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OPTIMAL CONTROL OF MATHEMATICAL MODELS IN BIOENERGY SYSTEMS AS EMPOWERMENT OF SUSTAINABLE ENERGY SOURCES

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Abstract. Energy has a very important role in everyday life. Dependence on non-renewable energy increases its vulnerability to supply instability, making it important to seek alternative energy sources to overcome this dependence. Bioenergy is an alternative energy produced from organic materials such as biomass. Control of renewable energy is needed to increase production and empowerment. In this research, a mathematical model of biogas production growth in the form of differential equations formed with optimal control modifications is proposed. Completion of the model is carried out by forming an objective function, as well as determining the Hamilton function and Lagrange function. Numerical simulations in the model show that providing control can increase biogas production as a sustainable energy source.
Keywords: Biogas, Energy, Optimal Control, Mathematical Models

I. Introduction

The presence of energy in human life is essential. Relying on non-renewable energy sources could lead to a potential energy crisis in the future, with adverse effects on the environment. Energy use in Indonesia is still predominantly dominated by non-renewable sources derived from fossil fuels, especially petroleum, natural gas, and coal. This dependence on fossil-based fuels is vulnerable to supply shortages, making alternative energy sources from renewable materials crucial to addressing this dependency.

One such alternative is biogas—a renewable, cost-effective, and environmentally friendly energy source. Biogas is generated through the decomposition of organic materials by microorganisms under relatively low-oxygen (anaerobic) conditions, biogas utilizes materials like animal manure, aquatic weeds, and agricultural waste. The key constituents of biogas include methane (CH4) and carbon dioxide (CO2), both of which can be burned, releasing energy for various human purposes. The composition of these gases depends on the anaerobic process and the raw materials used. Optimal biogas production requires careful selection and mixing of raw materials in appropriate proportions, considering the carbon-to-nitrogen ratio in the substance. Higher methane content results in greater energy production. Utilizing organic waste materials for biogas ensures a sustainable supply. However, maintaining a balance in animal and plant populations becomes crucial for the ongoing production of biogas, necessitating specific strategies.

Biogas is considered safer for the environment because the combustion of biogas can reduce greenhouse gas emissions. Biogas can also reduce odors, insects, and pathogens originating from traditional waste heaps. The advantages of biogas can be applied for diverse applications, similar to natural gas. Moreover, the biogas production process generates a secondary product called bio-slurry, which functions as organic fertilizer for plants [5]. As a result, the plants that thrive can serve as a food source for livestock. This clarification underscores the interconnected relationship among animals, plants, and biogas.

One method for assessing the impact of each factor involves expressing it through mathematical formulations. Mathematical models serve as tools to interpret diverse phenomena in everyday life, aiding in the identification of underlying issues, finding solutions, and even predicting future events [6-7]. To understand the relationships between animals, plants, and biogas, it is crucial to develop a dynamic model that incorporates these three variables. Al Seadi [4] established a relational framework depicting the connections between animals, plants, and biogas. Using this framework, a mathematical model is created to investigate how changes in one variable influence the others. The model was developed by providing optimal control to increase the interaction between these three variables.

Matheri [6] delved into the realm of mathematical modeling for biogas production, employing biochemical kinetic models. In 2019, Antoine [7] conducted research aimed at developing control strategies and bioreactor designs that optimize biogas production. In this study, we establish a model for bioenergy systems incorporating control mechanisms to enhance production and empowerment. The research proposes a mathematical model describing the growth of biogas production in the form of differential equations, incorporating optimal control modifications. The completion of the model involves the formulation of an objective function, along with the determination of the Hamilton and Lagrange functions. Numerical simulations within the model demonstrate that implementing control measures can effectively increase biogas production, positioning it as a sustainable energy source.

II. Mathematical Model

The formulation of a mathematical model for the dynamic interaction among populations of living organisms during the process of biogas production, as a method of enhancing energy sources, can be expressed in the following manner

$$
\begin{aligned}
\frac{dP}{dt} &= a_1 + l + v_1 T(t) - (p+e)P(t), \\
\frac{dE}{dt} &= eP(t) - (c+h)E(t) + oT(t), \\
\frac{dT}{dt} &= (c+h)E(t) + pP(t) - aT(t) - (v_2+o)T(t).
\end{aligned}
\tag{1}
$$

with $P(0) \geq 0, E(0) \geq 0, T(0) \geq 0.$ The system operates under the following model assumptions, Concentration on the formation of biogas as an alternative renewable energy $E(t)$, Concentration of processing in digestion (anaerobic) which can obtain biogas $P(t)$, Concentration of plant that are influenced by the surrounding environment $T(t)$.The parameters utilized in the models are detailed in Table 1. Based on the dynamical analysis in [8], model (1) possesses an equilibrium point that will be locally asymptotically stable only if

$$(p+e+h+a+v_1+o+c)(pc+ph+ap+op+ec+ae+v_1e+oe+ac+v_1c+ah+v_1h+eh)>$$
$$+apc+aph+eca+eha.$$

the equation is satisfied.

In this scenario, control needs to be implemented to increase biogas production, positioning it as a sustainable energy source. Two time-dependent controls, denoted as u_1 and u_2 are introduced in model (2), each associated with changing constant treatment rates e dan p respectively. The u_1 control is applied to concentration on the formation of biogas as an alternative to renewable energy while the u_2 control is directed concentration of plant that are influenced by the surrounding environment. Where the condition for the control above states, if u_1 and u_2 have a value of 0 then it means that the control has no effect on optimizing the objective function, and vice versa if u_1, u_2 has a value of 1, it means that the control has the maximum effect on optimizing the objective function. Thus, the model (2) is now augmented with controls,

$$\begin{aligned}
\frac{dP}{dt} &= a_1 + l + v_1T(t) - (u_1(t)+u_2(t))P(t),\\
\frac{dE}{dt} &= u_2(t)P(t) - (c+h)E(t) + oT(t),\\
\frac{dT}{dt} &= (c+h)E(t) + u_1(t)P(t) - aT(t) - (v_2+o)T(t).
\end{aligned} \tag{2}$$

with $P(0)=P_0, E(0)=E_0, T(0)=T_0,$ as initial conditions.

III. Optimal Control

The aim of the optimal control problem described by model (2) is to maximize biogas production while minimizing costs, utilizing a given cost function. This is achieved by optimizing the concentration of decomposition products, which serve as fertilizers for plants and as a renewable energy source. The relationship between the amount of processing in anaerobic decomposition, which yields biogas, and the cost function is nonlinear. Therefore, the objective function is formulated in a nonlinear manner, with a quadratic function (u^2) selected for this purpose. In summary, the objective function can be expressed as follows: [The actual objective function should be inserted here, as it's not provided in the paragraph.]

$$F[u_1,u_2] = \int_{t_0}^{t_f}\left[B_1P(t) - \frac{B_2}{2}u_1^2(t) - \frac{B_3}{2}u_2^2(t)\right]dt,$$

with equation system (2) as the constraint. The optimal control is determined such that

$$F[u_1,u_2] = \max\{F[u_1,u_2] \mid u_1,u_2 \in U\}$$

With t_0 as the initial time and t_f as the final time, and B_i as the weight variable correlated with the cost of control usage, which is the balancing factor of the system control cost, with $B_i > 0$ for each $i = 1,2,3, U = \{0 \le u_1, u_2 \le 1\}$. Next, we formulate the Hamiltonian function.

$$\begin{aligned} H = {} & B_1 P(t) - \frac{B_2}{2} u_1^2(t) - \frac{B_3}{2} u_2^2(t) + \lambda_p \left(a_1 + l + v_1 T(t)\right) - \left(u_1(t) + u_2(t) P(t)\right) \\ & + \lambda_E \left(u_2(t) P(t) - (c+h) E(t)\right) \\ & + oT(t) + \lambda_T \left((c+h) E(t) + u_1(t) P(t) - aT(t) - (v_2 + o) T(t)\right) \end{aligned}$$

where $\lambda_p, \lambda_E, \lambda_T$ are costate variables.

1. Maximizing H with respect to u. The equation H is differentiated with respect to u_1.

$$\begin{gathered} \frac{\partial H}{\partial u_1} = 0, \\ -B_2 u_1 - \lambda_p P + \lambda_T P = 0, \\ -B_2 u_1 = \lambda_p P - \lambda_T P, \\ u_1^* = \frac{\lambda_p P - \lambda_T P}{-B_2}, \\ u_1^* = \frac{(\lambda_T - \lambda_p) P}{B_2}. \end{gathered}$$

The equation H is differentiated with respect to u_2.

$$\begin{gathered} \frac{\partial H}{\partial u_2} = 0, \\ -B_3 u_2 - \lambda_p P + \lambda_E P = 0, \\ -B_3 u_2 = \lambda_p P + \lambda_E P, \\ u_2^* = \frac{\lambda_p P - \lambda_E P}{-B_3}, \\ u_2^* = \frac{(\lambda_E - \lambda_p) P}{B_3}, \end{gathered}$$

Since $u_1(t)$ and $u_2(t)$ are defined in $0 \le u_1 \le 1$ dan $0 \le u_2 \le 1$, so we get

$$u_1^* = \begin{cases} 0, & \frac{(\lambda_T - \lambda_p) P}{B_2} \le 0 \\ \frac{(\lambda_T - \lambda_p) P}{B_2}, & 0 < \frac{(\lambda_T - \lambda_p) P}{B_2} < 1 \\ 1, & \frac{(\lambda_T - \lambda_p) P}{B_2} \ge 1 \end{cases}$$

and

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$$u_2^{*}=\begin{cases}0, & \dfrac{(\lambda_E-\lambda_p)P}{B_3}\le 0\\ \dfrac{(\lambda_E-\lambda_p)P}{B_3}, & 0<\dfrac{(\lambda_E-\lambda_p)P}{B_3}<1\\ 1, & \dfrac{(\lambda_E-\lambda_p)P}{B_3}\ge 1\end{cases}$$

So, the optimal control value can be expressed as follows.,

$$u_1^{*}=\min\left\{1,\max\left(0,\left(\frac{(\lambda_T-\lambda_p)P}{B_2}\right)\right)\right\},$$

$$u_2^{*}=\min\left\{1,\max\left(0,\left(\frac{(\lambda_E-\lambda_p)P}{B_3}\right)\right)\right\}.$$

Next, a second derivative test is carried out to show that H has the maximum value $u(t)$ as follows.

$$\frac{\partial^2 H}{\partial u_1^{2}}=-B_2<0,$$

$$\frac{\partial^2 H}{\partial u_2^{2}}=-B_3<0.$$

Because B_2 and B_3 is negative, it means that the second derivative test is satisfied so H has a maximum value in u_1 and u_2.

2. Formation of the optimal Hamiltonian function.

   The optimal Hamiltonian function can be written as follows.

$$\begin{aligned}H^{*}=&B_1P(t)-\frac{B_2}{2}u_1^{*2}(t)-\frac{B_3}{2}u_2^{*2}(t)+\lambda_p\left(a_1+l+v_1T(t)-\left(u_1^{*}(t)+u_2^{*}(t)\right)P(t)\right)\\ &\lambda_E\left(u_2^{*}(t)P(t)-(c+h)E(t)+oT(t)\right)+\lambda_T\left((c+h)E(t)+u_1^{*}(t)P(t)-aT(t)+(v_2+o)T(t)\right)\end{aligned}$$

$$\begin{aligned}H^{*}=&B_1P(t)-\frac{B_2}{2}\left(\min\left\{1,\max\left(0,\left(\frac{(\lambda_T-\lambda_p)P}{B_2}\right)\right)\right\}\right)^2-\frac{B_3}{2}\left(\min\left\{1,\max\left(0,\left(\frac{(\lambda_E-\lambda_p)P}{B_3}\right)\right)\right\}\right)^2\\ &+\lambda_p\left(a_1+l+v_1T(t)-\left(\left(\min\left\{1,\max\left(0,\left(\frac{(\lambda_T-\lambda_p)P}{B_2}\right)\right)\right\}\right)+\left(\min\left\{1,\max\left(0,\left(\frac{(\lambda_E-\lambda_p)P}{B_3}\right)\right)\right\}\right)\right)P(t)\right)\\ &\lambda_E\left(\left(\left(\min\left\{1,\max\left(0,\left(\frac{(\lambda_E-\lambda_p)P}{B_3}\right)\right)\right\}\right)\right)P(t)-(c+h)E(t)+oT(t)\right)+\lambda_T\left((c+h)E(t)+(v_2+o)T(t)\right)\\ &+\left(\min\left\{1,\max\left(0,\left(\frac{(\lambda_T-\lambda_p)P}{B_2}\right)\right)\right\}\right)P(t)-aT(t)\end{aligned}$$

3. Solve the state equation

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The state equation in optimal conditions can be obtained by deriving the optimal Hamiltonian function λ as follows.

$$\dot{x}^* = \frac{\partial H^*}{\partial \lambda}.$$

Based on this equation, a state equation is obtained for the amount of processing in (anaerobic) decomposition that can produce biogas.

$$\dot{P}^*(t) = +\left(\frac{\partial H^*}{\partial \lambda_p}\right),$$

$$\dot{P}^*(t) = a_1 + l + v_1 T - \left(\min\left\{1, \max\left(0, \left(\frac{(\lambda_T - \lambda_p)P}{B_2}\right)\right)\right\} + \min\left\{1, \max\left(0, \left(\frac{(\lambda_E - \lambda_p)P}{B_3}\right)\right)\right\}\right)P,$$

The state equation for the amount of renewable (plants, animals, water, wind, sun, etc) can be written as follows.

$$\dot{E}^*(t) = +\left(\frac{\partial H^*}{\partial \lambda_E}\right),$$

$$\dot{E}^*(t) = \min\left\{1, \max\left(0, \left(\frac{(\lambda_E - \lambda_p)P}{B_3}\right)\right)\right\}P - (c+h)E, +oT$$

The state equation for the number of plants influenced by the surrounding environment can be written as follows.

$$\dot{T}^*(t) = +\left(\frac{\partial H^*}{\partial \lambda_T}\right),$$

$$\dot{T}^*(t) = (c+h)E + \min\left\{1, \max\left(0, \left(\frac{(\lambda_T - \lambda_p)P}{B_2}\right)\right)\right\}P - aT - (v_2 + o)T,$$

4. Solve the costate equation
   The costate equation in optimal conditions is obtained by deriving the optimal (negative value) Hamiltonian function for the state, as follows

$$\dot{\lambda}^* = -\frac{\partial H^*}{\partial x}.$$

Based on this equation, a costate equation is obtained for the amount of processing in decomposition (anaerobic) that can obtain biogas.

$$\dot{\lambda}_p{}^*(t) = -\left(\frac{\partial H^*}{\partial P}\right),$$

$$\dot{\lambda}_p{}^*(t) = -P + \lambda_p\left(\min\left\{1, \max\left(0, \left(\frac{(\lambda_T - \lambda_p)P}{B_2}\right)\right)\right\} + \min\left\{1, \max\left(0, \left(\frac{(\lambda_E - \lambda_p)P}{B_3}\right)\right)\right\}\right)$$

$$-\lambda_E\left(\min\left\{1, \max\left(0, \left(\frac{(\lambda_T - \lambda_p)P}{B_2}\right)\right)\right\}\right) - \min\left\{1, \max\left(0, \left(\frac{(\lambda_E - \lambda_p)P}{B_3}\right)\right)\right\},$$

The costate equation for the amount of renewable energy (coming from fossil fuels, renewable natural resources (plants, animals, water, wind, sun, etc.), and nuclear) can be written as follows.

$$\dot{\lambda}_E^{\,*} = -\left(\frac{\partial H^*}{\partial E}\right).$$

$$\dot{\lambda}_E^{\,*} = -(\lambda_T - \lambda_E)(c + h).$$

The costate equation for the number of plants influenced by the surrounding environment can be written as follows:

$$\dot{\lambda}_T^{\,*} = -\left(\frac{\partial H^*}{\partial T}\right). \text{ and } \dot{\lambda}_T^{\,*} = -\lambda_p v_1 - \lambda_E o + \lambda_T (a + (v_1 + o)).$$

IV. Numerical Simulations

To address the optimal control problem mentioned earlier, the state equations and costate equations will be solved numerically using the Forward-Backward Sweep Runge Kutta Order 4 method implemented in MATLAB software. The state equation is solved using the Forward Sweep Runge Kutta Order 4 method. This method is suitable because the initial value is known in the state equation. Conversely, the costate equation is solved using the Backward Sweep Runge Kutta Order 4 method. This approach is preferable since the final value is known in the costate equation. Throughout all numerical simulations, the parameters specified in Table 1 are consistently employed.

Table 1: Parameters of The Model

Parameter	Description	Value	Source
a_1	The growth rate of processing in decomposition is influenced by animal manure which is taken for decomposition (animal manure)	0.26	Nugraheni, 2021
l	The concentration industrial or domestic waste that can be decomposed	0.3	Nugraheni, 2021
v_1	The concentration falling leaves which can become plant waste	0.5	Nugraheni, 2021
p	The concentration of the decomposition product which can be used as fertilizer for plants	0.4	Nugraheni, 2021
e	The concentration of the decomposition product which is a source of renewable energy (biogas)	0.2	Nugraheni, 2021
c	The concentration of CO_2	0.35	Nugraheni, 2021
h	The concentration of H_2O	0.3	Nugraheni, 2021
o	The concentration of oxygen needed to utilize renewable energy	0.1	Nugraheni, 2021
a	The rate of reduced plant growth is influenced by the concentration of animals that eat plants for their survival	0.54	Nugraheni, 2021
v_2	The concentration of aging plants or the process of rejuvenation	0.7	Nugraheni, 2021

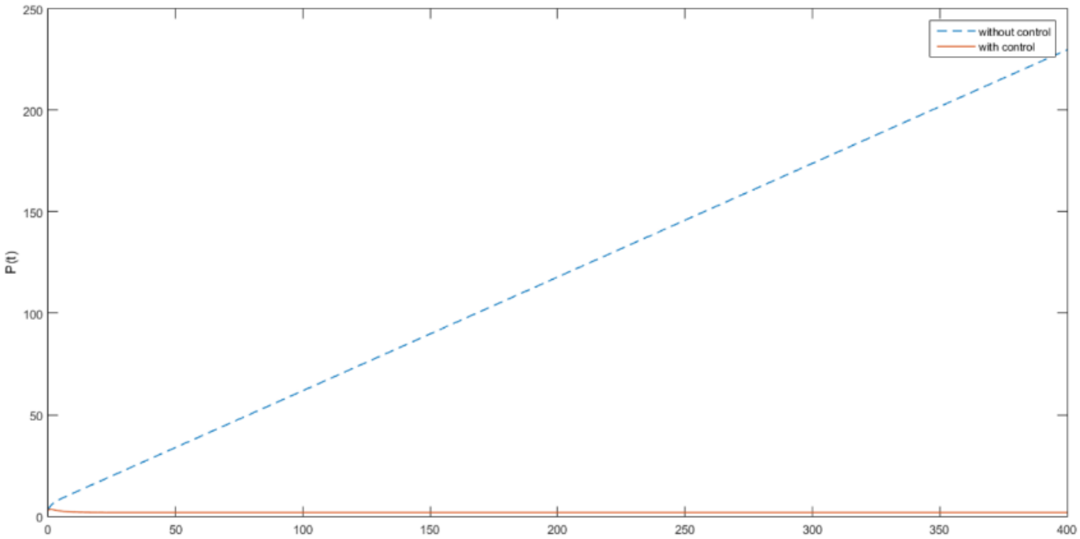

Figure 1. Illustration for optimal trajectory of $P(t)$ with control and without control

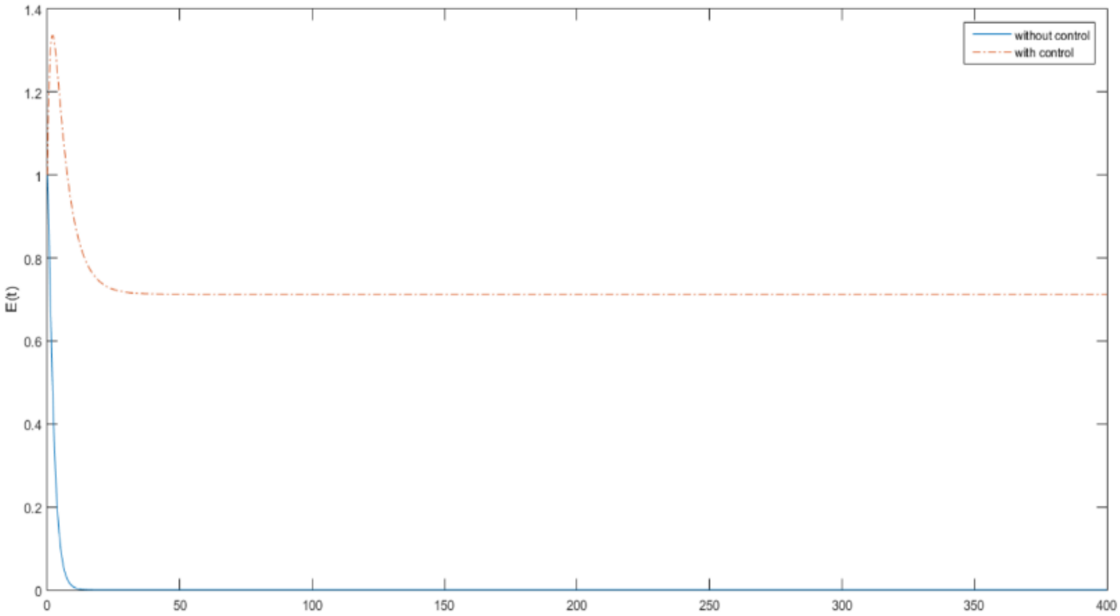

Figure 2. Illustration for optimal trajectory of $E(t)$ with control and without control

The numerical simulations depicted in Figures 1-3 illustrate the interaction of living populations in the production of biogas as a renewable energy alternative, both with and without control. The simulation outcomes lead us to the conclusion that the integration of control mechanisms $u_1(t)$ and $u_2(t)$ can effectively increase biogas production, positioning it as a sustainable energy source.

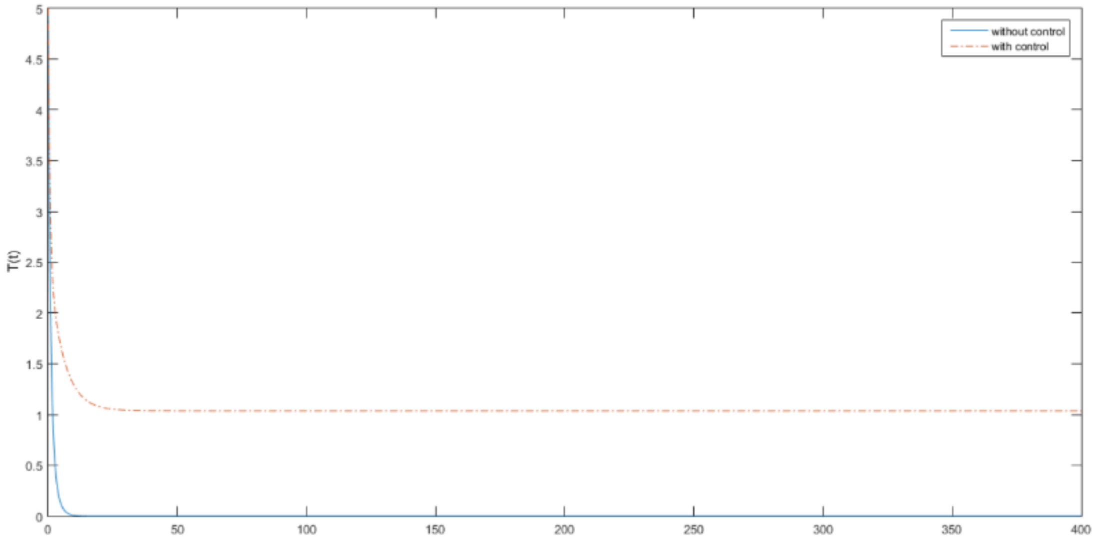

Figure 3. Illustration for optimal trajectory of $T(t)$ with control and without control

V. Conclusion

In this paper, a mathematical model of biogas production growth have been constructed with optimal control modifications. Completion of the model is carried out by forming an objective function, as well as determining the Hamilton function and Lagrange function. Numerical simulations in the model show that providing control can increase biogas production as a sustainable energy source.

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