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THE CHEMICAL TOPOLOGICAL GRAPH ASSOCIATED WITH THE NILPOTENT GRAPH OF A MODULO RING OF PRIME POWER ORDER

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Abstract. Chemical topological graph theory constitutes a subdomain within mathematical chemistry that leverages graph theory to model chemical molecules. In this context, a chemical graph serves as a graphical representation of molecular structures. Specifically, a chemical molecule is portrayed as a graph wherein atoms are denoted as vertices, and the interatomic bonds are represented as edges within the graph. Various molecular properties are intricately linked to the topological indices of these molecular graphs. Notably, commonly employed indices encompass the Wiener Index, the Gutman Index, and the Zagreb Index. This study is directed towards elucidating the numerical invariance and topological indices inherent to a nilpotent graph originating from a modulo integer ring with prime order. Consequently, the investigation seeks to discern how the Wiener Index, the Zagreb Index, and other characteristics of the nilpotent graph manifest within a ring of integers modulo prime order powers.

Keywords: topological indices, nilpotent graph, integer modulo ring

I. INTRODUCTION

Graph theory is one of the recent subjects in mathematics that has been frequently discussed, especially regarding the functions of graphs in various fields [1]. In the field of chemical topology graph, graph theory is employed to represent models of chemical molecules, where the atoms are treated as a set of vertices, and the bonds between atoms form the set of edges in the graph [2]. Chemical topology graph is commonly used to determine the topological indices of a molecular structure, which are widely utilized to understand quantitative relationships between structure-activity and structure-property in chemical molecule models [3]. Furthermore, in the field of mathematics itself, graphs are used to represent the geometric shapes in algebraic theories. This has inspired the author to explore the relationship between topological indices derived from these graph representations. In the realm of mathematics, numerous previous studies have discussed various aspects of graph representation, such as representations of graphs in group theory, rings, and modules. A graph itself is denoted as Γ and consists of a non-empty set, each element of which is referred to as a vertex, and an unordered pair of these vertices forms an edge [4].

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In previous study, several types of graph representations have been defined, such as identity graphs, inverse graphs, commuting and non-commuting graphs, coprime graphs, non-coprime graphs, power graph, intersection graph, nilpotent graphs, and various other graphs [5]–[16]. Specifically, in this study we will focus on nilpotent graphs, where the term 'nilpotent graph' first appeared in 2010 and was introduced by Basnet et al. A nilpotent graph is a graph derived from a finite group, with its vertices representing all the elements of the group, and two vertices are adjacent if the binary operation between them results in a nilpotent element [17].

Therefore, in this research, the author will delve into the topological indices, specifically the Wiener index, Zagreb index, Gutman index, and various properties of nilpotent graphs derived from the ring of integers modulo with prime power order.

II. PRELIMINARIES

First and foremost, we will provide various fundamental terminology and definitions that will be used throughout this article. Additionally, it is essential to establish a solid foundation of understanding before delving deeper into the subject matter. A graph is a mathematical data structure comprising vertices and edges that represent connections or relationships between these vertices. In the subsequent definition, we present the formal description of the Wiener Index of a graph.

Definition 1 [18] *The Wiener Index of a graph Γ is denoted as $W(\Gamma)$, and its definition is as follows:*

$$W(\Gamma) = \sum_{\{u,v\}\subseteq V(\Gamma)} d_{\Gamma}(u,v)$$

Where u and v are vertices in graph Γ, and $d_{\Gamma}(u,v)$ represents the distance between the two adjacent vertices.

The second index to be discussed is the First Zagreb Index, denoted as such to represent a measure defined as the summation of the squares of the degrees of each vertex, as formally defined in Definition 2 as follows:

Definition 2 [18] *The first Zagreb Index of a graph Γ is denoted as $M_1(\Gamma)$, and its definition is as follows:*

$$M_1 = \sum_{v\in V(\Gamma)} d_v^2$$

With v being a vertex of graph Γ, and d_v representing the degree of vertex v.

The third index under consideration is the Gutman Index, which quantifies the relationship between two contiguous vertices, as stipulated in Definition 3 as follows:

Definition 3 [19] *The Gutman Index of a graph Γ, denoted as $Gut(\Gamma)$, is defined as follows:*

$$Gut(\Gamma) = \sum_{u,v \in V(\Gamma)} d(u)d(v)d(u,v)$$

Where u and v are vertices of graph Γ, and $d(u)$ and $d(v)$ represent the degrees of vertices u and v, respectively, with $d(u,v)$ being the distance between vertices u and v.

Within this section, the author will expound upon the graph representation of the modulo integer ring within nilpotent graphs and delve into certain attributes and properties associated with these graph representations. The ring of integers modulo n $(n = p^k)$ is represented in the form $\mathbb{Z}_{p^k} = \{\bar{0}, \bar{1}, \dots, \overline{p^k - 1}\}$.

Definition 4 [17] *An element r in the ring R is called a nilpotent element if $r^k = 0_R$, for some $k \in \mathbb{N}$. The set of all nilpoten elements of R is denoted as $N(R)$, and the set off all non-nilpotent element of R denoted by $N^C(R)$.*

It is easy to recognize that for the ring $\mathbb{Z}_n$ with $n = p^k$, where p is prime and $k \in \mathbb{N}$, the set of nilpotent elements consists of $\overline{mp}$ for $m \in \mathbb{N}$.

Definition 5 [17] *For a ring R, the nilpotent graph of R, denoted as Γ_R, is a graph with its set of vertices being R, and $u, v \in R$ are considered adjacent if $uv \in N(R)$.*

III. RESULTS AND DISCUSSION

In this section, we shall engage in a discourse concerning certain outcomes pertaining to the topological indices associated with the nilpotent graph corresponding to the ring of integers modulo prime power with prime power order. In the next theorem, we will provide conditions for an element of the ring $\mathbb{Z}_n$ to be a nilpotent element.

Theorem 1 *The ring $\mathbb{Z}_n$, where $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, is represented by $\Gamma_N(\mathbb{Z}_n)$. Let $\bar{x} \in \mathbb{Z}_n$, then $\bar{x} \in N(\mathbb{Z}_n)$ if and only if $\bar{x} = \overline{q(p_1 p_2 \dots p_m)}$ for $q \in \mathbb{N}$* ■.

Proof: $(\Longrightarrow)$ Let $\bar{x} \in N(\mathbb{Z}_n)$, then $\overline{x^k} = \bar{0}$ for some $k \in \mathbb{N}$, as a result, $p_1^{k_1} p_2^{k_2} \dots p_m^{k_m} | x^k$. Because $p_1, p_2, \dots, p_m$ are prime, then p_i divides x for $i = 1, 2, \dots, m$. In other words, $p_1 p_2 \dots p_m$ divides x, or $x = q(p_1 p_2 \dots p_m)$ for some $q \in \mathbb{N}$. Thus, $\bar{x} = \overline{q(p_1 p_2 \dots p_m)}$ is obtained.
$(\Longleftarrow)$ Let $\bar{x} \in \mathbb{Z}_n$ such that $\bar{x} = \overline{q(p_1 p_2 \dots p_m)}$. Choose $k = \sup\{k_1, k_2 \dots k_m\}$, consequently, $\overline{x^k} = \overline{q^k (p_1^{k_1} p_2^{k_2} \dots p_m^{k_m})} = \overline{s p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}}$ with $s = q^k (p_1^{k-k_1} p_2^{k-k_2} \dots p_m^{k-k_m})$. As a result, $n | x^k$ and $\overline{x^k} = \bar{0}$, thus $\bar{x}$ is a nilpotent element. ■

In accordance with Theorem 1, vertices representing nilpotent elements exhibit adjacency to all other vertices within the graph. The subsequent theorem presented herein constitutes a specific instance of Theorem 1, upon which the subsequent portions of the article are based on its implications. Based on the previous theorem, now we can determine all nilpotent elements of the ring $\mathbb{Z}_n$ with prime power order.

Theorem 2 *If $\mathbb{Z}_n$ is the ring of integers modulo with $n = p^k$ where $k \in \mathbb{N}$, then $N(\mathbb{Z}_n) = \{\bar{0}, \bar{p}, \overline{2p}, \overline{3p}, \dots \overline{p^k - p}\}$*

Proof: Let $\bar{x} \in \mathbb{Z}_n$ be a nilpotent element, then by definition, $\bar{x}^m = 0$ for some $m \in \mathbb{N}$. Consequently, $p^k \mid x^m$, which implies $p|x^m$. Since p is a prime number, $p|x$, and thus, x is a multiple of p. Therefore, it is proven that $N(\mathbb{Z}_n) = \{\bar{0}, \bar{p}, \overline{2p}, \overline{3p}, \dots \overline{p^k - p}\}$. ■

According to Theorem 2, the nilpotent graph of the ring of integers modulo with prime power order, we can now prove that the graph should be a complete graph in the next theorem.

Theorem 3 *If $\mathbb{Z}_n$ is the ring of integers modulo with $n = p^k$, where $k \in \mathbb{N}$, then $\Gamma_{\mathbb{Z}_n}$ contains a complete subgraph $K_{p^{k-1}}$.*

Proof: It will be shown that the mentioned subgraph is $N(\mathbb{Z}_n)$. Let $\bar{x}$ and $\bar{y}$ be any elements in $N(\mathbb{Z}_n)$. Since $N(\mathbb{Z}_n)$ is an ideal of the ring $\mathbb{Z}_n$, it follows that $\overline{xy} \in N(\mathbb{Z}_n)$. Consequently, any distinct $\bar{x}$ and $\bar{y}$ in $N(\mathbb{Z}_n)$ are two adjacent vertices, thus, $\Gamma_{\mathbb{Z}_n}$ contains the complete subgraph $K_{p^{k-1}}$. ■

An additional attribute of nilpotent graphs within the ring of integers modulo with prime power order is the presence of a 'twin star' subgraph as stated in the next theorem.

Theorem 4 *If $\mathbb{Z}_n$ is the ring of integers modulo with $n = p^k$, where $k \in \mathbb{N}$, then $\Gamma_N(\mathbb{Z}_n)$ contains p^{k-1} twin star subgraphs $K_{1,n-1}$.*

Proof: There are p^{k-1} elements that are nilpotent. It will be shown that every $\bar{x} \in N(\mathbb{Z}_n)$ with all other elements form a twin star subgraph with $\bar{x}$ at its center. Let $\bar{y} \in \mathbb{Z}_n$ and $\bar{x} \neq \bar{y}$, since $N(\mathbb{Z}_n)$ is an ideal, then $\overline{xy} \in N(\mathbb{Z}_n)$. Based on this, $\bar{x} \in N(\mathbb{Z}_n)$ is adjacent to all other elements in $\mathbb{Z}_n$. So, $\Gamma_N\left(\mathbb{Z}_{p^k}\right)$ contains p^{k-1} twin star subgraphs $K_{1,n-1}$. ■

The number of edges in the nilpotent graph for integers modulo with prime power order can be calculated using the formula provided in the following theorem.

Theorem 5 *A nilpotent graph of the ring $\mathbb{Z}_n$ for $n = p^k$, where p is a prime and $k \in \mathbb{N}$, denoted as $\Gamma_N\left(\mathbb{Z}_{p^k}\right)$, has the following number of edges:*

$$\left|E\left(\Gamma_N\left(\mathbb{Z}_{p^k}\right)\right)\right| = (p^{2k-2}(p-1) + \frac{(p^{2k-2} - p^{k-1})}{2}$$

Proof: In the graph, there are two distinct types of edges based on the pairs of vertices they connect: edges connecting nilpotent vertices with non-nilpotent vertices and edges connecting one nilpotent vertex to another nilpotent vertex in the graph. As a result, the total number of edges in the graph is the sum of the edges connecting nilpotent vertices to non-nilpotent vertices and the edges connecting nilpotent vertices to other nilpotent vertices.

i. The number of edges connecting nilpotent vertices to non-nilpotent vertices is the product of the count of nilpotent elements p^{k-1} and the count of non-nilpotent elements $p^k - p^{k-1}$ in the ring $\mathbb{Z}_{p^k}$. Thus, we obtain the number of edges as:

$$\left(p^{k-1}\right)\left(p^k - p^{k-1}\right) = p^{2k-2}(p-1)$$

ii. The number of edges connecting nilpotent vertices to other nilpotent vertices is equal to the number of edges in the complete graph $K_{|N|}$, which is:

$$\sum_{i=1}^{|N|-1} i = 1 + 2 + \cdots + (|N| - 1)$$

Since $|N(\mathbb{Z}_n)| = p^{k-1}$, then:

$$\sum_{i=1}^{|N|-1} i = \sum_{i=1}^{p^{k-1}-1} i = 1 + 2 + \cdots + \left(p^{k-1} - 1\right) = \frac{(p^{2k-2} - p^{k-1})}{2}$$

Based on (i) and (ii), we obtain:

$\left|E\left(\Gamma_N\left(\mathbb{Z}_{p^k}\right)\right)\right| = \left(p^{2k-2}(p-1)\right) + \frac{(p^{2k-2}-p^{k-1})}{2}$. ■

To determine the topological indices of a graph, it is essential that the graph under consideration is connected. The following theorem explains that nilpotent graphs are connected graph.

Theorem 6 *The graph $\Gamma_N(\mathbb{Z}_n)$ for $n \in \mathbb{N}$ and $n > 2$, is a connected graph*

Proof: Consider any $\bar{u}, \bar{v} \in V(\Gamma_N(\mathbb{Z}_n))$ such that for $\bar{u} = \bar{0}$ or $\bar{v} = \bar{0}$, it is evident that $(\bar{u}, \bar{v}) \in E(\Gamma_N(\mathbb{Z}_n))$. Now, in the case where $\bar{u}, \bar{v} \neq \bar{0}$, it follows that $\bar{0} \in V(\Gamma_N(\mathbb{Z}_n))$ and $\overline{0.u} = \bar{0}$. Consequently, $(\bar{0}, \bar{u}) \in E(\Gamma_N(\mathbb{Z}_n))$, and similarly, $(\bar{0}, \bar{v}) \in E(\Gamma_N(\mathbb{Z}_n))$. As a result, there exists a path from $\bar{u}$ to $\bar{v}$, which is $\bar{u} \to \bar{0} \to \bar{v}$. Thus, the graph $\Gamma_N(\mathbb{Z}_n)$ is connected because every two distinct vertices in the graph are connected. ■

In determining several topological indices, the vertex degree in the graph is crucial. Here is the vertex degree of the vertices in the nilpotent graph of the ring $\mathbb{Z}_{p^k}$.

Theorem 7 *Let $\bar{v} \in V\left(\Gamma_N\left(\mathbb{Z}_{p^k}\right)\right)$, then:*

a. *for $\bar{v} \in N\left(Z_{p^k}\right)$, then $deg(\bar{v}) = p^k - 1$, and*

b. *for $\bar{v} \notin N\left(\mathbb{Z}_{p^k}\right)$, then $deg(\bar{v}) = p^{k-1}$.*

Proof:

a. For $\forall \bar{v} \in V\left(\Gamma_N\left(\mathbb{Z}_{p^k}\right)\right)$, where $\bar{v} \in N\left(\mathbb{Z}_{p^k}\right)$

Based on **Theorem 1**, the vertices representing nilpotent elements are adjacent to all vertices in the graph $\Gamma_N\left(\mathbb{Z}_{p^k}\right)$. Since the total number of vertices in the graph is p^k, the degree of nilpotent vertices is $p^k - 1$.

b. For $\forall \bar{v} \in V\left(\Gamma_N\left(\mathbb{Z}_{p^k}\right)\right)$, where $\bar{v} \notin N\left(\mathbb{Z}_{p^k}\right)$,

It is evident that, based on **Theorem 1**, vertice $\bar{v}$ is adjacent to all nilpotent elements of the ring $\mathbb{Z}_{p^k}$. Therefore, $\bar{v}$ is adjacent to p^{k-1} vertices in the graph. However, according to Theorem 1, $\bar{v}$ is not adjacent to non-nilpotent elements, so $\bar{v}$ is only adjacent to nilpotent elements. The vertex $\bar{v}$ has a degree of p^{k-1}. ■

Here are the theorems stating various different topological indices of the nilpotent graph of the ring $\mathbb{Z}_{p^k}$. The formula for the Wiener index is provided in the following theorem.

Theorem 8 *Let $\Gamma_N(\mathbb{Z}_{p^k})$ be the nilpotent graph of the ring $\mathbb{Z}_{p^k}$ with p as a prime number and $k \in N$, then:*

$$W\left(\Gamma_N(\mathbb{Z}_{p^k})\right) = \frac{p^{2k-2} + 2p^{2k} - 2p^{2k-1} - 2p^k + p^{k-1}}{2}$$

Proof: For this graph, it can be divided into several cases, namely the diameter between nilpotent vertices with nilpotent vertices, the diameter between non-nilpotent vertices with non-nilpotent vertices, and the diameter between nilpotent vertices with non-nilpotent vertices, resulting in the following:

Case 1: $\Sigma d(v_i, v_j)$ for $v_i, v_j \in N(\mathbb{Z}_{p^k})$ and $i \neq j$, so that:

Here, $d(v_i, v_j) = 1$ holds because every nilpotent element is adjacent to all vertices in the graph, and $\left|N(\mathbb{Z}_{p^k})\right| = p^{k-1}$. Therefore, we have $\Sigma d(v_i, v_j) = C_2^{p^{k-1}} \times d(v_i, v_j) = \left(\frac{(p^{k-1})(p^{k-1}-1)}{2}\right)(1) = \frac{(p^{k-1})(p^{k-1}-1)}{2}$,

Case 2: $\Sigma d(v_i, v_j)$ for $v_i, v_j \notin N(\mathbb{Z}_{p^k})$ and $i \neq j$, so that:

Here, $d(v_i, v_j) = 2$ applies because every pair of non-nilpotent elements are connected but not adjacent, and every non-nilpotent vertex is connected to a nilpotent vertex. Then, $\left|N^C(\mathbb{Z}_{p^k})\right| = p^k - p^{k-1}$, Therefore, we obtain $d(v_i, v_j) = C_2^{p^k - p^{k-1}} \times d(v_i, v_j) = \left(\frac{(p^k-p^{k-1})(p^k-p^{k-1}-1)}{2}\right)(2) = (p^k - p^{k-1})(p^k - p^{k-1} - 1)$, and

Case 3: $\Sigma d(v_i, v_j)$ for $v_i \in N(\mathbb{Z}_{p^k})$, $v_j \notin N(\mathbb{Z}_{p^k})$, and $i \neq j$, so that:

In this case, $d(v_i, v_j) = 1$ applies because every nilpotent element is adjacent to all vertices in the graph. Then, $\left|N^C(\mathbb{Z}_{p^k})\right| = p^k - p^{k-1}$ and $\left|N(\mathbb{Z}_{p^k})\right| = p^{k-1}$, therefore we obtain:

$\Sigma d(v_i, v_j) = C_1^{p^{k-1}} \times C_2^{p^k - p^{k-1}} \times d(v_i, v_j) = ((p^{k-1})(p^k - p^{k-1}))(1) = (p^{k-1})(p^k - p^{k-1})$.

We obtain the Wiener index as the sum of the results from these 3 cases:

$$W\left(\Gamma_N(\mathbb{Z}_{p^k})\right) = \frac{(p^{k-1})(p^{k-1}-1)}{2} + \left((p^k - p^{k-1})(p^k - p^{k-1} - 1)\right) + ((p^{k-1})(p^k - p^{k-1}) = \frac{p^{2k-2}-p^{k-1}}{2} + p^{2k} - 2p^{2k-1} - p^k + p^{2k-2} + p^{k-1} + p^{2k-1} - p^{2k-2} = \frac{p^{2k-2}+2p^{2k}-2p^{2k-1}-2p^k+p^{k-1}}{2}. \blacksquare$$

The formula for the first Zagreb index is provided in the following theorem

Theorem 9 *Let $\Gamma_N(\mathbb{Z}_{p^k})$ be the nilpotent graph of the ring $\mathbb{Z}_{p^k}$ with p as a prime number and $k \in N$, then $M_1\Gamma_N(\mathbb{Z}_{p^k}) = \left((p^{k-1})(p^k - 1)^2\right) + \left((p^k - p^{k-1})(p^{k-1})^2\right)$.*

Proof: In the nilpotent graph, it is divided into 2 cases based on vertices, namely:

Case 1: For the Nilpotent Vertices

Since every nilpotent vertex is adjacent to all other vertices, then we have $\deg(\bar{v}) = p^k - 1$ for $\forall \bar{v} \in N(\mathbb{Z}_{p^k})$ and the number of nilpotent vertices in the graph is $|N(\mathbb{Z}_{p^k})| = p^{k-1}$. Therefore, the Zagreb Index restricted to all nilpotent vertices is $\left((p^{k-1})(p^k - 1)^2\right)$.

Case 2: For the Non-Nilpotent Vertices

Because every two non-nilpotent vertices are connected but not adjacent to each other, yet every non-nilpotent vertex is adjacent to every nilpotent vertex, $\deg(\bar{v}) = p^k - 1$ for $\forall \bar{v} \notin N(\mathbb{Z}_{p^k})$ and the number of non-nilpotent vertices in the graph is $|N(\mathbb{Z}_{p^k})| = p^k - p^{k-1}$. Therefore, the Zagreb Index restricted to the non-nilpotent vertices is $(p^k - p^{k-1})(p^{k-1})^2$.

As a result, we obtain: $M_1\Gamma_N(\mathbb{Z}_{p^k}) = \left((p^{k-1})(p^k - 1)^2\right) + \left((p^k - p^{k-1})(p^{k-1})^2\right)$. ■

The formula for the Gutman index is provided in the following theorem

Theorem 10 *Let $\Gamma_N(\mathbb{Z}_{p^k})$ be the nilpotent graph of the ring $\mathbb{Z}_{p^k}$ with p as a prime number and $k \in \mathbb{N}$, then:*

$$Gut\left(\Gamma_N(\mathbb{Z}_{p^k})\right) = \frac{(p^{k-1})(p^{k-1} - 1)}{2}(p^k - 1)^2 +$$
$$(p^{2k-2})\left((p^k - p^{k-1})^2 - (p^k - p^{k-1})\right) + (p^k - p^{k-1})(p^{k-1})^2(p^k - 1)$$

Proof: The calculation of the Gutman Index can be divided into three cases based on the connection of vertices: nilpotent vertices with nilpotent vertices, non-nilpotent vertices with non-nilpotent vertices, and nilpotent vertices with non-nilpotent vertices, resulting in the following:

Case 1: $\Sigma d(v_i, v_j)$ for $v_i, v_j \in N(Z_{p^k})$ and $i \neq j$. In this case, $d(v_i, v_j) = 1$ because every nilpotent element is adjacent to all vertices in the graph, with a degree of each nilpotent vertex being $p^k - 1$, and $|N(\mathbb{Z}_{p^k})| = p^{k-1}$. Therefore,

$$Gut_1\left(\Gamma_N(\mathbb{Z}_{p^k})\right) = \sum_{\{x,y\} \in V(\mathbb{Z}_{p^k})} \deg(x)\deg(y)\, d(x,y)$$

$$= \binom{p^{k-1}}{2}(p^k - 1)(p^k - 1)(1)$$

$$= \frac{(p^{k-1})(p^{k-1} - 1)}{2}(p^k - 1)^2$$

Case 2: $\Sigma d(v_i, v_j)$ for $v_i, v_j \notin N(\mathbb{Z}_{p^k})$ and $i \neq j$. In this case, $d(v_i, v_j) = 2$ because every pair of non-nilpotent elements are connected but not adjacent, and each non-nilpotent vertex is connected to a nilpotent vertex. Then, $|N^C(\mathbb{Z}_{p^k})| = p^k - p^{k-1}$. Therefore,

$$Gut_3\left(\Gamma_N(\mathbb{Z}_{p^k})\right) = \sum_{\{x,y\} \in V(\mathbb{Z}_{p^k})} \deg(x)\deg(y)\, d(x,y)$$

$$= \binom{p^k - p^{k-1}}{2}(p^{k-1})(p^{k-1})(2)$$

$$= \frac{(p^k - p^k - 1)(p^k - p^{k-1} - 1)}{2}(p^k - 1)^2(2)$$

$$= (p^{2k-2})((p^k - p^{k-1})^2 - (p^k - p^{k-1}))$$

Case 3: $\Sigma d(v_i, v_j)$ for $v_i \in N(\mathbb{Z}_{p^k}), v_j \notin N(\mathbb{Z}_{p^k})$ and $i \neq j$, with $d(v_i, v_j) = 1$ because every nilpotent element is adjacent to all vertices in the graph. Then, $\left|N^C(\mathbb{Z}_{p^k})\right| = p^k - p^{k-1}$ and $\left|N(\mathbb{Z}_{p^k})\right| = p^{k-1}$. Therefore, based on cases 1, 2, and 3, the Gutman Index of $\Gamma_N(\mathbb{Z}_{p^k})$ is obtained as follows,

$$\begin{aligned} \mathrm{Gut}\left(\Gamma_N(\mathbb{Z}_{p^k})\right) &= \mathrm{Gut}_1\left(\Gamma_N(\mathbb{Z}_{p^k})\right) + \mathrm{Gut}_2\left(\Gamma_N(\mathbb{Z}_{p^k})\right) + \mathrm{Gut}_3\left(\Gamma_N(\mathbb{Z}_{p^k})\right) \\ &= \frac{(p^{k-1})(p^{k-1} - 1)}{2}(p^k - 1)^2 + (p^{2k-2})\left((p^k - p^{k-1})^2 - (p^k - p^{k-1})\right) \\ &\quad + (p^k - p^{k-1})(p^{k-1})^2(p^k - 1). \end{aligned}$$

This completes the proof. ■

IV. CONCLUSION

In this study, it is shown that the nilpotent graph of the ring $\mathbb{Z}_n$ for *n* being a power of $k \in \mathbb{N}$ of the prime number p contains a complete subgraph. It is also demonstrated that this nilpotent graph contains p^{k-1} twin star subgraphs $K_{1,n-1}$. Additionally, we have successfully formulated the Weiner Index, the first Zagreb index, and the Gutman Index.

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