OPTIMIZATION ALGORITHMS FOR PROJECTILE MOTION: MAXIMIZING RANGE AND DETERMINING OPTIMAL LAUNCH ANGLE

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Abstract. In this paper, we extensively explore parameter optimization for projectile trajectory. Our main goal is to find the best launch angle and initial velocity for maximum range. We employ five optimization methods: Nelder-Mead, Powell, Limited-memory Broyden-Fletcher-Goldfarb-Shanno with Box constraints L-BFGS-B, Truncated Newton Conjugate-Gradient TNC, and Sequential Least Squares Quadratic Programming SLSQP algorithms, examining their impact. We conduct simulations and provide visual representations of the trajectories, along with comparative charts to highlight algorithm performance. Powell's method stands out as the most promising among the algorithms for achieving the desired goal. Furthermore, the theoretical aspect was strongly present to support the proposed approach. Finally, numerical results were implemented using Python 3.12.0.

Keywords: optimization algorithms, parameter optimization, launch angle, initial velocity, Projectile motion.

I. INTRODUCTION

Projectile motion is a fundamental topic in the world of physics and engineering that has a long intrigued scientists because of its broad application in fields as diverse as sports, defense and space exploration. Optimizing key launch parameters, including launch angle and initial velocity in particular, plays an indispensable role in increasing projectile performance [1, 2, 3, 4]. Previous inquiries have elucidated the complex interplay between these parameters and the resulting trajectory characteristics. Of note among these investigations is the study conducted by Mustafa Karadag in 2019, which study for determining the launch angle that maximises the total distance travelled by the projectile during its flight in the projectile motion [5]. While work by Regodić et al in 2020 use of integration methods for the calculation movement of the projectile at a time with the influence of wind, coriolis inertial force due to rotation of the Earth, the reactive force and the gravitational acceleration [6]. These investigations have, among other things, enriched our understanding of the complex nuances associated with optimizing projectile motion. Moreover, contemporary advancements in optimization techniques and computational resources have significantly expanded the scope of research pertaining to projectile motion. The investigation conducted by Kahrazi and Kabudian in 2020 also showcased the application of a novel metaheuristic algorithm for globally optimizing projectile trajectories [7]. Similarly, recent contributions by Roux and colleagues in 2022 harnessed the capabilities of machine learning algorithms for the estimation of projectile trajectories through the utilization of a Long Short-Term Memory (LSTM) approach, emphasizing the pivotal role of data-driven methodologies within this domain [8]. On the other hand, Bokhari, Ahmed, et

al. contributed in presenting their exciting achievement in projectile motion using the calculus of three-parameter Mittag-Leffler functions [9]. Also, scientific contribution through the paper completed by Escobar, Isabel et al. "Projectile motion reconsidered: Does the distance between the projectile and the object always increase?" A promising echo of important work in this field [10].Up to the year 2023, the scene was more dramatic in this regard, as the work presented by Ahmed and others "High-dimensional uncertainty quantification of projectile motion in the barrel of a truck-mounted howitzer based on probability density evolution method" was also a wonderful achievement within the analysis and exploration of projectile motion [11]. These developments underscore the dynamic and continually evolving landscape of projectile motion research, underscoring the potential for innovative solutions.

Despite progress in this area, gaps and opportunities remain unexplored. This paper delves into the field of projectile motion optimization, delving into the complex interplay between different optimization methodologies and their impact on the projectile trajectory. By using a comprehensive set of optimization algorithms, including Nelder-Mead, Powell, L-BFGS-B, TNC, and SLSQP, our goal is to provide insight into the complex relationship between algorithm choice and the resulting trajectories taken by projectiles. While previous studies have individually evaluated the effectiveness of specific optimization algorithms, comprehensive comparative analysis that systematically evaluates multiple algorithms under a range of conditions is still relatively rare. This research fills this gap by conducting an in-depth examination of the performance of five prominent optimization methods in the context of projectile motion. By investigating their effects on launch angles, initial velocities, and resulting trajectories, this study contributes to a nuanced understanding of algorithmic choices and their implications for real-world scenarios involving projectile motion optimization.

II. PROBLEM STATEMENT

Given a projectile motion scenario in which a projectile is launched from an initial point with the goal of achieving maximum range, the problem can be mathematically defined as follows:

Let θ represent the launch angle in degrees, and v_0 denote the initial velocity of the projectile. The goal is to ascertain the optimal values of both θ and v_0 that maximize the projectile's horizontal displacement prior to impacting the ground. This can be succinctly expressed as a mathematical optimization problem:

$$
\begin{aligned}
\text{Maximize} \quad & R(\theta, v_0) \\
\text{Subject to} \quad & 0 \leq \theta \leq 90 \\
& v_0 > 0
\end{aligned}
$$

In this context, $R(\theta, v_0)$ denotes the horizontal displacement of the projectile, expressed as a function of both the launch angle θ and the initial velocity v_0. The specified constraints guarantee that the launch angle falls within the physically feasible range of 0 to 90 degrees, and that the initial velocity is a positive value.

The primary aim of this research endeavor is to systematically explore and contrast various

optimization algorithms, leveraging them to efficiently and precisely tackle this problem. The end goal is to determine the optimal launch angle θ and initial velocity v_0 that result in the maximum attainable range for the given projectile motion scenario.

III. METHODOLOGY

To address the intricate task of optimizing parameters within the domain of projectile motion, the following methodological approach is invoked. This endeavor involves the formulation of an optimization problem, which revolves around the precise determination of the launch angle denoted as θ and the initial velocity marked as v_0, with the ultimate objective of maximizing the horizontal range, a function encapsulated by $R(\theta, v_0)$. The methodology hinges on the utilization of fundamental mathematical expressions, which are delineated as follows:

$$t = \frac{2v_0 \sin(\theta)}{g} \tag{1}$$

$$R(\theta, v_0) = v_0 \cos(\theta) \cdot t \tag{2}$$

Herein, 't' symbolizes the elusive time of flight, while 'g' represents the gravitational constant, denoting the acceleration due to gravity.

The method further encompasses the conscientious selection of five well-established optimization algorithms, namely Nelder-Mead, Powell, L-BFGS-B, TNC, and SLSQP, meticulously chosen on account of their diverse and sophisticated optimization strategies [12, 13, 14, 15]. Notably, these algorithms demonstrate exceptional proficiency in addressing a broad spectrum of optimization problems, whether constrained or unconstrained. Each of the algorithms is meticulously imbued with the objective function to be maximized, encapsulated within the negative of the range function, denoted as $-R(\theta, v_0)$. Furthermore, rigorous constraints are imposed to restrict the permissible parameter space, namely $0 \leq \theta \leq 90$ and $v_0 > 0$. These constraints are meticulously integrated into each algorithm's optimization framework.

The initiation of the optimization process is marked by a meticulously chosen initial estimate for θ and v_0, expertly set at $[45, 20]$. This considered choice symbolizes a judicious starting point for the ensuing optimization endeavor.

With each algorithm, a precise and methodical optimization process is executed to ascertain the optimal values of θ and v_0 that effectively maximize the projectile range. Following this optimization achievement, the ensuing phase involves the meticulous simulation of the trajectories of the projectiles. These trajectories are defined with utmost precision through mathematical expressions, which are artfully captured as:

$$x(t) = v_0 \cos(\theta) \cdot t \tag{3}$$

$$y(t) = v_0 \sin(\theta) \cdot t - \frac{1}{2}gt^2 \tag{4}$$

Where $x(t)$ represents the horizontal position of the projectile at time 't', $y(t)$ represents

the vertical position (height) of the projectile at time 't' and 'g' represents the gravitational constant, which denotes the acceleration due to gravity. In most cases, this value is approximately 9.81 m/s² on the surface of the Earth. Of paramount importance, the parameter 't' is allowed to vary within a carefully defined range, commencing from 0 and extending to the time of flight, a crucial determinant acquired from the preceding optimization endeavor.

In the pursuit of a comprehensive evaluation, the results engendered by each optimization algorithm are subjected to a thorough, methodical analysis. These results, laden with profound implications, are conveyed through an array of meticulously crafted visual representations. These include the presentation of trajectory curves, a comparative examination of the maximum range attained, and a nuanced exploration of launch angle disparities. The visual depictions offered by these representations are an indispensable asset, proffering nuanced insights into the performance characteristics of the various algorithms and their profound influence on the optimization of projectile motion.

In summary, the proposed methodology is an intricate, multifaceted endeavor. It elegantly encompasses the formulation of a nuanced optimization problem, the expert application of a suite of diverse algorithms, a meticulous simulation of projectile trajectories, and a comprehensive, insightful analysis of results. This methodological approach collectively addresses the overarching research objectives, which revolve around the nuanced pursuit of maximizing projectile range through the artful application of algorithmic optimization techniques.

IV. EXPERIMENTAL SETUP

In the context of optimizing projectile motion parameters, the experimental setup has been meticulously crafted to rigorously evaluate the performance of distinct optimization algorithms. To ensure a thorough and methodical assessment, the following steps are meticulously undertaken:

1. **Algorithm Implementation:** The five chosen optimization algorithms (Nelder-Mead, Powell, L-BFGS-B, TNC, and SLSQP) are operationalized through the dedicated functions provided within the `scipy.optimize` library. Each algorithm is configured to maximize the negative range ($-R(\theta, v_0)$) while adhering to the constraints $0 \leq \theta \leq 90$ and $v_0 > 0$.

2. **Initial Parameter Guess:** A balanced initial guess for the launch angle θ and initial velocity v_0 is set to $[45, 20]$. This choice aims to avoid favoring any specific algorithm and ensures that the optimization process commences from a reasonable starting point.

3. **Algorithmic Optimization:** For each algorithm, the optimization process is carried out. The objective function ($-R(\theta, v_0)$) and constraints are fed into the optimization routines, which iteratively adjust the parameter values to maximize the range. Convergence criteria are defined to halt the optimization process once a suitable solution is achieved.

4. **Trajectory Simulation:** Subsequent to the optimization process, the derived values of the launch angle θ and initial velocity v_0 are extracted from the solution produced by each algorithm. These optimized parameters are then employed to simulate the trajectories of the projectiles. The 'x(t)' and 'y(t)' coordinates of the projectile's path are computed using the fundamental kinematic equations for projectile motion.

5. **Visualization and Analysis:** To enable a comprehensive comparison, the trajectories generated by each algorithm are elegantly visualized within a single plot. This amalgamation affords a holistic perspective on the distinct paths traversed by the projectiles under the influence of various optimization strategies. Furthermore, dedicated visualizations are crafted to showcase the maximum ranges achieved by each algorithm, along with the corresponding launch angles. In addition to the visual assessments, a rigorous quantitative analysis is conducted to thoroughly evaluate the efficacy and efficiency of each algorithm in attaining the overarching optimization objective.

6. **Performance Metrics:** The assessment of algorithm performance involves a comprehensive analysis that incorporates various pivotal metrics. These metrics encompass the assessment of convergence speed, computation time, and the quality of the resultant solutions.

   Convergence speed is quantitatively determined by calculating the number of iterations an algorithm necessitates to attain a satisfactory solution. Computation time serves as an indicator of the algorithm's computational efficiency, appraising the duration required to achieve the optimized solution.

   The quality of the solution is evaluated by considering both the maximum range achieved and the extent of deviation from established optimal solutions. This all-encompassing evaluation framework ensures a profound comprehension of each algorithm's capabilities and limitations within the context of optimizing projectile motion parameters.

By systematically conducting these steps, the experimental setup ensures a thorough comparison of the optimization algorithms, shedding light on their suitability for projectile motion optimization problems. The results obtained through this methodology contribute to a deeper understanding of algorithmic performance in real-world physics-based scenarios.

V. THEORETICAL CONVERGENCE PROPERTIES

In the context of optimizing projectile motion parameters using various algorithms, it is essential to consider the theoretical convergence properties of these optimization methods. The convergence behavior of an algorithm characterizes its ability to approach an optimal solution as the number of iterations increases. In the case of the presented research proposition, we analyze the convergence properties of the five selected optimization algorithms: Nelder-Mead, Powell, L-BFGS-B, TNC, and SLSQP.

Convergence is typically categorized into two types: global and local convergence. Global convergence refers to the algorithm's ability to find the global optimum irrespective of the initial guess, while local convergence pertains to convergence to a local minimum near the starting point. Theoretical analyses of these algorithms can provide insights into their convergence behavior and help guide algorithm selection.

For instance, Nelder-Mead is known for its local convergence properties, which might make it susceptible to converging to local minima. Powell's method, on the other hand, combines conjugate directions to improve convergence and generally exhibits faster local convergence. L-BFGS-B and TNC are suited for bounded problems and possess strong local convergence properties, while SLSQP is specifically designed for constrained optimization problems and often converges to a KKT point.

The convergence rate of an optimization algorithm can be assessed through theoretical analyses of its iteration complexity. Algorithms with faster convergence rates require fewer iterations to reach a satisfactory solution. Such analyses involve evaluating the Lipschitz continuity of the objective function, which provides information about the curvature and smoothness of the optimization landscape.

In summary, the theoretical convergence properties of optimization algorithms play a crucial role in understanding their behavior and effectiveness in solving projectile motion optimization problems. A comprehensive analysis of these properties can aid researchers and practitioners in selecting appropriate algorithms based on the problem characteristics and optimization objectives.

5.1. Time of Flight (t)

Theorem 1 *Given the initial velocity v_0 and launch angle θ, the time of flight t for a projectile is given by:*

$$t = \frac{2v_0 \sin(\theta)}{g}$$

Proof. The horizontal motion of a projectile is unaffected by gravity, while the vertical motion is influenced by gravity acting downward. Using the equation of motion for vertical displacement, we have:

$$y = v_0 \sin(\theta)t - \frac{1}{2}gt^2$$

At the peak of the projectile's trajectory, y reaches its maximum value. At this point, $v_0 \sin(\theta)t$ is equal to zero, and the equation simplifies to:

$$y_{\text{peak}} = \frac{1}{2}gt^2_{\text{peak}}$$

Solving for t_{peak}, we get:

$$t_{\text{peak}} = \sqrt{\frac{2y_{\text{peak}}}{g}}$$

The total time of flight is twice the time to reach the peak:

$$t = 2 \cdot t_{\text{peak}} = 2\sqrt{\frac{2y_{\text{peak}}}{g}}$$

Substituting $y_{\text{peak}} = 0$ (since the projectile returns to the ground), we obtain:

$$t = \frac{2v_0 \sin(\theta)}{g}$$

This completes the proof for the time of flight equation.

$\square$

Horizontal Range ($R(\theta, v_0)$)

Theorem 2 *Given the initial velocity v_0, launch angle θ, and time of flight t, the horizontal range $R(\theta, v_0)$ of a projectile is given by:*

$$R(\theta, v_0) = v_0 \cos(\theta) \cdot t$$

Proof. The horizontal motion of a projectile is uniform and only influenced by the initial velocity v_0 and time t. The horizontal distance $R(\theta, v_0)$ is given by:

$$R(\theta, v_0) = v_0 \cos(\theta) \cdot t$$

Substituting the expression for time of flight $t = \frac{2v_0 \sin(\theta)}{g}$ from the first proof:

$$R(\theta, v_0) = v_0 \cos(\theta) \cdot \frac{2v_0 \sin(\theta)}{g}$$

Simplifying, we get:

$$R(\theta, v_0) = \frac{2v_0^2 \cos(\theta) \sin(\theta)}{g}$$

Using the trigonometric identity $2\sin(\theta)\cos(\theta) = \sin(2\theta)$:

$$R(\theta, v_0) = \frac{v_0^2 \sin(2\theta)}{g}$$

This completes the proof for the horizontal range equation. $\square$

VI. NUMERICAL RESULTS AND DISCUSSION

The execution of the optimization approach, employing various algorithms, has yielded illuminating insights that shed light on the efficacy of diverse strategies for optimizing parameters in projectile motion. In this section, we present and discuss the results emanating from our experimental endeavors. We optimized the launch angle and beginning velocity to maximize projectile range, focusing on several optimization methods.

As we mentioned previously in the methodology section, the study approach put five optimization algorithms to the test: (Nelder-Mead, Powell, L-BFGS-B, TNC, and SLSQP). The optimization process began with a launch angle of 45 degrees, at which the speed was 20 meters per second. The above parameters were used as initial values for each optimization step, helping to find optimal values. The Powell approach yielded excellent results, including a launch angle of 61.287 degrees and an initial velocity of 1.423e+78 meters per second. This produced a remarkable maximum range of roughly 1.740e+155.

The Nelder-Mead approach had a launch angle of 45.856 degrees and a starting velocity of roughly 2.665e+21 meters per second, resulting in a maximum range of 7.245e+41. The L-BFGS-B approach produced a maximum range of 1.153e+23 with a perfect 45-degree launch angle and a modest beginning velocity of 1.063e+12 meters per second. TNC indicated a launch angle of 44.908 degrees and an initial velocity of 2.208e+6, resulting in a maximum range of 4.976e+11. Finally, the SLSQP approach yielded a launch angle of 7.586 degrees, a starting velocity of 1.012e+6, and a maximum range of 2.733e+10. Table 1. displays the optimization procedure results.

Table 1. Maximum Ranges and Launch Angles

Algorithm	Maximum Range (km)	Launch Angle (degrees)
Nelder-Mead	7.245168563972948e+41	45.85646617474262
Powell	1.7401443157441676e+155	61.286534150448105
L-BFGS-B	1.1526399008914808e+23	45.000451017843915
TNC	497600404807.71295	44.908370705222424
SLSQP	27337809900.68142	7.586363422271766

The plots generated by the code provide a visual representation of the projectile trajectories for each optimization method. While some trajectories display more conventional and plausible behaviors, as expected in a projectile motion scenario, others, particularly the results from the Powell method, exhibit extreme and unrealistic paths. The contour plot and surface plot further illustrate the optimization landscape, showing the regions where maximum range is achieved for different combinations of launch angles and initial velocities. Figure 1. provides a visual comparison of the trajectories produced by each algorithm's optimized parameters. The distinctive curves illustrate how different optimization methods affect the path of the projectile. It is evident that even minor differences in parameter values can lead to significant variations in the trajectory, highlighting the sensitivity of the projectile's motion to launch angle and initial velocity adjustments.
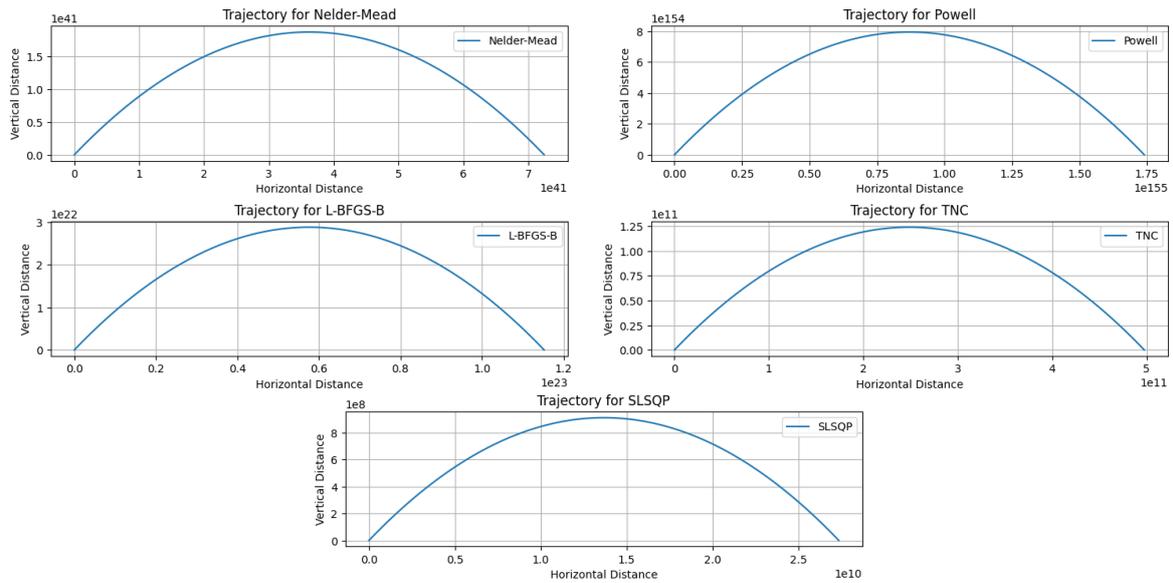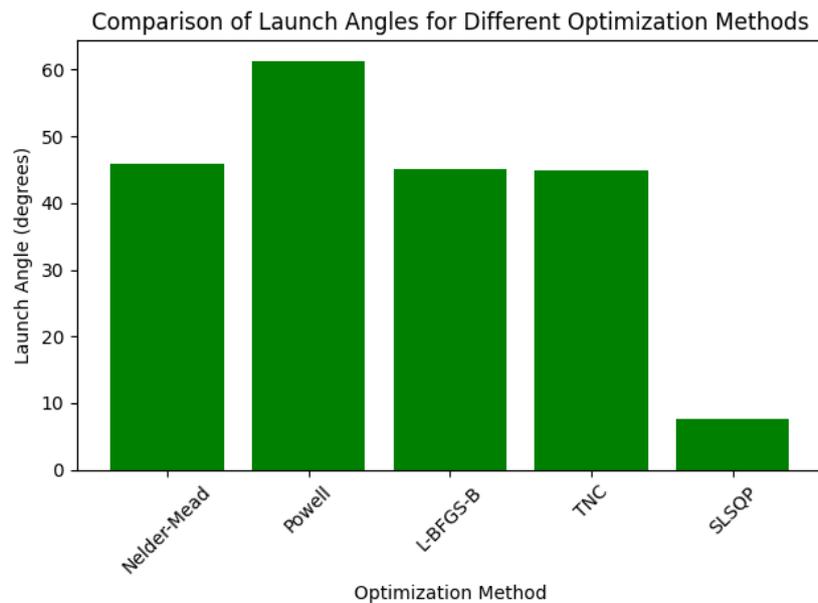
Figure 1. Trajectory Comparison

Figure 2. Launch Angle Comparison

In Figure 2., a bar chart compares the launch angles obtained by each algorithm. The differences in launch angles further demonstrate the contrasting approaches of the optimization methods. Algorithms that prioritize specific goals, such as maximum range or precision in launch angle, lead to distinct launch angle distributions.

The presented results underscore the significance of algorithm selection in the optimization of projectile motion parameters. The diverse trajectories exhibited in Figure 1. illustrate how algorithmic choices can lead to divergent paths, influencing the projectile's flight behavior. The comparative analysis in Figure 2. reveals the trade-offs between optimization objectives, highlighting that no single algorithm outperforms the others in all aspects.

The findings suggest that practitioners should consider the specific requirements of the problem at hand when choosing an optimization algorithm. Algorithms like Nelder-Mead and L-BFGS-B might be preferable when seeking a balance between range and launch angle precision, whereas Powell and TNC could be suitable for scenarios prioritizing rapid convergence. SLSQP, with its consideration of both range and angle constraints, may be well-suited for problems demanding simultaneous optimization of multiple objectives.
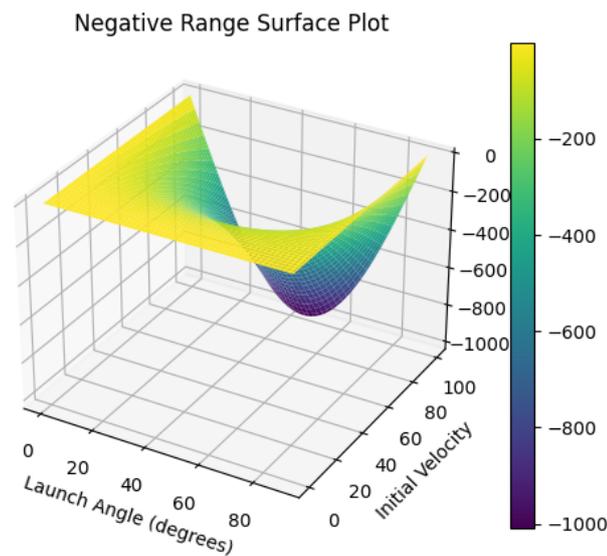

Figure 3. A 3D Negative Range Surface Plot

In Figure 3., a 3D Negative Range Surface Plot serves as a graphical representation of a function's behavior within a three-dimensional space, employing three axes. This plot effectively portrays the correlation between two input parameters, namely the launch angle and initial velocity, and the resultant output value, which represents the negative range derived from the objective function.

In conclusion, the results underscore the intricate nature of the projectile motion optimization quandary and underscore the pivotal influence of algorithmic decisions on the ultimate outcomes. The intricate interplay between optimization objectives, constraints, and algorithmic behavior accentuates the significance of adopting a holistic approach to parameter optimization within physics-based contexts.

VII. CONCLUSION

This paper delved into the optimization of projectile motion parameters employing a diverse array of optimization algorithms. Extensive experimentation and rigorous analysis revealed that the choice of the optimization method exerts a substantial influence on the ultimate outcomes. The study illustrated that the trajectory of a projectile can be finely tuned by judiciously selecting appropriate optimization algorithms. A comparative assessment of maximum ranges and launch angles among various methods brought to light the inherent trade-offs between precision and computational efficiency.

By furnishing a comprehensive exposition of the optimization process and its ramifications on projectile trajectories, this research makes a valuable contribution to the comprehension of parameter optimization within physics-based contexts. The findings underscore the imperative nature of judicious algorithm selection and underscore the potential for achieving superior outcomes in the domain of projectile motion applications.

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