define the rough torsion element and rough torsion module.

Definition 3.1 *Given a rough module M over a rough ring R. An element $m \in M$ is called a rough torsion element if there exists $r \in \overline{Apr}(R)\backslash 0$ such that $rm = 0$, with $0 \in \overline{Apr}(M)$. Furthermore, the set of all rough torsion elements in M is denoted by $M_T = \{m \in M \mid rm = 0$, for some $r \in \overline{Apr}(R)\backslash 0$, and $0 \in \overline{Apr}(M)\}$.*

Definition 3.2 *Given a rough module M over a rough ring R with a unit element, the module M is called a rough torsion module if every element in M is a rough torsion element, or in other words $M = M_T$.*

The following is an example of a rough torsion element and a rough torsion module on the rough quotient module M'.

Example 3.5 Based on Example 3.4, a rough quotient module M' over rough ring R, with

$$\begin{aligned} R &= \{\overline{3}, \overline{6}, \overline{10}, \overline{14}, \overline{22}, \overline{26}, \overline{30}, \overline{33}\}, \\ \overline{Apr}(R) &= \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, ...\overline{35}\}, \\ M^{'} &= \{\overline{1} +_{36} E_1, \overline{2} +_{36} E_1\}, \\ \overline{Apr}(M^{'}) &= \{\overline{0} +_{36} E_1, \overline{1} +_{36} E_1, \overline{2} +_{36} E_1\}. \end{aligned}$$

The rough torsion elements of the module $M^{'}$ over R are all elements in $M^{'}$, $M_T = \{\overline{1} +_{36} E_1, \overline{2} +_{36} E_1\}$. This is due to the following reasons:

i) for $\overline{1} +_{36} E_1 \in M^{'}$, there exists $\overline{3} \in \overline{Apr}(R)\backslash 0$, such that $\overline{3}(\overline{1} +_{36} E_1) = (\overline{3} \cdot \overline{1}) +_{36} E_1 = \overline{0} +_{36} E_1$;
ii) for $\overline{2} +_{36} E_1 \in M^{'}$, there exists $\overline{3} \in \overline{Apr}(R)\backslash 0$, such that $\overline{3}(\overline{2} +_{36} E_1) = (\overline{3} \cdot \overline{2}) +_{36} E_1 = \overline{0} +_{36} E_1$.

Thus, the set of all rough torsion elements in $M^{'}$ is every element in $M^{'}$ or $M_T = M^{'}$. Therefore, $M^{'}$ is a rough torsion module.

In the following, we will discuss the proposition that the set of all rough torsion elements or M_T is a rough submodule of a rough module and its proof.

Proposition 3.3 *Given a rough module M over a rough integral domain R. Then, the set of all rough torsion elements M_T is a rough submodule of M.*

Proof. Based on Definition 3.1, we obtain $M_T \subseteq M$ and $0 \in \overline{Apr}(M_T)$. In other words, $M_T \neq \emptyset$.

a) We will show that $m - n \in \overline{Apr}(M_T)$, for every $m, n \in M_T$. This means that there must be $t \in \overline{Apr}(R)\backslash 0$ such that $t(m - n) = 0$.
Given any $m, n \in M_T$. There exist $r, s \in \overline{Apr}(R)\backslash 0$ such that $rm = 0, sn = 0$. Given rough integral domain R, we have $rs \in \overline{Apr}(R)$, $rs \neq 0$. Using axiom (1) from Definition 3.1, which is related to the rough module, choose $t = rs$, then
$$\begin{aligned} t(m - n) &= rs(m - n) = (rs)m - (rs)n = (sr)m - r(sn) \\ &= s(rm) - r(sn) = s \cdot 0 - r \cdot 0 = 0. \end{aligned}$$

b) We will show that $sm \in \overline{Apr}(M_T)$, for every $m \in M$ and $s \in \overline{Apr}(R)$. This means that there must be $r \in \overline{Apr}(R)\backslash 0$ such that $r(sm) = 0_M$.
Given any $s \in \overline{Apr}(R)$, there exists $r \in \overline{Apr}(R)\backslash 0$, such that

$$r(sm) = (rs)m = (sr)m = s(sr) = 0.$$

Thus, for every $s \in \overline{Apr}(R)$, $m \in M_T$, there exists $r \in \overline{Apr}(R)$, such that $r(sm) = 0$ with $sm \in \overline{Apr}(M_T)$

From a) and b), it is proved that M_T is a rough submodule of M. □

Next, we give an example of Proposition 3.3 using its rough module, namely the rough quotient module M'.

Example 3.6 Based on Example 3.5 given a rough quotient module M' over a rough integral region R, and the set of all rough torsion elements M_T, $M_T \subseteq M'$. We will prove that M_T is a rough submodule of M'.

i) For every $a +_{36} E_1, b +_{36} E_1 \in M_T$, it holds that $(a +_{36} E_1) -_{36} (b +_{36} E_1) = (a -_{36} b) +_{36} E_1 \in \overline{Apr}(M_T)$.
ii) For every $r \in R, a +_{36} E_1 \in M_T$, it holds that $r(a +_{36} E_1) = (ra) +_{36} E_1 \in \overline{Apr}(M_T)$.

Thus, by i) and ii) it is proved that M_T is a rough submodule of M'.

IV. CONCLUSIONS

Based on the results and discussion, we conclude that the rough quotient group $\langle {}^{M}/_{S}, +\rangle$ is a commutative rough group, where M is a rough module over the rough ring R and S is a rough submodule of M. Moreover, the scalar multiplication operation in $\langle {}^{M}/_{S}, +\rangle$ is well-defined, ensuring that $\langle {}^{M}/_{S}, +\rangle$ forms a rough quotient module over the rough ring. Since S is a rough submodule of M, its upper approximation $\overline{Apr}(S)$ is also a submodule. This implies that S is an arbitrary nonempty subset of E such that $\overline{Apr}(S) = E$, where E is a submodule of U.

Furthermore, if every element of the rough quotient module $\langle {}^{M}/_{S}, +\rangle$ over the rough ring R is a torsion element, then $\langle {}^{M}/_{S}, +\rangle$ is called a rough torsion module. In this case, the set of all rough torsion elements, denoted as M_T, forms a rough submodule of $\langle {}^{M}/_{S}, +\rangle$.

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