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MODIFIED HOUSEHOLDER METHOD OF FIFTH ORDER OF CONVERGENCE AND ITS DYNAMICS ON COMPLEX PLANE

Ayunda Putri[1*], M. Imran[2], Rike Marjulisa[3]

[1,2,3]*Department of Mathematics, University of Riau, Pekanbaru, Indonesia*
Email: [1]ayundaputri@lecturer.unri.ac.id, [2]mimran@lecturer.unri.ac.id, [3]rikemarjulisa@lecturer.unri.ac.id
*Corresponding author

Abstract. In this paper, a modified Householder method of fifth order is proposed for solving nonlinear equations. The modification is done by adapting a cubic interpolation polynomial to approximate the second derivative in the Householder method. We provide a theorem to prove the order of convergence of the proposed method. The simulations reveals that the proposed method needs fewer iterations, even with challenging initial guesses, and excels in sending large portion of initial points to convergence and exhibits rapid convergence.
Keywords: Iterative methods, Householder method, order of convergence, basins of attraction

I. INTRODUCTION

There is an abundant phenomena in nature that can be modeled via nonlinear equations. However finding the solution through analytical methods does not consistently yield success. The development of computational technology has catalyzed a significant progress in the field of applied mathematics, leading to the emergence of a highly significant and widely explored domain within mathematics. This domain revolves around the pursuit of numerical solutions to nonlinear equations through the application of numerical techniques employing computational tools. The goal is to find the solution of a nonlinear equation

$$\xi(x) = 0, \tag{1}$$

where $\xi : \mathbb{R} \longrightarrow \mathbb{R}$, by employing an efficient iterative method. According to Traub [1], there are two measures of an efficient iterative method. One of which is called computational efficiency. If a method is convergent to a simple root of (1), say α, with order of convergence d, then the computational efficiency of the method is given by $I = d^{1/p}$ where p is the number of function evaluations required in each iteration.

The subject of root-finding methods has witnessed extensive research, with Newton's method being a classic approach known for its quadratic convergence and $I = 1.414$. Another popular method was proposed by Householder [2], defined as:

$$x_{i+1} = x_i - \frac{\xi(x_i)}{\xi'(x_i)} - \frac{\xi(x_i)^2\xi''(x_i)}{2\xi'(x_i)^2}, \quad i = 0, 1, 2, \ldots. \tag{2}$$

This method is of third order of convergence and efficiency index $I = 1.442$. However, it should be noted that the method involves the computation of a second derivative, which can be a draw-

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back. The introduction of higher order derivative often presents challenges including increased computational costs and practical application difficulties. Consequently, many researchers seek to find free second derivatives iterative methods with various approaches as evidenced in works such as [3–8].

An alternative perspective for assessing the performance of an iterative method involves studying its basins of attraction, which was introduced by [1]. Researchers have delved into this subject, exploring it in various studies, including [3, 9–16]. An example of extensive analysis regarding the performance of iterative methods through their basins of attraction and its relation to several efficiency measures is given in [17].

To introduce a novel approach for solving nonlinear equations, we have employed polynomial interpolation that approximates the second derivative within the Householder method. Additionally, in this paper, we examine the basins of attraction associated with our proposed method and conduct comparisons with various existing methods to establish the superiority of our method in terms of convergence speed, efficiency index, and the number of points where convergence is achieved.

The organization of this paper is as follows: The derivation of modified Householder method of fifth order of convergence by approximating the second derivative using a cubic interpolating polynomial is presented in the subsequent section. The third section is dedicated to an exploration of the behavior of the proposed method through some simulations on several transcendental functions. We also display the dynamics of the discussed method on complex plain in the penultimate section. Finally, the conclusion of our research is given in the last section of this article.

II. MODIFIED HOUSEHOLDER METHOD OF FIFTH ORDER OF CONVERGENCE

In this section, we present our modified householder method. The modification is done by approximating the second derivative in (2) using polynomial interpolation. we consider an interpolating polynomial given by:

$$\phi(x) = a + b(x - x_i) + c(x - x_i)^2 \tag{3}$$

where a, b and c are coefficients. This polynomial is chosen to satisfy the interpolation conditions: $\xi(x_i) = \phi(x_i), \xi(y_i) = \phi(y_i), \xi'(x_i) = \phi'(x_i)$, and $\xi''(y_i) = \phi''(y_i)$, where y_i is obtained from:

$$y_i = x_i - \frac{\xi(x_i)}{\xi'(x_i)}$$

By imposing these conditions on (3) we obtain:

$$\begin{aligned} \phi(y_i) &= \xi(y_i) = a + b(y_i - x_i) + c(y_i - x_i)^2 \\ \phi(x_i) &= \xi(x_i) = a \\ \phi'(x_i) &= \xi'(x_i) = b \end{aligned} \tag{4}$$

By solving system order three in (4) for the three unknowns, we have

$$c = -\frac{\xi'(y_i - x_i) + \xi(x_i) - \xi(y_i)}{(y_i - x_i)^2} \tag{5}$$

Ultimately, by substituting to the last interpolation condition $\xi''(y_i) = \phi''(y_i)$, we obtain the following expression:

$$\xi''(y_i) = \frac{2}{y_i - x_i}\left(\frac{\xi(y_i) - \xi(x_i)}{y_i - x_i} - \xi'(x_i)\right) = \phi_2(x_i, y_i) \tag{6}$$

By applying (6) to (2), we derive a new fifth order householder method free from second derivative that can be expressed as follows:

$$y_i = x_i - \frac{\xi(x_i)}{\xi'(x_i)} \tag{7}$$

$$x_{i+1} = y_i - \frac{\xi(y_i)}{\xi'(y_i)} - \frac{\xi(y_i)^3\xi'(x_i)^2}{\xi(x_i)^2\xi'(y_i)^3} \tag{8}$$

For the remainder of this paper, we will refer to these equations as Modified Householder Method (MHM). The convergence analysis of this method is presented in the following theorem:

Theorem 1 *Suppose $\xi : X \longrightarrow \mathbb{R}$ where $X \subseteq \mathbb{R}$ is an open interval. Let $\alpha \in X$ be the simple root of* (1) *where ξ is sufficiently differentiable around α. Then the method described by* (7) *and* (8) *(MHM) is of fifth order.*

Proof. Let α represent a simple root of $\xi(x) = 0$. By expanding $\xi(x)$ around $x = \alpha$ using Taylor series, we obtain:

$$\begin{aligned}\xi(x) = \xi(\alpha) + \xi'(\alpha)(x-\alpha) + \frac{1}{2!}\xi''(\alpha)(x-\alpha)^2 + \frac{1}{3!}\xi'''(x)(x-\alpha)^3 \\ + \frac{1}{4!}\xi^{(4)}(x)(x-\alpha)^4 + \frac{1}{5!}\xi^{(5)}(x)(x-\alpha)^5 + \frac{1}{6!}\xi^{(6)}(x)(x-\alpha)^6 \\ + O(x-\alpha)^7.\end{aligned} \tag{9}$$

By evaluating $\xi(x)$ at x_i, we have

$$\xi(x_i) = \xi'(\alpha)\big(e_i + C_2e_i^2 + C_3e_i^3 + C_4e_i^4 + C_5e_i^5 + C_6e_i^6 + O(e_i^7)\big), \tag{10}$$

where $e_i = x_i - \alpha$ denotes error at the i-th iteration and $C_i = (1/i!)(\xi^{(i)}(\alpha)/\xi'(\alpha)), i = 1, 2, 3, \cdots$. By differentiating (9) and evaluating it at x_i, we obtain:

$$\xi'(x_i) = \xi'(\alpha)\big(1 + 2C_2e_i^2 + 3C_3e_i^2 + 4C_4e_i^3 + 5C_5e_i^4 + 6C_6e_ki^5 + O(e_i^6)\big). \tag{11}$$

Substituting (10) and (11) into (7) results in

$$\begin{aligned} y_i = \alpha + C_2e_i^2 + (-2C_2^2 + 2C_3)e_i^3 + (-4C_2^3 - 7C_2C_3 + 3C_4)e_i^4 + (-16C_2^2C_3 \\ - 10C_2C_4 - 6C_3^2 + 4C_5)e_i^5 + (-20C_2^2C_4 - 21C_2C_3^2 - 13C - 2C_5 \\ - 17C_3C_4 + 5C_6)e_i^6 + O(e_i^7) \end{aligned} \tag{12}$$

Similarly, by expanding $\xi(y_i)$ around α, we obtain

$$\begin{aligned} \xi(y_i) = \xi'(\alpha)\big(C_2e_i^2 + (-2C_2^2 + 2C_3)e_i^3 + (-3C_2^3 - 7C - 2C_3 + 3C - 4)e_i^4 \\ + (-4C_2^4 - 12C_2^2C_3 - 10C_2C_4 - 6C_3^2 + 4C_5)e_i^5 + (-4C_2^5 - 22C_2^3C_3 \\ - 14C_2^2C_4 - 17C_2C_3^2 - 13C_2C_5 - 17C_3C_4 + 5C_6)e_i^6 + O(e_i^7)\big) \end{aligned} \tag{13}$$

Furthermore, by differentiating (9) and evaluating it at y_i, we get:

$$\begin{aligned} \xi'(y_i) = \xi'(\alpha)\Bigg(1 + 2C_2\big(C_2e_i^2 + (-2C_2^2 + 2C - 3)e_i^3 + (-4C_2^3 - 7C_2C_3 + 3C_4)e_i^4 \\ (-16C_2^2C_3 - 10C_2C_4 - 6C_3^2 + 4C_5)e_i^5 + (-20C_2^2C_4 - 21C_2C_3 \\ - 13C_2C_5 - 17C_3C_4 + 5C_6)e_i^6 + O(e_i)^7\big)\Bigg) \end{aligned} \tag{14}$$

Now, inserting (10), (11), (13) and (14) into (8) and simplifying yields

$$e_{i+1} = -2C_2^2C_3e_i^5 + (10C_2^5 + 7C_2^3C_3 - 3C_2^2C_4 - 8C_2C_3^2)e_i^6 + O(e_i^7) \tag{15}$$

Based on the definition of order of convergence [18], we conclude that the method described by (7) and (8) is of fifth order. □

Number of functions evaluations for each iteration of this method is four. Hence, the efficiency index of the method is $I = 1.495$.

III. NUMERICAL SIMULATIONS

In this section, we test MHM method on eight transcendental functions. Furthermore, we compare our proposed method with several modified householder methods, including householder method (HM3) with an efficiency index $I = 1.442$, householder method of fourth order (NHM4) from the work of Naeem, et al. [3] with $I = 1.414$ and householder method of fifth order (NHM5) introduced by Nazeer et al. [4] with $I = 1.495$. The followings are the testing functions:

- $\xi_1(x) = x^2 - \exp(x) - 3x + 2$
- $\xi_2(x) = \cos(x) - x$
- $\xi_3(x) = (x-1)^3 - 1$
- $\xi_4(x) = x^3 + x^2 - 10$
- $\xi_5(x) = x^2 - x\exp(x) + \cos(x)$
- $\xi_6(x) = \sin(x)^2 - x^2 + 1$
- $\xi_7(x) = x\exp(x^2) - \sin(x)^2 + 3\cos(x) + 5$
- $\xi_8(x) = \ln(x\exp(x) + 1)$

The stopping criteria for the iterations are as follows:

1. Maximum iterations: The iterations will stop if the maximum allowed iterations exceed 100.
2. Convergence based on the difference between consecutive approximations: $|x_{i+1} - x_i| \leq 10^{-15}$
3. Convergence based on the proximity to the true solution α. The iteration will stop if the absolute difference between the current approximation and the true root is less than 10^{-15}: $|x_{i+1} - \alpha| \leq 10^{-15}$

The following table presents the comparison of the discussed methods and their performance, considering the specified stopping criteria. The first column of the table corresponds to the tested functions, the second column indicates the exact root of each tested function, and the third column reveals the initial guesses. The final four columns depict the number of iterations generated for each testing method.

Table 1: Comparison of number of iterations produced by testing methods for functions $\xi_1(x)$ through $\xi_8(x)$

Function	Exact Root	x_0	HM3	NHM4	NHM5	MHM
$\xi_1(x)$	0.257530285439861	-2.0	4	19	5	3
		2.0	3	3	5	3
$\xi_2(x)$	0.739085133215161	1.0	3	2	4	2
		2.0	4	3	4	2
$\xi_3(x)$	2.00000000000000	-0.9	13	12	6	8
		3.5	5	13	7	3
$\xi_4(x)$	1.86746002460432	-1.9	4	24	10	20
		5.0	5	$*$	7	3
$\xi_5(x)$	0.639154096332008	3.5	6	5	5	4
		0.1	4	3	4	3
$\xi_6(x)$	1.40449164821534	4.5	4	$*$	5	3
		0.01	39	$*$	13	11
$\xi_7(x)$	-1.20764782713092	-2.5	7	NaN	6	6
		1.0	74	1	6	6
$\xi_8(x)$	0.00000000000000	0.9	4	3	5	3
		3.4	4	3	5	3

In Table 1 we provide two initial guesses for our observations. There is a case where a method exceeds the fixed maximum iterations, hence we denote this case with $*$. The symbol NaN is used when the iterations diverge. In order to see the accuracy of the approximation of the root, we present relative errors produced by of each testing method in Table 2 below.

Based on Table 1, it appears that the MHM exhibits faster convergence across various function. For function ξ_1, MHM requires fewest iterations while NHM4 demands higher iterations. NHM5 and HM3 show competitive performance but MHM stands out for efficiency.

Table 2: Comparison of relative errors given by testing methods for functions $\xi_1(x)$ through $\xi_8(x)$

Function	x_0	HM3	NHM4	NHM5	MHM
$\xi_1(x)$	-2.0	$9.30E-16$	$9.30E-16$	$9.30E-16$	$9.30E-16$
	2.0	$9.30E-16$	$9.30E-16$	$9.30E-16$	$9.30E-16$
$\xi_2(x)$	1.0	$4.85E-16$	$4.90E-16$	$4.85E-16$	$4.85E-16$
	2.0	$4.85E-16$	$4.85E-16$	$4.85E-16$	$4.85E-16$
$\xi_3(x)$	-0.9	$2.22E-35$	$1.81E-28$	$5.67E-24$	$1.71E-28$
	3.5	$6.12E-32$	$7.67E-47$	$1.88E-18$	$1.22E-28$
$\xi_4(x)$	-1.9	$2.68E-15$	$2.67E-15$	$2.67E-15$	$6.67E-16$
	5.0	$2.67E-15$	$4.10E-02$	$2.66E-15$	$2.67E-15$
$\xi_5(x)$	3.5	$6.55E-16$	$6.55E-16$	$6.57E-16$	$6.55E-16$
	0.1	$6.55E-16$	$6.55E-16$	$6.56E-16$	$6.55E-16$
$\xi_6(x)$	4.5	$8.73E-16$	$1.52E+02$	$8.73E-16$	$8.73E-16$
	0.01	$2.00E+00$	$1.51E+02$	$2.00E+00$	$8.73E-16$
$\xi_7(x)$	-2.5	$2.00E+00$	$1.51E+02$	$2.00E+00$	$8.73E-16$
	1.0	$2.00E+00$	$1.51E+02$	$2.00E+00$	$8.73E-16$
$\xi_8(x)$	0.9	Inf	Inf	Inf	Inf
	3.4	Inf	Inf	Inf	Inf

In the case of function ξ_2, NHM5 and MHM demonstrate the best performance, requiring the fewest iterations for both initial guesses where the same case happens for function ξ_3 as well. In the case of ξ_4 and ξ_5, MHM and NHM5 perform well with MHM having slight edge efficiency. MHM show efficient convergence for both initial guesses while NHM5 exceeds the fixed maximum iterations in function ξ_6. Again, MHM require the fewest iterations together with NHM5 in function ξ_7 while HM3 shows competitive performance but iterations give by NHM4 are diverge. Finally, MHM and NHM4 converge faster for both initial guesses in function ξ_8. MHM consistently achieves convergence within a relatively low number of iterations, even when starting from various initial guesses. In contrast, some of the other methods experience divergence, exceeded maximum iterations, or require more iterations to achieve convergence.

Table 2 exhibits that MHM often stands out with consistently low relative errors for most functions, indicating its accuracy in obtaining solutions. For the first two functions and both initial guesses, all methods achieved low relative errors. In function ξ_3, for the first initial guess, all methods achieve extremely low relative errors, with MHM having slightly lower error. While for the second initial guess, MHM again exhibits a lower relative errors compared to other methods. In function ξ_4, MHM and NHM5 have the lowest errors while NHM4 failed to provide higher accuracy. All methods produced very low and consistent relative errors ranging from $6.55e^{-16}$ to $6.57e^{-16}$ for both initial guesses in function ξ_5. For both initial guesses in function ξ_6 and xi_7, MHM stands out providing high precision and accuracy. Ultimately, relative errors for function xi_8 are reported as ”Inf” for all methods for both initial guesses. This suggests challenges in handling this specific function, leading to infinite relative errors.

IV. THE DYNAMICS OF THE METHOD ON COMPLEX PLANE

In order to gain a deeper insight on the performance of the discussed method, we present an observation on their basins of attraction. The analysis pf these basins offers a valuable information on the convergence and stability of the examined function when subjected to a root-finding method.

In this analysis, we consider a complex function $\xi(z) = 0$ where $\xi : \mathbb{C} \longrightarrow \mathbb{C}$ is a complex plane. The figure of basins of attraction of the tested function is generated from a uniform grid of $[-1, 1] \times [1, 1] \subset \mathbb{C}$. This gives us 1000000 initial points to be tested. Each point is associated with a specific color that marks its convergence. In this work, we fixed error tolerance to be 10^{-15}. In order to see the speed of convergence of iterative method, we set the maximum iterations to be just 10. The time required to generate the basins of attraction on our computer is measured in seconds.

We construct basins of attraction using four distinct methods in simulation sections, specifically HM3, NHM4, NHM5, and MHM. These methods are applied to evaluate the behavior of four test functions:

- $\xi_1(z) = z^2 - z + 1$
- $\xi_2(z) = z^3 - 1$
- $\xi_3(z) = z^4 - 10z^2 + 9$
- $\xi_4(z) = z^5 - 5z^3 + 4z$

Table 3 presents the simulation results. In the first column, the test functions are identified, while the second column specifies the roots of each tested function. The ”divergent” category indicates the number of points that fail to converge to the roots, and the ”time” column denotes the time taken by each method to process one million initial points.

The data presented in Table 3 clearly demonstrate that MHM outperforms the other methods by successfully guiding a significantly higher percentage of initial points to convergence. To elaborate, for function $\xi_1(z)$ through $\xi_3(z)$, MHM exhibits the lowest proportion of divergent points, namely $0.01\%, 2.1\%$, and 0.8% of consecutively. In all three cases, NHM5 reports the highest number of divergent points. In the context of $\xi_4(z)$, MHM remains advantageous with only 1.4% of divergent points, while NHM4 struggles to achieve convergence for nearly 47% of initial points. Furthermore, it is evident that MHM consistently generates basins of attraction at a faster rate compared to the other methods under consideration which is showed in Table 3 that it needs less time compared to all other methods.

Figures below display the basins of attraction of the discussed functions. Figure 1 illustrates the basins of attraction for function $\xi_1(z)$. This function has two complex roots, denoted by colors blue and red, while the black color remarks the divergent initial guesses.Notably, NHM5 and MHM exhibit the expansive areas of convergence. Figure 2 depicts the area of convergence for each testing method applied to $\xi_2(z)$ which features two complex roots and a real root. The colors blue and red represent the two complex roots successively, yellow remarks the third root and black indicates divergence. Remarkably, MHM showcases the largest area of convergence for all three roots. Figure 3 presents the basins of attraction for function $\xi_3(z)$, displaying the convergence regions for four testing functions with four distinct colors for each root. NHM5 and MHM provide the largest area of convergence where the black areas are not as spread across as in HM and NHM4. Lastly, Figure 4 presents basins of attraction for the solution of function $\xi_4(z)$, featuring five different colors representing each real roots, namely

Table 3: Comparison of number of convergent, divergent and time required of four methods in solving $\xi(z) = 0$

Function	Roots	HM3	NHM4	NHM5	MHM
$\xi_1(z)$	$0.5000000000 - 0.8660254038i$	489257	412064	312902	499943
	$0.5000000000 - 0.8660254038i$	489257	412064	312902	499943
	divergent	175872	21486	374196	114
	time	1969.402	5323.196	5206.946	1875.426
$\xi_2(z)$	$-0.5000000000 - 0.8660254038i$	291887	301676	256200	316599
	$-0.5000000000 + 0.8660254038i$	291887	301676	256200	316599
	1	151726	207802	302188	345648
	divergent	114038	244922	279798	21154
	time	3314.902	9621.241	6483.919	2408.467
$\xi_3(z)$	-3	112754	96730	41600	123678
	-1	348898	363250	394448	372550
	1	348898	360012	394448	372550
	3	112754	91132	41600	123678
	divergent	76696	88876	127904	7544
	time	3857.295	11295.855	7009.016	3917.205

Table 4: Comparison of number of convergent, divergent and time required of four methods in solving $\xi(z) = 0$

Function	Roots	HM3	NHM4	NHM5	MHM
$\xi_4(z)$	$-2.$	99554	52654	13382	111266
	-1	246996	145286	293654	266420
	0	199896	134560	285604	230708
	1	246996	145286	293654	266420
	2	99554	52654	13382	111266
	divergent	107004	469560	100324	13920
	time	5306.896	15930.315	7979.46	4113.141

blue for first root, red for second, yellow for the third root, and green and purple for the last two root consecutively. MHM and NHM5 exhibit notable prominence, displaying large the areas of convergence, while HM3 and NHM4 produces the most extensive regions of divergence.

These figures serve as valuable tools for identifying regions of divergence, enabling the identification of potentially challenging initial guesses. Moreover, they facilitate the collection of initial guesses leading to convergence towards specific roots, thereby ensuring the reliability of the convergence.

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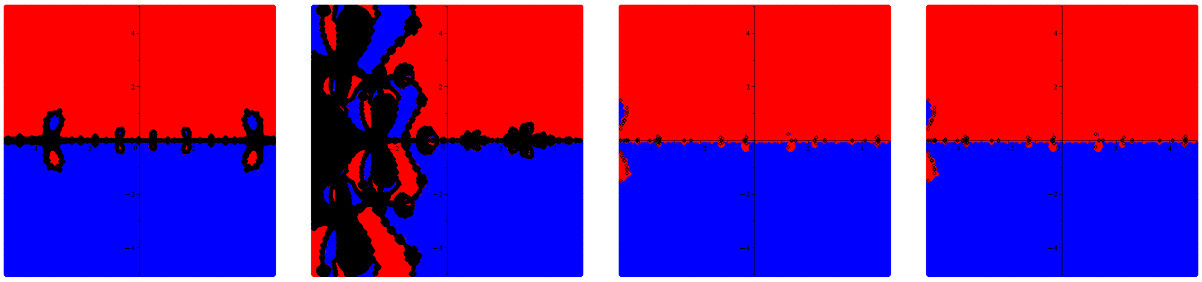
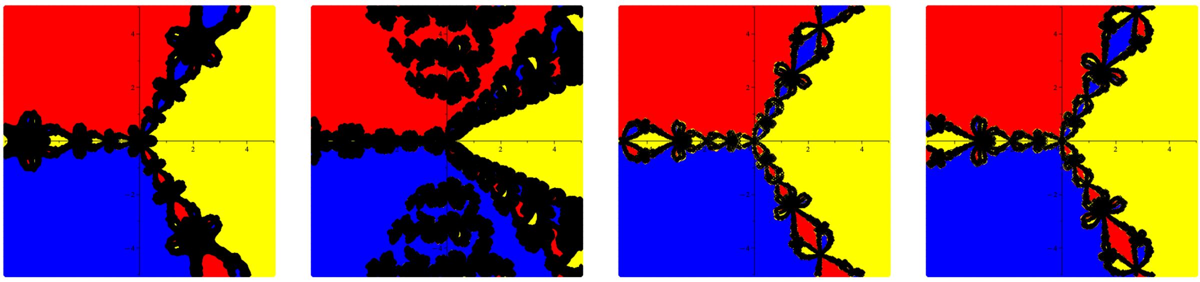
V. CONCLUSIONS

In this research, we have introduced a novel approach, a modified Householder method where the second derivative is approximated by cubic polynomial interpolation. We have conducted extensive simulations on the proposed method and have given comparisons with various Householder methods with different order of convergence. Our analysis encompasses the construction of the basins of attraction, the calculation of the number of initial guesses that converge and diverge and the measurement of time needed to generate the basins of attraction. Notably, MHM outperforms other methods in terms of computational efficiency, requiring fewer iterations for certain functions. It is also evident that MHM is adaptive and maintains stability to challenging initial guesses while some methods encountered issus or anomaly in specific scenarios. When given a million initial guesses to simulate solution fo several functions, MHM succeeds in sending a large portions of them to converge rapidly. These findings suggest that MHM is capable to handle diverse mathematical function effectively, thereby making it valuable tool for numerical optimization tasks.



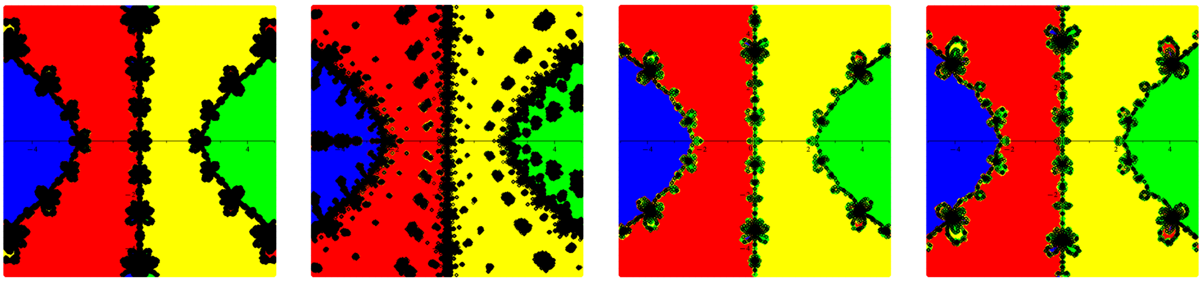
(a) HM3 (b) NHM4 (c) NHM5 (d) MHM

Figure 1: Basins of attraction of iterative methods for $\xi_1(z) = z^2 - z + 1$



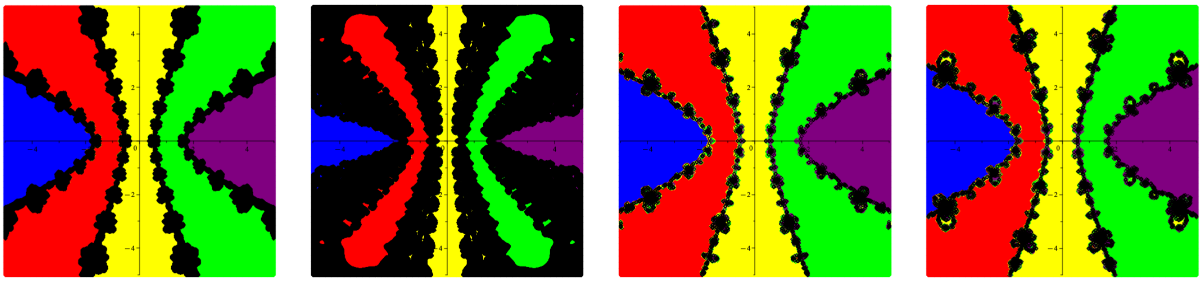
(a) HM3 (b) NHM4 (c) NHM5 (d) MHM

Figure 2: Basins of attraction of iterative methods for $\xi_2(z) = z^3 - 1$



(a) HM3 (b) NHM4 (c) NHM5 (d) MHM

Figure 3: Basins of attraction of iterative methods for $\xi_3(z) = z^4 - 10z^2 + 9$



(a) HM3 (b) NHM4 (c) NHM5 (d) MHM

Figure 4: Basins of attraction of iterative methods for $\xi_4(z) = z^5 - 5z^3 + 4z$

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