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SOME PROPERTIES OF ALMOST JOINTLY PRIME (R,S)-SUBMODULES

Dian Ariesta Yuwaningsih[1*], Burhanudin Arif Nurnugroho[2],
Diniar Mar'atus Sholiha[3], Lolita[4]

[1,3,4]*Department of Mathematics Education, Universitas Ahmad Dahlan, Yogyakarta, Indonesia*

[2]*Department of Magister Mathematics Education, Universitas Ahmad Dahlan, Yogyakarta, Indonesia*

Email: [1]dian.ariesta@pmat.uad.ac.id
*Corresponding Author

Abstract. Let R and S be rings with identity. The definition of prime submodule has been generalized to the almost prime submodule. In addition, the definition of prime submodule has also been carried over to the (R,S)-module structure, which is called jointly prime (R,S)-submodules. However, as a generalization of prime submodules, the concept of almost prime submodules has not been carried over to (R,S)-module structures. In this paper, we construct the definition of almost jointly prime (R,S)-submodules as the generalization of jointly prime (R,S)-submodules. We also present several necessary and sufficient conditions for an (R,S)-submodule to be an almost jointly prime (R,S)-submodule.

Keywords: almost prime, jointly almost prime, jointly prime, (R,S)-modules

I. INTRODUCTION

Every non-zero element is not a zero divisor in an integral domain. If we take any elements a and b of an integral domain with $ab = 0$, we have $a = 0$ or $b = 0$. A prime ideal definition emerges in a commutative ring with identity by generalizing the properties of element 0 (or the singleton $\{0\}$). Let R be a commutative ring with identity and I a proper ideal of R. Referring to [1] and [2], the ideal I is said to be prime if for each $a, b \in R$ with $ab \in I$ implies $a \in I$ or $b \in I$. As it develops, the prime ideal has been generalized into an almost prime ideal over commutative ring by Bhatwadekar and Sharma [3]. According to [3], a non-zero proper ideal I of R is called an almost prime ideal if for each $a, b \in R$ with $ab \in I \setminus I^2$ implies $a \in I$ or $b \in I$. Moreover, Abouhalaka and Fındık [4] have introduced the definition of almost prime ideals in non-commutative ring structures.

We already know that the primeness in the ring has been carried over into the module structure. The concept of the prime ideal has been carried over to the module structure into prime submodules. Following [5], a proper submodule P of R-module M is said to be prime if foe every $m \in M$ and $r \in R$ with $rm \in P$, implies $m \in P$ or $rM \subseteq P$. Khashan [6] has generalized a prime submodule into an almost prime submodule in a module over a commutative ring with identity. A proper submodule N of R-module M is said to be almost prime if for each $r \in R$ and $m \in M$ with $rm \in N \setminus (N:_R M)N$ implies $m \in N$ or $r \in (N:_R M)$. We recall that $(N:_R M) = \{r \in R \mid rM \subseteq N\}$ is the annihilator set of M/N. Furthermore, Beiranvand and Beyrandvand in [7] have defined an almost prime submodule in a module over the non-commutative ring. Following [7], a proper submodule N of M is said to be almost prime if for every submodule K of M and ideal I of R with $KI \subseteq N$ and $KI \nsubseteq (N:_R M)N$ implies $K \subseteq N$ or $I \subseteq (N:_R M)$. Furthermore, the concept of almost prime submodules has been generalized, such that almost semiprime submodules [8], n-almost prime submodules [9], almost strongly prime submodules [10], and $S(N)$-almost prime submodules [11], and almost S-rime submodules [12].

On the other hand, the primeness in the module structure has been carried over to the (R,S)-module structure by Khumprapussorn et al. [13]. In [13], Khumprarpussorn et al. have generalized the prime submodule to jointly prime (R,S)-submodules. Furthermore, several other researchers have generalized the jointly prime (R,S)-submodule, such as left R-prime (R,S)-submodules [14], left weakly jointly prime (R,S)-submodules [15]–[17], jointly β-prime (R,S)-submodules [18], jointly α-prime (R,S)-submodules [19]–[21], and jointly second (R,S)-submodules [22].

The concept of almost prime submodules has never been brought to (R,S)-module structures. Therefore, through this research, we will develop the almost prime submodule into an (R,S)-module structure, which is after this called almost jointly prime (R,S)-submodules. We show that this almost jointly prime (R,S)-submodule is a generalization of the jointly prime (R,S)-submodule in [13]. Moreover, we also present some properties of the almost jointly prime (R,S)-submodule.

II. THE DEFINITION OF ALMOST JOINTLY PRIME (R,S)-SUBMODULES

Throughout this paper, R and S are rings with identity and M an (R,S)-modules, unless otherwise stated. Before we define the almost jointly prime (R,S)-submodule, here we give a proposition in [13] which will used in this research.

Proposition 1 [13] *Let R and S be rings, N an (R,S)-submodule of M, $X \subseteq R$ a non-empty set, and $Y \subseteq S$ a non-empty set. If M satisfy $a \in RaS$ for every $a \in M$, then:*

1. *If $(RX)MS \subseteq N$, then $XMS \subseteq N$. Moreover, $XMS \subseteq (XR)MS$.*
2. *If $RM(YS) \subseteq N$, then $RMY \subseteq N$. Moreover, $RMY \subseteq RM(SY)$.*
3. *$W \subseteq RWS$ for every non-empty set W of M. If W is an (R,S)-submodule of M, then $W = RWS$.*

Furthermore, let M be an (R,S)-module and N an (R,S)-submodule $of\ M$. Then, we define the annihilator of factor (R,S)-module M/N is the set $(N:_R M) = \{r \in R \mid rM \subseteq N\}$. Following [13], if the ring S satisfy $S^2 = S$ then we can show that the set $(N:_R M)$ form an ideal of R. Below we give the definition of jointly prime (R,S)-submodule.

Definition 1 [13] *Let R and S be rings and M an (R,S)-module. A proper (R,S)-submodule N of M is called jointly prime if for each (R,S)-submodule K of M, left ideal I of R, and right ideal J of S with $IKJ \subseteq N$ implies $IMJ \subseteq N$ or $K \subseteq N$.*

Next, here we give another version of the definition of jointly prime (R, S)-submodule.

Definition 2 [13] *Let R and S be rings and M an (R,S)-module satisfy $RMS = M$. A proper (R,S)-submodule N of M is called jointly prime if for each (R,S)-submodule K of M, ideal I of R, and ideal J of S with $IKJ \subseteq N$ implies $IMJ \subseteq N$ or $K \subseteq N$.*

In the following, we define almost jointly prime (R, S)-submodules.

Definition 3 *Let M be an (R, S)-module and N a proper (R, S)-submodule of M. The submodule N is called an almost jointly prime if for every (R, S)-submodule K of M, left ideal I of R, and right ideal J of S with $IKJ \subseteq N$ and $IKJ \nsubseteq (N:_R M)NS$ implies $K \subseteq N$ or $IMJ \subseteq N$.*

We can show that the almost jointly prime (R, S)-submodule is a generalization of the jointly prime (R, S)-submodule in the following proposition.

Proposition 1 *Let M be an (R, S)-module and P a proper (R, S)-submodule of M. If P is a jointly prime (R, S)-submodule of M, then P is an almost jointly prime (R, S)-submodule of M.*

Proof: Let any left ideal I of R, right ideal J of S, and (R, S)-submodule N of M with $INJ \subseteq P$ and $INJ \nsubseteq (P:_R M)PS$. Since P is a jointly prime (R, S)-submodule of M, then from $INJ \subseteq P$ we obtain $IMJ \subseteq P$ or $N \subseteq P$. Hence, P is an almost jointly prime (R, S)-submodule of M. □

Next, we give several examples of almost jointly prime (R, S)-submodules.

Example 1 Let $\mathbb{Z}$ be an $(2\mathbb{Z}, 3\mathbb{Z})$-module and $6\mathbb{Z}$ an $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$. We show that $6\mathbb{Z}$ is an almost jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$. Let any ideal $I = (2k)\mathbb{Z}$ of $2\mathbb{Z}$, ideal $J = (3l)\mathbb{Z}$ of $3\mathbb{Z}$, and $(2\mathbb{Z}, 3\mathbb{Z})$-submodule $(3m + 1)\mathbb{Z}$ di $\mathbb{Z}$, for an element $k, l, m \in \mathbb{Z}^+$. Considering the set $(6\mathbb{Z}:_{2\mathbb{Z}} \mathbb{Z}) = \{r \in 2\mathbb{Z} \mid r\mathbb{Z}(3\mathbb{Z}) \subseteq 6\mathbb{Z}\}$, so that $(6\mathbb{Z}:_{2\mathbb{Z}} \mathbb{Z}) = (2n)\mathbb{Z}$, for an element $n \in \mathbb{Z}^+$. Thus, we obtain $(6\mathbb{Z}:_{2\mathbb{Z}} \mathbb{Z})(6\mathbb{Z})(3\mathbb{Z}) = (36n)\mathbb{Z}^3$. From this, we obtain $I(3m + 1)\mathbb{Z}J = (18klm + 6kl)\mathbb{Z}^3 \subseteq 6\mathbb{Z}$ and $I(3m + 1)\mathbb{Z}J = (18klm + 6kl)\mathbb{Z}^3 \nsubseteq (6\mathbb{Z}:_{2\mathbb{Z}} \mathbb{Z})(6\mathbb{Z})(3\mathbb{Z})$ but $(3m + 1)\mathbb{Z} \nsubseteq 6\mathbb{Z}$. Consequently, we get $I\mathbb{Z}J = (6kl)\mathbb{Z}^3 \subseteq 6\mathbb{Z}$. Thus, $6\mathbb{Z}$ is an almost jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$. □

Example 2 Let $\mathbb{Z}$ be an $(2\mathbb{Z}, 2\mathbb{Z})$-module and $4\mathbb{Z}$ an $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$. We show that $4\mathbb{Z}$ is an almost jointly prime $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$. Let any ideal $I = (2k)\mathbb{Z}$ of $2\mathbb{Z}$, ideal $J = (2l)\mathbb{Z}$ of $2\mathbb{Z}$, and $(2\mathbb{Z}, 2\mathbb{Z})$-submodule $(2m + 1)\mathbb{Z}$ of $\mathbb{Z}$, for an element $k, l, m \in \mathbb{Z}^+$. Considering the set $(4\mathbb{Z}:_{2\mathbb{Z}} \mathbb{Z}) = \{r \in 2\mathbb{Z} \mid r\mathbb{Z}(2\mathbb{Z}) \subseteq 4\mathbb{Z}\}$, so that $(4\mathbb{Z}:_{2\mathbb{Z}} \mathbb{Z}) = (2n)\mathbb{Z}$, for an element $n \in \mathbb{Z}^+$. Thus, we obtain $(4\mathbb{Z}:_{2\mathbb{Z}} \mathbb{Z})(4\mathbb{Z})(2\mathbb{Z}) = (16n)\mathbb{Z}^3$. From this, we obtain $I(2m + 1)\mathbb{Z}J = (8klm + 4kl)\mathbb{Z}^3 \subseteq 4\mathbb{Z}$ and $I(2m + 1)\mathbb{Z}J = (8klm + 4kl)\mathbb{Z}^3 \nsubseteq (4\mathbb{Z}:_{2\mathbb{Z}} \mathbb{Z})(4\mathbb{Z})(2\mathbb{Z})$ but $(2m + 1)\mathbb{Z} \nsubseteq 4\mathbb{Z}$. Consequently, we get $I\mathbb{Z}J = (4kl)\mathbb{Z}^3 \subseteq 4\mathbb{Z}$. Hence, $4\mathbb{Z}$ is an almost jointly prime $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$. □

III. SOME PROPERTIES OF ALMOST JOINTLY PRIME (R, S)-SUBMODULES

In this section, we present some properties of the almost jointly prime (R, S)-submodule. First, below are given the necessary and sufficient conditions for an (R, S)-submodule to be an almost jointly prime (R, S)-submodule.

Proposition 1 *Let M be an (R, S)-module with $RMS = M$ and P a proper (R, S)-submodule of M. The submodule P is an almost jointly prime (R, S)-submodule if and only if for each ideal I of R, ideal J of S, and (R, S)-submodule N of M such that $INJ \subseteq P$ and $INJ \nsubseteq (P:_R M)PS$, implies $IMJ \subseteq P$ or $N \subseteq P$.*

Proof: Let any ideal I of R, ideal J of S, and (R, S)-submodule N of M such that $INJ \subseteq P$ and $INJ \nsubseteq (P:_R M)PS$. Since P is an almost jointly prime (R, S)-submodule, we have $IMJ \subseteq P$ or $N \subseteq P$. Conversely, let any left ideal I of R, right ideal J of S, and (R, S)-submodule N of M such that $INJ \subseteq P$ and $INJ \nsubseteq (P:_R M)PS$. It is clear that IR is an ideal of R and SJ is an ideal of S. Moreover, we obtain $(IR)N(SJ) = I(RNS)J = INJ \subseteq P$ and $(IR)N(SJ) \nsubseteq (P:_R M)PS$.

Referring to the hypothesis, we obtain $(IR)M(SJ) \subseteq P$ or $N \subseteq P$. Thus, $IMJ \subseteq P$ or $N \subseteq P$. Hence, P is an almost jointly prime (R,S)-submodule of M. □

Example 1 Let $\mathbb{Z}$ be an $(\mathbb{Z},\mathbb{Z})$-module and $3\mathbb{Z}$ an $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}$. We show that $3\mathbb{Z}$ is an almost jointly prime $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}$. Obviously that $\mathbb{Z}\mathbb{Z}\mathbb{Z} = \mathbb{Z}$. Let any ideal $I = (2k+1)\mathbb{Z}$ of $\mathbb{Z}$, ideal $J = l\mathbb{Z}$ of $\mathbb{Z}$, and $(\mathbb{Z},\mathbb{Z})$-submodule $(2m)\mathbb{Z}$ of $\mathbb{Z}$, for an element $k,l,m \in \mathbb{Z}^+$. Considering the set $(3\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z}) = \{r \in \mathbb{Z} \mid r\mathbb{Z}(\mathbb{Z}) \subseteq 3\mathbb{Z}\}$, so that $(3\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z}) = (3n)\mathbb{Z}$, for an element $n \in \mathbb{Z}^+$. Thus, we obtain $(3\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z})(3\mathbb{Z})(\mathbb{Z}) = (9n)\mathbb{Z}^3$. From this, we obtain $I(2m)\mathbb{Z}J = (4klm + 2ml)\mathbb{Z}^3 = (2k+1)(2ml)\mathbb{Z}^3 \subseteq 3\mathbb{Z}$ and $I(2m+1)\mathbb{Z}J \nsubseteq (4\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z})(4\mathbb{Z})(\mathbb{Z})$ but $(2m)\mathbb{Z} \nsubseteq 3\mathbb{Z}$. Consequently, we get $I\mathbb{Z}J = (2k+1)l\mathbb{Z}^3 \subseteq 3\mathbb{Z}$. Hence, $3\mathbb{Z}$ is an almost jointly prime $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}$. □

Next, we present several necessary and sufficient conditions for an (R,S)-submodule to be almost jointly prime.

Proposition 2 *Let M an (R,S)-module satisfy $a \in RaS$ for each $a \in M$ and P a proper (R,S)-submodule of M. The following statements are equivalent:*

1. *P is an almost jointly prime (R,S)-submodule.*
2. *For every right ideal I of R, $m \in M$, and left ideal J of S such that $ImJ \subseteq P$ and $ImJ \nsubseteq (P:_R M)PS$, implies $m \in P$ or $IMJ \subseteq P$.*
3. *For every right ideal I of R, (R,S)-submodule N of M, and left ideal J of S such that $INJ \subseteq P$ and $INJ \nsubseteq (P:_R M)PS$, implies $N \subseteq P$ or $IMJ \subseteq P$.*
4. *For every left ideal I of R, $m \in M$, and right ideal J of S such that $(IR)m(SJ) \subseteq P$ and $(IR)m(SJ) \nsubseteq (P:_R M)PS$, implies $m \in P$ or $IMJ \subseteq P$.*
5. *For every element $a \in R$, $m \in M$, and $b \in S$ such that $(aR)m(Sb) \subseteq P$ and $(aR)m(Sb) \nsubseteq (P:_R M)PS$, implies $aMb \subseteq P$ or $m \in P$.*

Proof: $(1 \Rightarrow 2)$. Let any right ideal I of R, $m \in M$, and left ideal J of S such that $ImJ \subseteq P$ and $ImJ \nsubseteq (P:_R M)PS$. We have $(RI)(RmS)(JS) = R(IR)m(SJ)S \subseteq R(ImJ)S$. Since $ImJ \subseteq P$, we obtain $R(ImJ)S \subseteq P$. So, we get $(RI)(RmS)(JS) \subseteq P$. On the other hand, we have $R(ImJ)S \nsubseteq R(P:_R M)PSS \subseteq (P:_R M)PS$, so $(RI)(RmS)(JS) \nsubseteq (P:_R M)PS$. Since P is an almost jointly prime (R,S)-submodule, we obtain $(RI)M(JS) \subseteq P$ or $RmS \subseteq P$. Thus, we get $IMJ \subseteq P$ or $m \in P$.

$(2 \Rightarrow 3)$. Let any right ideal I of R, (R,S)-submodule N of M, and left ideal J of S such that $INJ \subseteq P$ and $INJ \nsubseteq (P:_R M)PS$ but $N \nsubseteq P$. Next, let any element $n \in N \setminus P$. We have $InJ \subseteq INJ \subseteq P$ and $InJ \nsubseteq (P:_R M)PS$. Referring to the hypothesis, we obtain $IMJ \subseteq P$.

$(3 \Rightarrow 4)$. Let any left ideal I of R, $m \in M$, and right ideal J of S such that $(IR)m(SJ) \subseteq P$ and $(IR)m(SJ) \nsubseteq (P:_R M)PS$. It is clear that IR is an ideal of R and SJ is an ideal of S. So, we have $(IR)(RmS)(SJ) = (IRR)m(SSJ) \subseteq (IR)m(SJ) \subseteq P$ and $(IR)(RmS)(SJ) \nsubseteq (P:_R M)PS$. Referring to the hypothesis, we obtain $(IR)M(SJ) \subseteq P$ or $RmS \subseteq P$. Thus, we get $IMJ \subseteq P$ or $m \in P$.

$(4 \Rightarrow 1)$. Let any left ideal I of R, right ideal J of S, (R,S)-submodule N of M such that $INJ \subseteq P$ and $INJ \nsubseteq (P:_R M)PS$ but $N \nsubseteq P$. Next, let any element $n \in N \setminus P$. Since N is an (R,S)-submodule of M, we obtain $N = RNS$ so that we get $(IR)N(SJ) = INJ \subseteq P$. Hence, we obtain $(IR)n(SJ) \subseteq (IR)N(SJ) \subseteq P$ and $(IR)n(SJ) \nsubseteq (P:_R M)PS$. Referring to the hypothesis, we get $IMJ \subseteq P$. Thus, P is an almost jointly prime (R,S)-submodule of M.

$(2 \Rightarrow 5)$. Let any element $a \in R$, $m \in M$, and $b \in S$ such that $(aR)m(Sb) \subseteq P$ and $(aR)m(Sb) \nsubseteq (P:_R M)PS$. Since aR is a right ideal of R, Sb is a left ideal of S, then based on the hypothesis, we obtain $(aR)M(Sb) \subseteq P$ or $m \in P$. Thus, we get $aMb \subseteq P$ or $m \in P$.

$(5 \Rightarrow 3)$. Let any right ideal I of R, (R,S)-submodule N of M, and left ideal J of S such that $INJ \subseteq P$ and $INJ \nsubseteq (P:_R M)PS$ but $N \nsubseteq P$. Next, let any element $n \in N \setminus P$, we get $InJ \subseteq INJ \subseteq P$ and $InJ \nsubseteq (P:_R M)PS$. Let any element $a \in I$ and $b \in J$. We obtain $aR \subseteq I$ and $Sb \subseteq J$. Hence, we get $(aR)n(Sb) \subseteq InJ \subseteq P$ and $(aR)n(Sb) \nsubseteq (P:_R M)PS$. Referring to the hypothesis, we obtain $aMb \subseteq P$. Thus, we have $IMJ \subseteq P$. □

Proposition 3 L*et M an (R,S)-module satisfy $a \in RaS$ for every $a \in M$ and P a proper (R,S)-submodule of M. The following statements are equivalent:*

1. *P is an almost jointly prime (R,S)-submodule.*
2. *For every element $a \in R$, (R,S)-submodule L of M, dan element $b \in S$ with $(RaR)L(SbS) \subseteq P$ and $(RaR)L(SbS) \nsubseteq (P:_R M)PS$, implies $L \subseteq P$ or $aMb \subseteq P$.*
3. *For every element $a \in R$, (R,S)-submodule L of M, and element $b \in S$ with $(Ra)L(bS) \subseteq P$ and $(Ra)L(bS) \nsubseteq (P:_R M)PS$, implies $L \subseteq P$ or $aMb \subseteq P$.*
4. *For every element $a \in R$, (R,S)-submodule L of M, and element $b \in S$ with $(aR)L(Sb) \subseteq P$ and $(aR)L(Sb) \nsubseteq (P:_R M)PS$, implies $L \subseteq P$ or $aMb \subseteq P$.*

Proof: $(1 \Rightarrow 2)$. Let any element $a \in R$, (R,S)-submodule L of M, and element $b \in S$ such that $(RaR)L(SbS) \subseteq P$ and $(RaR)L(SbS) \nsubseteq (P:_R M)PS$. Since L is a submodule of M, we get $(Ra)L(bS) = (RaR)L(SbS) \subseteq P$ and $(Ra)L(bS) \nsubseteq (P:_R M)PS$. Since Ra is a left ideal of R, bS is a right ideal of S, and P is an almost jointly prime (R,S)-submodule, we have $L \subseteq P$ or $(Ra)M(bS) \subseteq P$. Since $aMb \subseteq (Ra)M(bS)$, we get $L \subseteq P$ or $aMb \subseteq P$.

$(2 \Rightarrow 3)$. Let any element $a \in R$, (R,S)-submodule L of M, and element $b \in S$ with $(Ra)L(bS) \subseteq P$ and $(Ra)L(bS) \nsubseteq (P:_R M)PS$. Since L is a submodule of M, we get $L = RLS$ so that $(RaR)L(SbS) \subseteq P$ and $(RaR)L(SbS) \nsubseteq (P:_R M)PS$. Referring to the hypothesis, we obtain $L \subseteq P$ or $aMb \subseteq P$.

$(3 \Rightarrow 1)$. Let any left ideal I of R, right ideal J of S, and (R,S)-submodule L of M such that $ILJ \subseteq P$ and $ILJ \nsubseteq (P:_R M)PS$. Let any element $a \in I$ dan $b \in J$. We obtain $Ra \subseteq I$, $bS \subseteq J$ so that $(Ra)L(bS) \subseteq ILJ \subseteq P$ and $(Ra)L(bS) \nsubseteq (P:_R M)PS$. Referring to the hypothesis, we get $L \subseteq P$ or $aMb \subseteq P$. Thus, we obtain $L \subseteq P$ or $IMJ \subseteq P$.

$(4 \Rightarrow 2)$. Let any element $a \in R$, (R,S)-submodule L of M, and element $b \in S$ such that $(RaR)L(SbS) \subseteq P$ and $(RaR)L(SbS) \nsubseteq (P:_R M)PS$. Since R and S are rings with identity, we get $(aR)L(Sb) \subseteq (RaR)L(SbS) \subseteq P$ and $(aR)L(Sb)) \nsubseteq (P:_R M)PS$. Referring to the hypothesis, we obtain $L \subseteq P$ or $aMb \subseteq P$.

$(2 \Rightarrow 4)$. Let any element $a \in R$, (R,S)-submodule L of M, and element $b \in S$ such that $(aR)L(Sb) \subseteq P$ and $(aR)L(Sb) \nsubseteq (P:_R M)PS$. Considering that $R(aR)L(Sb)S \subseteq RPS = P$ and $R(aR)L(Sb)S \nsubseteq (P:_R M)PSS \subseteq (P:_R M)PS$. Referring to the hypothesis, we obtain $L \subseteq P$ or $aMb \subseteq P$. □

IV. CONCLUSIONS AND FUTURE RESEARCH DIRECTION

Prime submodules have been generalized to almost prime submodules. The definition and properties of almost prime submodules have been extended to the structure of (R,S)-modules to almost jointly prime (R,S)-submodules. The results of this research will be used as a reference for further research on almost prime submodules in the left multiplication (R,S)-module and the almost jointly prime radical of (R,S)-module.

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