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MIXTURE PURIFICATION MODEL WITH CASCADING TANK CONFIGURATION

Yanuar Bhakti Wira Tama[1*], Robby[2]

[1]*Departement of Mathematics, Institut Teknologi Kalimantan, Balikpapan, Indonesia*
[2] *Department of Mathematics, Faculty of Information Technology and Science, Parahyangan Catholic University, Jalan Ciumbuleuit 94 Bandung 40141, West Java, Indonesia*

Email: [1]yanuar.bhakti@lecturer.itk.ac.id, [2]robby@unpar.ac.id
*Corresponding Author

Abstract. Consider mixing problems which are often found in Calculus or Differential Equation courses. Under some assumptions, this problem can be used to model the purification process in a polluted mixture. In this case, the cascading configuration will be investigated for modelling the spread of pollution from one mixture to another. There are two main problems: finding time needed so the amount of pollutant in mixture inside the certain tank does not exceed certain threshold and finding the number of tanks needed so that the amount of mixture in the last tank does not exceed certain threshold. The solution for the second problem will be simplified by using Stirling approximation, which approximates factorial into exponential term. For the first problem, the time needed depends on the number of tanks, initial value of the pollutant, the rate of flow, and the volume of solution inside the tanks. For the second problem, the number of tanks only depends on the initial value of the pollutant.
Keywords: mixing problem, system of differential equations, Stirling approximation

I. INTRODUCTION

Consider mixing problems which are often found in Calculus or Differential Equation courses. This problem also can be found in [1], [2], [3]. Consider one tank filled with brine. Solution with certain concentration flows into tank at a certain rate. At the same time, the solution inside the tank flows outside with a certain rate. This problem can be modified into some configuration of tanks: the solution flows outside the upper tank goes inside the lower tank, and so on. The illustration can be seen in Figure 2. The problem often asked in this is the amount of salt in each of the tanks after a certain time. The assumption that is often used is the solution inside the tank will be mixed instantly, so there is no delay time involved when mixing happens. Under this assumption, this problem can be used to model the purification process in a polluted mixture.

While [4] already gave one of the approach for this problem and [5] already gave the analytic and asymptotic behavior of the solution, in this paper, different perspective of the problem will be investigated, especially in the cascading configuration. This model will give us linear ordinary differential equation of order n. Solving this, analytically and numerically, some of them can be seen in [6], [7], [8], [9], [10], [11].

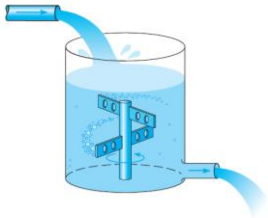

Figure 1. Illustration of One Tank Mixing Problem

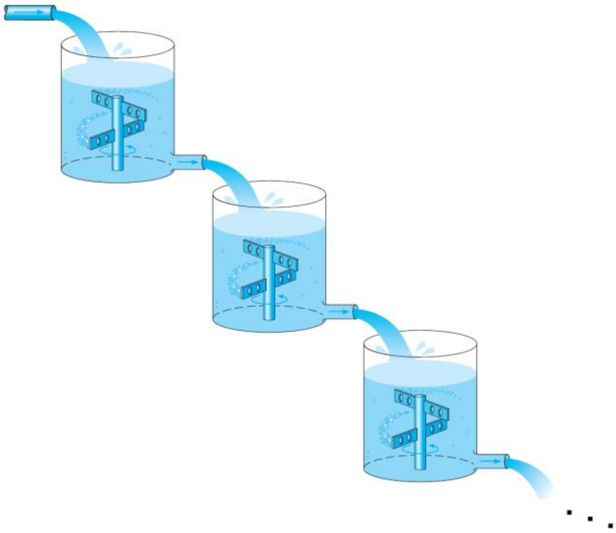

Figure 2. Illustration of Cascading Tanks Mixing Problem

The different perspectives are including the time needed to make the pollutant in a certain tank below a certain threshold and the number of tanks needed to make the last tank in the configuration have pollutant below some threshold any time. These two main problems can help water engineer to set up the cascading configuration to purify the lake with the efficient amount of cost and energy.

II. MATHEMATICAL MODEL

Consider N cascading tank mixing problem. Initially, tank 0 (the uppermost tank) is polluted with amount of a meanwhile other tanks are filled with pure water. Assume that the volume of solution inside all tanks is same, denoted by V. Pure water flows into tank 0 with rate r and the solution inside the tank flows outside with the same rate, so all volume inside the tank is constant. For simplicity, assume that the solution inside every tank is mixed instantly. Let $x_n(t)$ be the amount of salt inside the nth tank after time t, where $n \in \{0, 1, 2, \dots, N-1\}$.

Let $c \coloneqq \frac{r}{V}$. By the conservation of mass, the rate of amount of pollutant in tank 0 can be modelled as the initial value problem.

$$\frac{dx_0(t)}{dt} = 0 \cdot r - \frac{x_0(t)}{V} \cdot r = -c \cdot x_0(t), \qquad x_0(0) = a.$$

Moreover, the model for tank n, for $n \in \{1, 2, \dots, N-1\}$, are the following.

$$\frac{dx_n(t)}{dt} = \frac{x_{n-1}(t)}{V}\cdot r - \frac{x_n(t)}{V}\cdot r = -c\cdot\left(x_{n-1}(t) - x_n(t)\right), \qquad x_i(0) = 0,$$

where a denotes the amount of pollutant in the first tank, V is volume inside all tanks, and r is the flow rate of the solution between two tanks.

III. RESULTS

The amount of pollutant inside every tank can be solved inductively.

Theorem 1 *For $n \in \{0,1,2,\dots,N-1\}$ and $t \geq 0$, $x_n(t) = \frac{a(ct)^n e^{-ct}}{n!}$.*

Proof: For $n = 0$, the differential form is $\frac{dx_0(t)}{x_0(t)} = -c\,dt$. Integrating both sides gives

$$\ln\left(x_0(t)\right) = -ct + K_1 \Leftrightarrow x_0(t) = Ke^{-ct}$$

for some constant K. Using the information that $x_0(0) = a$, it can be obtained that $a = K$. Hence, $x_0(t) = ae^{-ct}$.

Assume that $x_k(t) = \frac{a(ct)^k e^{-ct}}{k!}$ holds for some $k \in \{1,2,\dots,N-1\}$. Notice this calculation.

$$\frac{dx_k(t)}{dt} = \frac{ac^k t^{k-1} e^{-ct}}{k!} - \frac{ac^k t^n c e^{-ct}}{k!} = c\cdot\left(\frac{a(ct)^{k-1}e^{-ct}}{(k-1)!} - \frac{a(ct)^k e^{-ct}}{k!}\right).$$

Therefore, $\frac{dx_k(t)}{dt} = c\cdot\left(x_{k-1}(t) - x_k(t)\right)$ and $x_k(0) = 0$ for all k. □

The first problem that will be investigated is the time needed for the certain tank reached the certain level of pollutant, which is finding $\tau > 0$ such that $x_n(\tau) = \varepsilon$ for some $n \in \{1,2,\dots,N-1\}$ and $\varepsilon > 0$. In this case, some special "function" needed to have the closed form.

Definition 1 [1] *Lambert W Function, denoted by $W(x)$, is the multi-valued inverse of the function $E(x) = xe^x$, which is $x = W(x)e^{W(x)}$.*

Other use of this function can be seen in [12], [13], [14], [15], [16], [17], [18], [19], [20], [21]. With the help of Lamber W Function, the first problem can be solved analytically.

Theorem 2 *The time needed to satisfy $x_n(\tau) = \varepsilon$ is $\tau = -\frac{n}{c}\cdot W\left(-\frac{1}{n}\sqrt[n]{\frac{\varepsilon\, n!}{a}}\right)$.*

Proof: From the equation $\frac{a(c\tau)^n e^{-c\tau}}{n!} = \varepsilon$, notice this calculation.

$$(c\tau)^n e^{-c\tau} = \frac{\varepsilon\, n!}{a} \Leftrightarrow -\frac{c\tau}{n}e^{-\frac{c\tau}{n}} = -\frac{1}{n}\sqrt[n]{\frac{\varepsilon\, n!}{a}}.$$

By the Lambert W Function, $-\frac{c\tau}{n} = W\left(-\frac{1}{n}\sqrt[n]{\frac{\varepsilon\, n!}{a}}\right)$. Therefore, $\tau = -\frac{n}{c}\cdot W\left(-\frac{1}{n}\sqrt[n]{\frac{\varepsilon\, n!}{a}}\right)$. □

Furthermore, if $W_k(t)$ denotes the kth brance of the Lambert W Function, the solution of the inequality $x_n(t) > \varepsilon$ is $t \in \left(-\frac{n}{c}\cdot W_0\left(-\frac{1}{n}\sqrt[n]{\frac{\varepsilon\, n!}{a}}\right), -\frac{n}{c}\cdot W_{-1}\left(-\frac{1}{n}\sqrt[n]{\frac{\varepsilon\, n!}{a}}\right)\right)$.

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For the second problem, the least number of tanks needed so all pollutant in the last tank does not exceed certain value will be investigated, which is finding N such that $x_{N-1}(t) < \varepsilon$ for $t > 0$. On the other side, the maximum value of pollutant in every tank also can be found.

Theorem 3 *The maximum value of* $x_n(t)$ *is* $M_n = \frac{an^n e^{-n}}{n!}$ *Attained when* $t = \frac{n}{c}$.
Proof: Differentiating $x_n(t)$ with respect to t gives the following.

$$x_n'(t) = \frac{ac^n t^{n-1} e^{-ct}}{n!} - \frac{ac^k t^n c e^{-ct}}{n!} = \frac{ac^n t^{n-1} e^{-ct}}{n!}(n - ct).$$

The stationary point (the only critical point in this case) is $n - ct = 0 \Leftrightarrow t = \frac{n}{c}$. By the first derivative test, this critical point yields (global) maximum value. Hence,

$$M_n = x_n\left(\frac{n}{c}\right) = \frac{a\left(c\cdot\frac{n}{c}\right)^n e^{-c\cdot n/c}}{n!} = \frac{an^n e^{-n}}{n!}. \ \square$$

For solving the second problem, a certain approximation is needed to simplify the computation.

Lemma 1 *(Stirling Approximation)* [22] $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.
Theorem 4 *The least number of tanks needed to satisfy* $x_{N-1}(t) < \varepsilon$ *is* $N = \left[\frac{a^2}{2\pi\varepsilon^2} + 1\right]$.
Proof: For all $t > 0$ and $N \in \mathbb{N}$, note that $x_{N-1}(t) \le M_{n-1} = \frac{a(N-1)^{N-1} e^{-(N-1)}}{(N-1)!}$. By Stirling Approximation, $M_{N-1} \approx \frac{a}{\sqrt{2\pi(N-1)}}$. So, $\frac{a}{\sqrt{2\pi(N-1)}} < \varepsilon \Leftrightarrow N > \frac{a^2}{2\pi\varepsilon^2} + 1$.
Therefore, choose $N = \left[\frac{a^2}{2\pi\varepsilon^2} + 1\right]$ so that the inequality still fulfilled. $\square$

Figure 4 gives an illustration for $a = 10$, $c = 1$, and $\varepsilon = 1$.

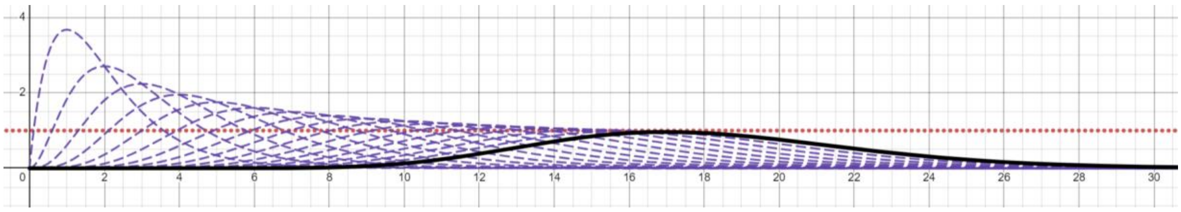

Figure 3. Illustration of Theorem 4 for $a = 10$, $c = 1$, and $\varepsilon = 1$

From Figure 3, for the given parameters, the plot of the solution for the amount of pollutant in tank 17 (thick black plot) are completely below the threshold $y = \varepsilon = 1$ (dotted red plot), which is consistent with the formula in Theorem 4.

IV. CONCLUSION AND FUTURE RESEARCH DIRECTION

Two main problems from the mixing problem of cascading tanks have already been discussed. This problem can help engineers to design some configuration to purify certain configuration of lake. For the first problem, if the configuration is fixed, then the time needed can be computed with the help of the formula $\tau = -\frac{n}{c} \cdot W\left(-\frac{1}{n}\sqrt[n]{\frac{\varepsilon\, n!}{a}}\right)$. For the second problem, if the configuration can be manipulated, then the route can be designed so that the least amount of lake can be solved by the help of the formula $N = \left\lceil \frac{a^2}{2\pi\varepsilon^2} + 1 \right\rceil$.

For the future research direction, other configurations can be considered, such as branched tanks illustrated in Figure 4. For this example, solution from tank 1 will be poured into two tanks below it with a certain proportion. Finding the amount of pollutant inside each tank will be more complicated than before.

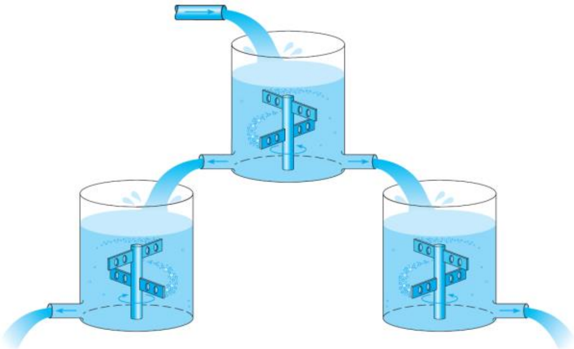

Figure 4. Illustration of Branched Tanks Mixing Problem

Another perspective that can be added to the model is considering the time needed for the pollutant to travel from one tank to another. This will modify the model into the system of delay differential equations, which can be solved by another method.

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