JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 7 NO. 2 (NOV 2024)
Available online at www.jfma.math.fsm.undip.ac.id

THE FRACTIONAL DERIVATIVE OF SOME FUNCTIONS

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Abstract. This paper aims to inquire, investigate, extend and generalize the order of the derivative from the set of integers into set of non-integers, known as the fractional derivative, of some functions. In particular, this paper introduces, determines and provides sufficient conditions in the existence of $\frac{p}{2}th$ derivative of the functions in the form $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)$ and $D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}$ where k, k_1, k_2 are nonzero real numbers, $m\in\mathbb{Z}$ and p is odd. This paper also presents and describes the graphs of $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)$ and $D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}$ where k, k_1, k_2 are nonzero real numbers, $m\in\mathbb{Z}^+\cup\{0\}$ and p is odd.

Keywords: fractional derivative, non-integer, nth derivative

I. INTRODUCTION

Fractional calculus has attracted a lot of attention lately, mostly as a result of the growing number of applied science research projects. Aside from that, the study of fractional calculus has advanced mathematically, resulting in connections with other areas of mathematics like probability and stochastic process analysis. It is the new calculus of modernity, where derivatives and integrals of any order can be performed because integration and differentiation procedures are no longer restricted to integer orders. The growing number of papers published in this field and its application to other applied science fields [1–9] demonstrate that it has been a focus of research worldwide and is currently regarded as a highly studied issue.

The relevance of fractional calculus was supported by the numerous definitions of fractional derivatives that have emerged over time. Some of these definitions have been driven by pure theoretical goals, but more significantly, many have been obtained with a judgement toward their potential use in various research domains. In the literature, several different definitions of fractional integrals and derivatives are present. Some of the most prominent definitions, each with its own significance and use: Hadamard, Weyl, Caputo–Hadamard, Hilfer, Riesz, Hilfer-Hadamard, Erdélyi–Kober, Caputo–Riesz, and Grünwald–Letnikov [1, 2, 10–12]. On the otherhand, some applications can be found in [13-20]. According to some research, taking into account derivatives of non-integer order helps us better fit a theoretical model to experimental data and forecast the dynamics of the processes it describes in the future.

In this paper, we aim to expose some mathematical theories on the fractional derivative of some functions. Specifically, it intends to devise the generalized formula for finding $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)$ and $D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}$ where p is odd, $m\in\mathbb{Z}$ and k,k_1,k_2 are nonzero real numbers. This work also will provide sufficient conditions in the existence of $\frac{p}{2}th$ derivative of the functions $f(x)=kx^{\frac{m}{2}}$ and $g(x)=\left(k_1x\pm k_2\right)^{\frac{m}{2}}$ to be a real number, a complex number and a complex infinity. It will also present and describe the graphs of $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)$ and $D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}$ where p is odd, $m\in\mathbb{Z}^+\cup\{0\}$ and k,k_1,k_2 are nonzero real numbers.

II. RESULTS AND DISCUSSION

We determine the sufficient conditions in the existence of $\frac{p}{2}th$ derivative of the functions $f(x)=kx^{\frac{m}{2}}$ and $f(x)=\left(k_1x\pm k_2\right)^{\frac{m}{2}}$ to be a real number, a complex number and a complex infinity. This presents and describes the graphs of $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)$ and $D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}$ where p is odd, $m\in\mathbb{Z}^+\cup\{0\}$ and k,k_1,k_2 are nonzero real numbers.

Theorem 1. *Let $f(x)=kx^{\frac{m}{2}}$ where $m\in\mathbb{Z}$ and $k,x\in\mathbb{R}$, $k\neq 0$. Then the fractional derivative $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)$ is given by*

(i) $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\dfrac{2^{m-p}\left(\frac{m-p-1}{2}\right)!\left(\frac{m}{2}\right)!}{(m-p)!}\sqrt{\dfrac{k^2x^{m-p}}{\pi}}$ *if $m\in\mathbb{Z}^+$ is even and $m-p>0$;*

(ii) $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\dfrac{2^{m-p+1}\left(\frac{m}{2}\right)!(-m+p)!}{(m-p)(-1)^{\left(\frac{-m+p+1}{2}\right)}\left(\frac{-m+p-1}{2}\right)!}\sqrt{\dfrac{k^2x^{m-p}}{\pi}}$ *if $m\in\mathbb{Z}^+$ is even and $m-p<0$; and*

(iii) $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\dfrac{m!}{2^m\left(\frac{m-p}{2}\right)!\left(\frac{m-1}{2}\right)!}\sqrt{k^2\pi x^{m-p}}$ *if $m\in\mathbb{Z}^+$ is odd and $m-p>0$*

where p is odd.

Proof: (i) Suppose $f(x)=kx^{\frac{m}{2}}$, $m\in\mathbb{Z}^+$ is even, $m-p>0$, p is odd and $k,x\in\mathbb{R}$ with $k\neq 0$. Then by definition, we know that

$$D_x\left(kx^{\frac{m}{2}}\right)=k\left(\frac{m}{2}\right)x^{\left(\frac{m}{2}-1\right)}$$

$$D_xD_x\left(kx^{\frac{m}{2}}\right)=D_x^2\left(kx^{\frac{m}{2}}\right)=D_x\left[k\left(\frac{m}{2}\right)x^{\left(\frac{m}{2}-1\right)}\right]=k\left(\frac{m}{2}\right)\left(\frac{m}{2}-1\right)x^{\left(\frac{m}{2}-2\right)}$$

Continuing this pattern we will obtain

$$\begin{aligned}D_x^n\left(kx^{\frac{m}{2}}\right)&=D_xD_x^{n-1}\left(kx^{\frac{m}{2}}\right)=D_xk\left(\frac{m}{2}\right)\left(\frac{m}{2}-1\right)\left(\frac{m}{2}-2\right)\left(\frac{m}{2}-3\right)\cdots\left(\frac{m}{2}-n+2\right)x^{\frac{m}{2}-(n-1)}\\&=k\left(\frac{m}{2}\right)\left(\frac{m}{2}-1\right)\left(\frac{m}{2}-2\right)\left(\frac{m}{2}-3\right)\cdots\left(\frac{m}{2}-n+2\right)\left(\frac{m}{2}-n+1\right)x^{\frac{m}{2}-(n-1)-1}\\&=\frac{k}{2^n}m(m-2)(m-4)(m-6)\cdots(m-2n+4)(m-2n+2)x^{\left(\frac{m}{2}-n\right)}\end{aligned}$$

Multiplying the numerator and denominator by $(m-2n)!!$, we will arrive

$$D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{k}{2^n}\frac{m(m-2)(m-4)(m-6)\cdots(m-2n+4)(m-2n+2)x^{\left(\frac{m}{2}-n\right)}}{1}\cdot\frac{(m-2n)!!}{(m-2n)!!}$$

$D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{m!!}{2^n(m-2n)!!}kx^{\frac{m}{2}-n}$. Now, if $m\in\mathbb{Z}^+$ is even then let $m=2r$ for any $r\in\mathbb{Z}^+$ implies that

$$\begin{aligned}m!!=2r!!&=2r(2r-2)(2r-4)(2r-8)\cdots\cdots4\cdot2\\&=\frac{2r(2r-1)(2r-2)(2r-3)(2r-4)\cdots\cdots3\cdot2\cdot1}{(2r-1)(2r-3)(2r-5)(2r-7)\cdots\cdots3\cdot1}\\&=2(r)2(r-1)(r-2)(r-3)(r-4)\cdots\cdots2(3)\cdot2(2)\cdot2(1)\\&=2^r r!\end{aligned}$$

Also,

$(m-2n)!!=(2r-2n)!!=(2(r-n))!!=(2(r-n))(2(r-n)-2)(2(r-n)-4)\cdots\cdots4\cdot2$
$=2^{r-n}(r-n)!$. It follows that $D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{2^r r!}{2^n2^{(r-n)}(r-n)!}kx^{\left(\frac{m}{2}-n\right)}=\frac{r!}{(r-n)!}kx^{\left(\frac{m}{2}-n\right)}$.

Substituting with $r=\frac{m}{2}$ and $n=\frac{p}{2}$, results in $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{\left(\frac{m}{2}\right)!}{\left(\frac{m-p}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}$. Note that $m\in\mathbb{Z}^+$ is even and p is odd which means that $\left(\frac{m-p}{2}\right)$ is non-integer in

$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{\left(\frac{m}{2}\right)!}{\left(\frac{m-p}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}$ and since $m-p>0$, then we can express $\left(\frac{m-p}{2}\right)!$, we have $\left(\frac{m-p}{2}\right)!=\Gamma\left(\left(\frac{m-p}{2}\right)+1\right)=\left(\frac{m-p}{2}\right)\Gamma\left(\frac{m-p}{2}\right)$. Then solving for $\Gamma\left(\frac{m-p}{2}\right)$ yields in $\Gamma\left(\frac{m-p}{2}\right)=\frac{2^{(1-(m-p))}\sqrt{\pi}\left((m-p)-1\right)!}{\left(\frac{m-p-1}{2}\right)!}$. Substituting,

$$\left(\frac{m-p}{2}\right)!=\left(\frac{m-p}{2}\right)\frac{2^{(1-m+p)}\sqrt{\pi}\left(m-p-1\right)!}{\left(\frac{m-p-1}{2}\right)!}=\frac{(m-p)(m-p-1)!2^{(-m+p+1)}\sqrt{\pi}}{2\left(\frac{m-p-1}{2}\right)!}=\frac{(m-p)!}{2^{(m-p)}\left(\frac{m-p-1}{2}\right)!}\sqrt{\pi}.$$

Substituting this result into $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{\left(\frac{m}{2}\right)!}{\left(\frac{m-p}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}$, hence, we have

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{2^{m-p}\left(\frac{m-p-1}{2}\right)!\left(\frac{m}{2}\right)!}{(m-p)!}\sqrt{\frac{k^2x^{m-p}}{\pi}}.$$

(ii) Suppose $f(x)=kx^{\frac{m}{2}}$, $m\in\mathbb{Z}^+$ is even, $m-p<0$, p is odd and $k,x\in\mathbb{R}$ with $k\neq 0$. By (i) $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{\left(\frac{m}{2}\right)!}{\left(\frac{m-p}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}$, we have $\left(\frac{m-p}{2}\right)!=\left(\frac{m-p}{2}\right)\Gamma\left(\frac{m-p}{2}\right)$. Since $m-p<0$, we can express $\Gamma\left(\frac{m-p}{2}\right)$, which means

$$\Gamma\left(\frac{m-p}{2}\right)=\Gamma\left(-\left(\frac{-m+p}{2}\right)\right)=\frac{(-1)^{\left(\frac{-m+p+1}{2}\right)}2^{(-m+p)}\left(\frac{-m+p-1}{2}\right)!\sqrt{\pi}}{(-m+p)!},$$

it follows that $\left(\frac{m-p}{2}\right)!=\left(\frac{m-p}{2}\right)\Gamma\left(\frac{m-p}{2}\right)=\frac{(m-p)\left(\frac{-m+p-1}{2}\right)!(-1)^{\left(\frac{-m+p+1}{2}\right)}2^{-m+p-1}\sqrt{\pi}}{(-m+p)!}$.

Thus, substituting this result into $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{\left(\frac{m}{2}\right)!}{\left(\frac{m-p}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}$, we obtain

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{2^{m-p+1}\left(\frac{m}{2}\right)!(-m+p)!}{(-1)^{\left(\frac{-m+p+1}{2}\right)}(m-p)\left(\frac{-m+p-1}{2}\right)!}\sqrt{\frac{k^2x^{m-p}}{\pi}}\ .$$

(iii) Suppose $f(x)=kx^{\frac{m}{2}}$, $m\in\mathbb{Z}^+$ is odd, $m-p>0$, p is odd and $k,x\in\mathbb{R}$ with $k\neq 0$. By (i) $D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{k}{2^n}\frac{m!!}{(m-2n)!!}$, subsequently, if $m\in\mathbb{Z}^+$ is odd, that is, $m=2r+1$ for any $r\in\mathbb{Z}^+$ which implies

$$\begin{aligned}m!!=(2r+1)!!&=(2r+1)(2r-1)(2r-3)(2r-5)\cdots\cdots 3\cdot 1\\&=\frac{(2r+1)!}{2^r r!}\end{aligned}$$

and $(m-2n)!!=(2r-2n+1)!!=(2r-2n+1)(2r-2n-1)(2r-2n-3)(2r-2n-5)\cdots\cdots 3\cdot 1$ $=\frac{(2r-2n+1)!}{2^{r-n}(r-n)!}$. Then substituting $m!!=\frac{(2r+1)!}{2^r r!}$ and $(m-2n)!!=\frac{(2r-2n+1)!}{2^{r-n}(r-n)!}$ into $D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{m!!}{2^n(m-2n)!!}kx^{\left(\frac{m-2n}{2}\right)}$, we now have

$$D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{\frac{(2r+1)!}{2^r r!}}{\frac{2^n(2r-2n+1)!}{2^{r-n}(r-n)!}}kx^{\left(\frac{m-2n}{2}\right)}=\frac{(2r+1)!(r-n)!}{2^{2n}r!(2r-2n+1)!}kx^{\left(\frac{m-2n}{2}\right)}$$

And replacing $r=\frac{m-1}{2}$ and $n=\frac{p}{2}$, we will get

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{m!\left(\frac{m-p-1}{2}\right)!}{2^p(m-p)!\left(\frac{m-1}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}.$$

Note that $m\in\mathbb{Z}^+$ is odd and p is odd implies that $\left(\frac{m-p-1}{2}\right)$ is non-integer, thus, we can express $\left(\frac{m-p-1}{2}\right)!$, since $m-p>0$, we have

$$\left(\frac{m-p-1}{2}\right)!=\Gamma\left(\frac{m-p-1}{2}+1\right)=\Gamma\left(\frac{m-p+1}{2}\right)=\frac{2^{-m+p}(m-p)!\sqrt{\pi}}{\left(\frac{m-p}{2}\right)!}.$$

Hence,

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{m!\left(\frac{m-p-1}{2}\right)!}{2^p(m-p)!\left(\frac{m-1}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}=\frac{m!\frac{2^{-m+p}(m-p)!\sqrt{\pi}}{\left(\frac{m-p}{2}\right)!}}{2^p(m-p)!\left(\frac{m-1}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}=\frac{m!}{2^m\left(\frac{m-p}{2}\right)!\left(\frac{m-1}{2}\right)!}\sqrt{k^2\pi x^{m-p}}\ .\ \blacksquare$$

Example 1. Let $f(x)=x$, $m=2$, $p=1$ and $k=1$. Since $m\in\mathbb{Z}^{+}$ is even, $m-p>0$, p is odd, we can apply (i) in Theorem 1 so we have

$$D_x^{\frac{1}{2}}x^{\frac{2}{2}}=D_x^{\frac{1}{2}}x=\frac{2^{2-1}\left(\frac{2-1-1}{2}\right)!\left(\frac{2}{2}\right)!}{(2-1)!}\sqrt{\frac{1^2x^{2-1}}{\pi}}=2\sqrt{\frac{x}{\pi}}.$$

Corollary 2. *Let $f(x)=kx^{\frac{m}{2}}$. Then the fractional derivative $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)$ is a complex number given by*

(i) $$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{(i)^p\,2^{m-p+1}(-m+p-1)!}{\left(\frac{-m+p-1}{2}\right)!\left(\frac{-m-2}{2}\right)!}\sqrt{k^2\pi x^{m-p}}$$ *if $m\in\mathbb{Z}^{-}$ is even and $m-p<0$;*

(ii) $$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^{\left(\frac{m}{2}\right)}i2^{m-p}\left(\frac{m-p-1}{2}\right)!}{(m-p)!\left(\frac{-m-2}{2}\right)!}\sqrt{k^2\pi x^{m-p}}$$ *if $m\in\mathbb{Z}^{-}$ is even and $m-p>0$; and*

(iii) $$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{(i)^p\left(\frac{-m+p}{2}\right)!\left(\frac{-m-1}{2}\right)!}{2^m(-m+p)(-m-1)!}\sqrt{\frac{k^2x^{m-p}}{\pi}}$$ *if $m\in\mathbb{Z}^{-}$ is odd and $m-p<0$*

where p is odd.

Proof: (i) Suppose $f(x)=kx^{\frac{m}{2}}$, $m\in\mathbb{Z}^{-}$ is even, $m-p<0$ and p is odd. By (i) in Theorem 1, we get

$$D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{k}{2^n}\left[m(m-2)(m-4)(m-6)\cdots\cdots(m-2n+4)(m-2n+2)x^{\frac{m}{2}-n}\right].$$

Since $m\in\mathbb{Z}^{-}$ is even, then $m, m-2, m-4, m-6,\ldots, m-2n+4, m-2n+2$ are negative. To make them positive, we multiply by $1=(-1)(-1)$ and obtain

$$D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{k}{2^n}(-1)(-1)\left[m(m-2)(m-4)(m-6)\cdots(m-2n+4)(m-2n+2)x^{\left(\frac{m}{2}-n\right)}\right]$$

$$D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{k}{2^n}(-1)^n(-m)(-m+2)(-m+4)(-m+6)\cdots(-m+2n-4)(-m+2n-2)x^{\left(\frac{m}{2}-n\right)}$$

Likewise multiplying the numerator and denominator by $(-m-2)!!$ to have

$$D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^n(-m)(-m+2)(-m+4)\cdots\cdots(-m+2n-4)(-m+2n-2)kx^{\left(\frac{m}{2}-n\right)}}{2^n}\cdot\frac{(-m-2)!!}{(-m-2)!!}$$

and

$$D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^n(-m+2n-2)!!}{2^n(-m-2)!!}kx^{\left(\frac{m-2n}{2}\right)}.$$

Since $m\in\mathbb{Z}^{-}$ is even, we let $m=-2s$ for $s\in\mathbb{Z}^{+}$, then by substitution, we get

$$\begin{aligned}(-m+2n-2)!! &= (2s+2n-2)!! \\ &= (2s+2n-2)(2s+2n-4)(2s+2n-6)(2s+2n-8)\cdots 4\cdot 2 \\ &= \frac{(2s+2n-2)(2s+2n-3)(2s+2n-4)(2s+2n-5)\cdots\cdots 3\cdot 2\cdot 1}{(2s+2n-3)(2s+2n-5)(2s+2n-7)(2s+2n-9)\cdots\cdots 3\cdot 1} \\ &= 2^{s+n-1}(s+n-1)!.\end{aligned}$$

Also,

$$\begin{aligned}(-m-2)!! = (2s-2)!! &= (2s-2)(2s-4)(2s-6)(2s-8)\cdots\cdots 4\cdot 2 \\ &= \frac{(2s-2)(2s-3)(2s-4)(2s-5)\cdots\cdots 3\cdot 2\cdot 1}{(2s-3)(2s-5)(2s-7)(2s-9)\cdots\cdots 3\cdot 1} \\ &= 2(s-1)2(s-2)2(s-3)2(s-4)\cdots\cdots 2(2)\cdot 2(1) \\ &= 2^{s-1}(s-1)!.\end{aligned}$$

Substituting $(-m+2n-2)!! = 2^{s+n-1}(s+n-1)!$ and $(-m-2)!! = 2^{s-1}(s-1)!$ into $D_x^n\left(kx^{\frac{m}{2}}\right) = \frac{(-1)^n(-m+2n-2)!!}{2^n(-m-2)!!}kx^{\left(\frac{m-2n}{2}\right)}$ yields

$$D_x^n\left(kx^{\frac{m}{2}}\right) = \frac{(-1)^n 2^{s+n-1}(s+n-1)!}{2^n 2^{s-1}(s-1)!}kx^{\left(\frac{m-2n}{2}\right)}$$

and thus

$$D_x^n\left(kx^{\frac{m}{2}}\right) = \frac{(-1)^n(s+n-1)!}{(s-1)!}kx^{\left(\frac{m-2n}{2}\right)}.$$

Replacing $s = -\frac{m}{2}$ and $n = \frac{p}{2}$ results in

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right) = \frac{(-1)^{\frac{p}{2}}\left(\frac{-m+p-2}{2}\right)!}{\left(\frac{-m-2}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}.$$

Since $m \in \mathbb{Z}^-$ is even and p is odd implies that $\left(\frac{-m+p-2}{2}\right)$ is non-integer in

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right) = \frac{(-1)^{\frac{p}{2}}\left(\frac{-m+p-2}{2}\right)!}{\left(\frac{-m-2}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}.$$

So, we can express $\left(\frac{-m+p-2}{2}\right)!$, and we obtain

$$\left(\frac{-m+p-2}{2}\right)! = \Gamma\left(\frac{-m+p-2}{2}+1\right) = \Gamma\left(\frac{-m+p}{2}\right) = \frac{2^{m-p+1}(-m+p-1)!}{\left(\frac{-m+p-1}{2}\right)!}.$$

Therefore, we get

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^{\frac{p}{2}}\left(\frac{-m+p-2}{2}\right)!}{\left(\frac{-m-2}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}=\frac{(-1)^{\frac{p}{2}}\frac{2^{m-p+1}(-m+p-1)!\sqrt{\pi}}{\left(\frac{-m+p-1}{2}\right)!}}{\left(\frac{-m-2}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}$$

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^{\frac{p}{2}}2^{m-p+1}(-m+p-1)!}{\left(\frac{-m+p-1}{2}\right)!\left(\frac{-m-2}{2}\right)!}\sqrt{k^2\pi x^{m-p}}=\frac{\left((-1)^{\frac{1}{2}}\right)^p 2^{m-p+1}(-m+p-1)!}{\left(\frac{-m+p-1}{2}\right)!\left(\frac{-m-2}{2}\right)!}\sqrt{k^2\pi x^{m-p}}$$

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{(i)^p 2^{m-p+1}(-m+p-1)!}{\left(\frac{-m+p-1}{2}\right)!\left(\frac{-m-2}{2}\right)!}\sqrt{k^2\pi x^{m-p}}\,.$$

This completes the proof of part (*i*).

(ii) Suppose $f(x)=kx^{\frac{m}{2}}$, $m\in\mathbb{Z}^-$ is even, $m-p>0$, and p is odd. By (i) $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^{\frac{p}{2}}\left(\frac{-m+p-2}{2}\right)!}{\left(\frac{-m-2}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}$ and since $-m+p<0$, we can express $\left(\frac{-m+p-2}{2}\right)!$ as

$$\left(\frac{-m+p-2}{2}\right)!=\Gamma\left(\frac{-m+p-2}{2}\right)=\text{b}$$

$$\Gamma\left(\frac{-m+p}{2}\right)=\Gamma\left(-\left(\frac{m-p}{2}\right)\right)=\frac{(-1)^{\frac{m-p+1}{2}}2^{m-p}\left(\frac{m-p-1}{2}\right)!\sqrt{\pi}}{(m-p)!}.$$

Henceforth,

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^{\frac{p}{2}}\left(\frac{-m+p-2}{2}\right)!}{\left(\frac{-m-2}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}=\frac{(-1)^{\frac{p}{2}}\frac{(-1)^{\left(\frac{m-p+1}{2}\right)}2^{m-p}\left(\frac{m-p-1}{2}\right)!\sqrt{\pi}}{(m-p)!}}{\left(\frac{-m-2}{2}\right)!}kx^{\left(\frac{m-p}{2}\right)}$$

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^{\left(\frac{m}{2}\right)}i2^{m-p}\left(\frac{m-p-1}{2}\right)!}{(m-p)!\left(\frac{-m-2}{2}\right)!}\sqrt{k^2\pi x^{m-p}}\,.$$

This completes the proof of part (*ii*).

(iii) Suppose $f(x)=kx^{\frac{m}{2}}$, $m\in\mathbb{Z}^-$ is odd, $m-p<0$, and p is odd. By (i), $D_x^{n}\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^n(-m+2n-2)!!}{2^n(-m-2)!!}kx^{\left(\frac{m-2n}{2}\right)}$. If $m\in\mathbb{Z}^-$ is odd so we let $m=-2t+1$ for any $t\in\mathbb{Z}^+$ which means that by double factorial, we have $(-m+2n-2)!!=(2t+2n-3)!!$

$=\left(2\left(\frac{2t+2n-2}{2}\right)-1\right)!!=\frac{(2t+2n-2)!}{2^{\left(\frac{2t+2n-2}{2}\right)}\left(\frac{2t+2n-2}{2}\right)!}$ and $(-m-2)!!=(2t-3)!!=(2(t-1)-1)!!=\frac{(2t-2)!}{2^{t-1}(t-1)!}$. By substitution, it follows that $D_x^n\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^n(2t+2n-2)!(t-1)!}{2^{2n}\left(\frac{2t+2n-2}{2}\right)!(2t-2)!}kx^{\left(\frac{m-2n}{2}\right)}$. Now, substituting $t=\frac{-m+1}{2}$ and $n=\frac{p}{2}$ then we will obtain

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^{\frac{p}{2}}(-m+p-1)!\left(\frac{-m-1}{2}\right)!}{2^p\left(\frac{-m+p-1}{2}\right)!(-m-1)!}kx^{\left(\frac{m-p}{2}\right)}=\frac{(-1)^{\frac{p}{2}}(-m+p-1)!\left(\frac{-m-1}{2}\right)!}{2^p\left(\frac{-m+p-1}{2}\right)!(-m-1)!}kx^{\left(\frac{m-p}{2}\right)}.$$

Since $m\in\mathbb{Z}^-$ is odd and p is odd which implies that $\left(\frac{-m+p-1}{2}\right)$ is non-integer. Since $-m+p>0$, thus, we have

$$\left(\frac{-m+p-1}{2}\right)!=\Gamma\left(\left(\frac{-m+p-1}{2}\right)+1\right)=\Gamma\left(\frac{-m+p+1}{2}\right)=\frac{2^{m-p}(-m+p)!\sqrt{\pi}}{\left(\frac{-m+p}{2}\right)!}.$$

Now, substituting

$$\left(\frac{-m+p-1}{2}\right)!=\frac{2^{m-p}(-m+p)!\sqrt{\pi}}{\left(\frac{-m+p}{2}\right)!}$$

into

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{(-1)^{\frac{p}{2}}(-m+p-1)!\left(\frac{-m-1}{2}\right)!}{2^p\left(\frac{-m+p-1}{2}\right)!(-m-1)!}kx^{\left(\frac{m-p}{2}\right)},$$

we have

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{\left((-1)^{\frac{1}{2}}\right)^p\left(\frac{-m+p}{2}\right)!\left(\frac{-m-1}{2}\right)!}{2^m(-m+p)(-m-1)!}\sqrt{\frac{k^2x^{m-p}}{\pi}}=\frac{(i)^p\left(\frac{-m+p}{2}\right)!\left(\frac{-m-1}{2}\right)!}{2^m(-m+p)(-m-1)!}\sqrt{\frac{k^2x^{m-p}}{\pi}}.\ \blacksquare$$

Example 2. Let $f(x)=x^{-1}$, $m=-2$, $p=1$ and $k=1$. Since $m\in\mathbb{Z}^-$ is even, $m-p<0$ and p is odd, we can apply (i) in Corollary 2 so we have

$$D_x^{\frac{1}{2}}\left(x^{-\frac{2}{2}}\right)=D_x^{\frac{1}{2}}\left(x^{-1}\right)=\frac{2^{-2-1+1}\left(-(-2)+1-1\right)!(i)^1}{\left(\frac{-2+1-1}{2}\right)!\left(\frac{2-2}{2}\right)!}\sqrt{1^2\pi x^{-2-1}}=\frac{1}{2}i\sqrt{\frac{\pi}{x^3}}.$$

Corollary 3. *Let $f(x)=kx^{\frac{m}{2}}$. Then the fractional derivative $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)$ is zero if $m\in\mathbb{Z}^+$ and $m-p<0$ where p and m are odd.*

Proof: Let $m \in \mathbb{Z}^{+}$ is odd, $m-p<0$ and p is odd. By (iii) in Theorem 1, we have

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{m!}{2^m\left(\frac{m-p}{2}\right)!\left(\frac{m-1}{2}\right)!}\sqrt{k^2\pi x^{m-p}},$$

and we can express $\left(\frac{m-p}{2}\right)!=\Gamma\left(\frac{m-p}{2}+1\right)=\Gamma\left(\frac{m-p+2}{2}\right)$ so we get

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{m!}{2^m\Gamma\left(\frac{m-p+2}{2}\right)\left(\frac{m-1}{2}\right)!}\sqrt{k^2\pi x^{m-p}} \text{ for } m-p+2>0.$$

Since $m \in \mathbb{Z}^{+}$ is odd, $m-p<0$ and p is odd, we let $m=2s-1$ and $p=2r+1$ where $s,r \in \mathbb{Z}^{+}$. Since

$$m-p<0 \Rightarrow 2s-1-(2r+1)<0 \Rightarrow 2s-1-2r-1<0 \Rightarrow 2s-2r-2<0$$
$$\Rightarrow 2(s-r-1)<0 \Rightarrow s-r-1<0 \Rightarrow s-r<1,$$

substituting $s=\frac{m+1}{2}$ and $r=\frac{p-1}{2}$, it follows that

$$s-r<1 \Rightarrow \frac{m+1}{2}-\left(\frac{p-1}{2}\right)<1 \Rightarrow \frac{m-p+2}{2}<1.$$

Then by definition, $\Gamma\left(\frac{m-p+2}{2}\right)=\tilde{\infty}$. So it implies that

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{m!}{2^m\tilde{\infty}\left(\frac{m-1}{2}\right)!}\sqrt{k^2\pi x^{m-p}}.$$

By arithmetical operations of complex infinity, $\frac{c}{\tilde{\infty}}=0$ where $c \in \mathbb{C}$, we get

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{1}{\tilde{\infty}}\cdot\frac{m!}{2^m\left(\frac{m-1}{2}\right)!}\sqrt{k^2\pi x^{m-p}}=0. \quad \blacksquare$$

Example 3. Let $f(x)=x^{\frac{3}{2}}$, $m=3$, $p=5$ and $k=1$. Then $D_x^{\frac{5}{2}}\left(x^{\frac{3}{2}}\right)=0$ by Corollary 3 since $m-p=3-5=-2<0$.

Corollary 4. *Let $f(x)=kx^{\frac{m}{2}}$. Then the fractional derivative $D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)$ is complex infinity $\tilde{\infty}$ if $m \in \mathbb{Z}^{-}$ and $m-p>0$ where p and m are odd.*

Proof: Let $m \in \mathbb{Z}^{-}$ is odd, $m-p>0$ and p is odd. Then by (iii) in Corollary 2, we have

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{i^p\left(\frac{-m+p}{2}\right)!\left(\frac{-m-1}{2}\right)!}{2^m\left(-m+p\right)\left(-m-1\right)!}\sqrt{\frac{k^2x^{m-p}}{\pi}},$$

and we get

$$\left(\frac{-m+p}{2}\right)!=\Gamma\left(\frac{-m+p}{2}+1\right)=\Gamma\left(\frac{-m+p+2}{2}\right)$$

so we have

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{i^p\Gamma\left(\frac{-m+p+2}{2}\right)\left(\frac{-m-1}{2}\right)!}{2^m\left(-m+p\right)\left(-m-1\right)!}\sqrt{\frac{k^2x^{m-p}}{\pi}} \text{ for } -m+p+2>0.$$

Since $m\in\mathbb{Z}^-$ is odd, $m-p>0$ and p is odd, we let $m=-2s+1$ and $p=-2r-1$ where $s,r\in\mathbb{Z}^+$. Since

$$m-p>0\Rightarrow -2s+1-(-2r-1)>0\Rightarrow -2s+1+2r+1>0\Rightarrow -2s+2r+2>0\Rightarrow$$
$$-2(s-r-1)>0\Rightarrow s-r-1<0\Rightarrow s-r<1,$$

substituting $s=\frac{-m+1}{2}$ and $r=\frac{-p-1}{2}$, it follows that

$$\frac{-m+1}{2}-\left(\frac{-p-1}{2}\right)<1\Rightarrow\frac{-m+p+2}{2}<1.$$

Then by definition, $\Gamma\left(\frac{-m+p+2}{2}\right)=\tilde{\infty}$. So it implies that

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\frac{i^p\,\tilde{\infty}\left(\frac{-m-1}{2}\right)!}{2^m\left(-m+p\right)\left(-m-1\right)!}\sqrt{\frac{k^2x^{m-p}}{\pi}}.$$

By arithmetical operations of complex infinity, $c\cdot\tilde{\infty}=\tilde{\infty}$ where $c\in\mathbb{C}$, hence,

$$D_x^{\frac{p}{2}}\left(kx^{\frac{m}{2}}\right)=\tilde{\infty}\cdot\frac{i^p\left(\frac{-m-1}{2}\right)!}{2^m\left(-m+p\right)\left(-m-1\right)!}\sqrt{\frac{k^2x^{m-p}}{\pi}}=\tilde{\infty}. \blacksquare$$

Example 4. Let $f(x)=\frac{1}{2}x^{-\frac{1}{2}}$, $m=-1$, $p=-3$ and $k=\frac{1}{2}$. Then $D_x^{\frac{-3}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)=\tilde{\infty}$ by Corollary 4, since $m-p=-1-(-3)=2>0$.

Theorem 5. *Let* $f(x)=(k_1x\pm k_2)^{\frac{m}{2}}$ *where* $m\in\mathbb{Z}$, $k_1,k_2,x\in\mathbb{R}$, $k_1,k_2\neq 0$. *Then the fractional derivative* $D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}$ *is given by*

(i) $D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}=\frac{2^{m-p}k_1^{\frac{p}{2}}\left(\frac{m-p-1}{2}\right)!\left(\frac{m}{2}\right)!}{(m-p)!}\sqrt{\frac{(k_1x\pm k_2)^{m-p}}{\pi}}$ *if* $m\in\mathbb{Z}^+$ *is even and* $m-p>0$;

(ii) $D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}=\frac{2^{m-p+1}k_1^{\frac{p}{2}}\left(\frac{m}{2}\right)!(-m+p)!}{(-1)^{\left(\frac{-m+p+1}{2}\right)}(m-p)\left(\frac{-m+p-1}{2}\right)!}\sqrt{\frac{(k_1x\pm k_2)^{m-p}}{\pi}}$ *if* $m\in\mathbb{Z}^+$ *is even and* $m-p<0$;

JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 7 NO. 2 (NOV 2024)
Available online at www.jfma.math.fsm.undip.ac.id

and $(iii)\ D_x^{\frac{p}{2}}(k_1x \pm k_2)^{\frac{m}{2}} = \dfrac{k_1^{\frac{p}{2}} m!}{2^m\left(\frac{m-p}{2}\right)!\left(\frac{m-1}{2}\right)!}\sqrt{\pi(k_1x \pm k_2)^{m-p}}$ *if* $m \in \mathbb{Z}^+$ *is odd and* $m-p>0$ *where* p *is odd.*

Proof: (i) Suppose that $f(x) = (k_1x \pm k_2)^{\frac{m}{2}}$, $m \in \mathbb{Z}^+$ is even, $m-p>0$ and $k_1, k_2, x \in \mathbb{R}$ with $k_1, k_2 \neq 0$. Then by definition, we know that

$$D_x(k_1x \pm k_2)^{\frac{m}{2}} = k_1\left[\frac{m}{2}(k_1x \pm k_2)^{\left(\frac{m}{2}-1\right)}\right]$$

$$D_xD_x(k_1x \pm k_2)^{\frac{m}{2}} = D_x^2(k_1x \pm k_2)^{\frac{m}{2}} = D_x\left[k_1\frac{m}{2}(k_1x \pm k_2)^{\left(\frac{m}{2}-1\right)}\right] = k_1k_1\left(\frac{m}{2}\right)\left(\frac{m}{2}-1\right)(k_1x \pm k_2)^{\left(\frac{m}{2}-2\right)}$$

and continuing this pattern we will arrive at

$$D_x^n(k_1x \pm k_2)^{\frac{m}{2}} = D_xD_x^{n-1}(k_1x \pm k_2)^{\frac{m}{2}}$$

$$= D_x\left[(k_1)^{n-2}(k_1)\left(\frac{m}{2}\right)\left(\frac{m-2}{2}\right)\left(\frac{m-4}{2}\right)\left(\frac{m-6}{2}\right)\cdots\left(\frac{m}{2}-n+2\right)(k_1x \pm k_2)^{\frac{m}{2}-(n-1)}\right]$$

$$= (k_1)^{n-1}(k_1)\left(\frac{m}{2}\right)\left(\frac{m-2}{2}\right)\left(\frac{m-4}{2}\right)\left(\frac{m-6}{2}\right)\cdots\left(\frac{m-2n+4}{2}\right)\left(\frac{m}{2}-n+1\right)(k_1x \pm k_2)^{\frac{m}{2}-n+1-1}$$

$$= \left(\frac{k_1^{\ n}}{2^n}\right)m(m-2)(m-4)(m-6)\cdots(m-2n+4)(m-2n+2)(k_1x \pm k_2)^{\left(\frac{m-2n}{2}\right)}.$$

Multiplying the numerator and denominator by $(m-2n)!!$ will yield to

$$D_x^n(k_1x \pm k_2)^n = \left(\frac{k_1^{\ n}}{2^n}\right)\frac{m(m-2)(m-4)(m-6)\cdots(m-2n+4)(m-2n+2)(k_1x \pm k_2)^{\left(\frac{m-2n}{2}\right)}}{1}\cdot\frac{(m-2n)!!}{(m-2n)!!}$$

$$D_x^n(k_1x \pm k_2)^{\frac{m}{2}} = \left(\frac{k_1^{\ n}}{2^n}\right)\frac{m!!}{(m-2n)!!}(k_1x \pm k_2)^{\left(\frac{m-2n}{2}\right)} \text{ where } m-2n \geq 0.$$

Accordingly from (i) in Theorem 1 for $m \in \mathbb{Z}^+$ is even which means $m-2n$ is also even then we let $m = 2r$ for any $r \in \mathbb{Z}^+$ and substituting $r = \frac{m}{2}$ it follows that $m!! = 2^{\frac{m}{2}}\left(\frac{m}{2}\right)!$ and $(m-2n)!! = 2^{\left(\frac{m-2n}{2}\right)}\left(\frac{m-2n}{2}\right)!$. It follows that

$$D_x^n(k_1x \pm k_2)^{\frac{m}{2}} = \frac{k_1^{\ n}m!!}{2^n(m-2n)!!}(k_1x \pm k_2)^{\left(\frac{m-2n}{2}\right)}$$

$$= \frac{k_1^{\ n}2^{\frac{m}{2}}\left(\frac{m}{2}\right)!}{2^n2^{\frac{m-2n}{2}}\left(\frac{m-2n}{2}\right)!}(k_1x \pm k_2)^{\left(\frac{m-2n}{2}\right)} = \frac{k_1^{\ n}\left(\frac{m}{2}\right)!}{\left(\frac{m-2n}{2}\right)!}(k_1x \pm k_2)^{\left(\frac{m-2n}{2}\right)}$$

then taking $n = \frac{p}{2}$, we get

$$D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}=\frac{k_1^{\frac{p}{2}}\left(\frac{m}{2}\right)!}{\left(\frac{m-p}{2}\right)!}(k_1x\pm k_2)^{\left(\frac{m-p}{2}\right)}.$$

Note that $m\in\mathbb{Z}^+$ is even and p is odd which means that $\left(\frac{m-p}{2}\right)$ is non-integer. Since $m-p>0$, then we can express $\left(\frac{m-p}{2}\right)!$, we have

$$\left(\frac{m-p}{2}\right)!=\frac{(m-p)!2^{(-m+p)}\sqrt{\pi}}{\left(\frac{m-p-1}{2}\right)!}=\frac{(m-p)!}{2^{(m-p)}\left(\frac{m-p-1}{2}\right)!}.$$

Hence, we get

$$D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}=\frac{k_1^{\frac{p}{2}}\left(\frac{m}{2}\right)!}{\left(\frac{m-p}{2}\right)!}(k_1x\pm k_2)^{\left(\frac{m-p}{2}\right)}=\frac{k_1^{\frac{p}{2}}\left(\frac{m}{2}\right)!}{\frac{(m-p)!\sqrt{\pi}}{2^{(m-p)}\left(\frac{m-p-1}{2}\right)!}}(k_1x\pm k_2)^{\left(\frac{m-p}{2}\right)}$$

$$D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}=\frac{k_1^{\frac{p}{2}}2^{m-p}\left(\frac{m-p-1}{2}\right)!\left(\frac{m}{2}\right)!}{(m-p)!}\sqrt{\frac{(k_1x\pm k_2)^{m-p}}{\pi}}.\quad\blacksquare$$

(ii) Suppose that $f(x)=(k_1x\pm k_2)^{\frac{m}{2}}$, $m\in\mathbb{Z}^+$ is even, $m-p<0$, p is odd and $k_1,k_2,x\in\mathbb{R}$ with $k_1,k_2\neq 0$. By (i), $D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}=\frac{k_1^{\frac{p}{2}}\left(\frac{m}{2}\right)!}{\left(\frac{m-p}{2}\right)!}(k_1x\pm k_2)^{\left(\frac{m-p}{2}\right)}$ and note that $\left(\frac{m-p}{2}\right)$ is non-integer since $m\in\mathbb{Z}^+$ is even and p is odd, so we can express $\left(\frac{m-p}{2}\right)!$, we have

$$D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}=\frac{k_1^{\frac{p}{2}}\left(\frac{m}{2}\right)!}{\left(\frac{m-p}{2}\right)\Gamma\left(\frac{m-p}{2}\right)}(k_1x\pm k_2)^{\left(\frac{m-p}{2}\right)}\text{ for } m-p>0.$$

Since $m-p<0$ then from (ii) in Theorem 1, we have

$$\Gamma\left(\frac{m-p}{2}\right)=\Gamma\left(-\left(\frac{-m+p}{2}\right)\right)=\frac{(-1)^{\left(\frac{-m+p+1}{2}\right)}2^{(-m+p)}\left(\frac{-m+p-1}{2}\right)!\sqrt{\pi}}{(-m+p)!}.$$

Then,

$$\left(\frac{m-p}{2}\right)!=\left(\frac{m-p}{2}\right)\Gamma\left(\frac{m-p}{2}\right)=\frac{(m-p)\left(\frac{-m+p-1}{2}\right)!(-1)^{\left(\frac{-m+p+1}{2}\right)}2^{-m+p-1}\sqrt{\pi}}{(-m+p)!}.$$

Therefore, by substitution, we have

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{k_1^{\frac{p}{2}}\left(\frac{m}{2}\right)!}{\left(\frac{m-p}{2}\right)!}\left(k_1x\pm k_2\right)^{\left(\frac{m-p}{2}\right)}=\frac{k_1^{\frac{p}{2}}(-m+p)!\left(\frac{m}{2}\right)!}{(-1)^{\left(\frac{-m+p+1}{2}\right)}2^{-m+p-1}(m-p)\left(\frac{-m+p-1}{2}\right)!}\sqrt{\frac{\left(k_1x\pm k_2\right)^{(m-p)}}{\pi}}.\ \blacksquare$$

(iii) Suppose $f(x)=\left(k_1x\pm k_2\right)^{\frac{m}{2}}$, $m\in\mathbb{Z}^+$ is odd, $m-p>0$, p is odd and $k_1,k_2,x\in\mathbb{R}$ with $k_1,k_2\neq 0$. By (i), we have $D_x^n\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\left(\frac{k_1^n}{2^n}\right)\frac{m!!}{(m-2n)!!}\left(k_1x\pm k_2\right)^{\left(\frac{m-2n}{2}\right)}$ for $m-2n\geq 0$. Since $m\in\mathbb{Z}^+$ is odd then we let $m=2t+1$ for any $t\in\mathbb{Z}^+$ which implies that

$$m!!=(2t+1)!!=\frac{(2t+1)!}{2^t t!}\text{ and }(m-2n)!!=(2t-2n+1)!!=\frac{(2t-2n+1)!}{2^{t-n}(t-n)!}.$$

By substitution, it follows that

$$D_x^n\left(k_1x+k_2\right)^{\frac{m}{2}}=\frac{\left(k_1\right)^n(2t+1)!(t-n)!}{2^{2n}t!(2t-2n+1)!}\left(k_1x+k_2\right)^{\left(\frac{m-2n}{2}\right)}.$$

Hence, substituting $n=\frac{p}{2}$ and $t=\frac{m-1}{2}$, we have

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{\left(k_1\right)^{\frac{p}{2}}m!\left(\frac{m-p-1}{2}\right)!}{2^p(m-p)!\left(\frac{m-1}{2}\right)!}\left(k_1x\pm k_2\right)^{\left(\frac{m-p}{2}\right)}.$$

Subsequently, by (iii) in Theorem 1 for $m-p>0$ then

$$\left(\frac{m-p-1}{2}\right)!=\Gamma\left(\frac{m-p-1}{2}+1\right)=\Gamma\left(\frac{m-p+1}{2}\right)=\frac{2^{-m+p}(m-p)!\sqrt{\pi}}{\left(\frac{m-p}{2}\right)!}.$$

Thus,

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{\left(k_1\right)^{\frac{p}{2}}m!\left(\frac{m-p-1}{2}\right)!}{2^p(m-p)!\left(\frac{m-1}{2}\right)!}\left(k_1x\pm k_2\right)^{\left(\frac{m-p}{2}\right)}$$

$$=\left(k_1\right)^{\frac{p}{2}}m!\frac{\frac{2^{-m+p}(m-p)!\sqrt{\pi}}{\left(\frac{m-p}{2}\right)!}}{2^p(m-p)!\left(\frac{m-1}{2}\right)!}\left(k_1x\pm k_2\right)^{\left(\frac{m-p}{2}\right)}=\frac{\left(k_1\right)^{\frac{p}{2}}m!}{2^m\left(\frac{m-p}{2}\right)!\left(\frac{m-1}{2}\right)!}\sqrt{\pi\left(k_1x\pm k_2\right)^{m-p}}.\quad\blacksquare$$

Example 5. Let $f(x)=x\pm 1$, $m=2$, $p=1$, $k_1=1$ and $k_2=1$. Since $m\in\mathbb{Z}^+$ is odd, $m-p>0$ and p is odd, we can apply (i) in Theorem 5 so we have

$$D_x^{\frac{1}{2}}(x\pm 1)^{\frac{2}{2}}=D_x^{\frac{1}{2}}(x\pm 1)=\frac{(1)^{\frac{1}{2}}2^{2-1}\left(\frac{2-1-1}{2}\right)!\left(\frac{2}{2}\right)!}{(2-1)!}\sqrt{\frac{(x\pm 1)^{2-1}}{\pi}}=2\sqrt{\frac{x\pm 1}{\pi}}.$$

Corollary 6. *Let* $f(x)=(k_1x\pm k_2)^{\frac{m}{2}}$. *Then the fractional derivative* $D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}$ *is a complex number given by*

(i) $D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}=\dfrac{(i)^p k_1^{\frac{p}{2}}2^{m-p+1}(-m+p-1)!}{\left(\frac{-m+p-1}{2}\right)!\left(\frac{-m-2}{2}\right)!}\sqrt{\pi(k_1x\pm k_2)^{m-p}}$ *if* $m\in\mathbb{Z}^-$ *is even and* $m-p>0$; (ii)

$D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}=\dfrac{(-1)^{\left(\frac{m}{2}\right)}i2^{m-p}k_1^{\frac{p}{2}}\left(\frac{m-p-1}{2}\right)!}{(m-p)!\left(\frac{-m-2}{2}\right)!}\sqrt{\pi(k_1x\pm k_2)^{m-p}}$ *if* $m\in\mathbb{Z}^-$ *is even and* $m-p>0$; *and* (iii)

$D_x^{\frac{p}{2}}(k_1x\pm k_2)^{\frac{m}{2}}=\dfrac{(i)^p k_1^{\frac{p}{2}}\left(\frac{-m+p}{2}\right)!\left(\frac{-m-1}{2}\right)!}{2^m(-m+p)(-m-1)!}\sqrt{\dfrac{(k_1x\pm k_2)^{m-p}}{\pi}}$ *if* $m\in\mathbb{Z}^-$ *is odd and* $m-p<0$ *where* p *is odd.*

Proof: (i) Suppose $f(x)=(k_1x\pm k_2)^{\frac{m}{2}}$, $m\in\mathbb{Z}^-$ is even, $m-p>0$, and p is odd. By (i) in Theorem 5, we have

$$D_x^n(k_1x\pm k_2)^{\frac{m}{2}}=\left(\frac{k_1^{\,n}}{2^n}\right)m(m-2)(m-4)(m-6)\cdots(m-2n+4)(m-2n+2)(k_1x\pm k_2)^{\left(\frac{m-2n}{2}\right)}.$$

Since $m\in\mathbb{Z}^-$ is even, it follows that $m, m-2, m-4, m-6,\ldots\ldots, m-2n+n, m-2n-2$ are all negative. Consequently we multiply $1=(-1)(-1)$ to make this term be positive. So, we have

$$D_x^n(k_1x\pm k_2)^{\frac{m}{2}}=\left(\frac{k_1}{2}\right)^n(-1)(-1)\left[m(m-2)(m-4)(m-6)\cdots(m-2n+4)(m-2n+2)(k_1x\pm k_2)^{\left(\frac{m}{2}-n\right)}\right]$$

$$=\left(\frac{k_1}{2}\right)^n(-1)^n-m(-(m-2))(-(m-4))(-(m-6))\cdots(-(m-2n+4))(-(m-2n+2))(k_1x\pm k_2)^{\left(\frac{m}{2}-n\right)}.$$

Likewise multiplying the numerator and denominator by $(-m-2)!!$ to have

$$=\frac{(k_1)^n(-1)^n(-m)(-m+2)(-m+4)\cdots\cdots(-m+2n-4)(-m+2n-2)}{2^n}\cdot\frac{(-m-2)!!}{(-m-2)!!}(k_1x\pm k_2)^{\left(\frac{m}{2}-n\right)}$$

$$=\left(\frac{k_1}{2}\right)^n(-1)^n\frac{(-m+2n-2)!!}{(-m-2)!!}(k_1x\pm k_2)^{\left(\frac{m}{2}-n\right)}=\left(\frac{k_1}{2}\right)^n(-1)^n\frac{2^{\frac{-m+2n-2}{2}}\left(\frac{-m+2n-2}{2}\right)!}{2^{\frac{-m-2}{2}}\left(\frac{-m-2}{2}\right)!}(k_1x\pm k_2)^{\left(\frac{m}{2}-n\right)}$$

$$D_x^n(k_1x\pm k_2)^{\frac{m}{2}}=\frac{(-1)^n(k_1)^n\left(\frac{-m+2n-2}{2}\right)!}{\left(\frac{-m-2}{2}\right)!}(k_1x\pm k_2)^{\left(\frac{m}{2}-n\right)}.$$

Then replacing $n=\dfrac{p}{2}$, we

have

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{(-1)^{\frac{p}{2}}\left(k_1\right)^{\frac{p}{2}}\left(-\frac{m}{2}+\frac{p}{2}-1\right)!}{\left(-\frac{m}{2}-1\right)!}\sqrt{\left(k_1x\pm k_2\right)^{m-p}}=\frac{(-1)^{\frac{p}{2}}\left(k_1\right)^{\frac{p}{2}}\left(\frac{-m+p-2}{2}\right)!}{\left(\frac{-m-2}{2}\right)!}\sqrt{\left(k_1x\pm k_2\right)^{m-p}}.$$

Note that $\left(\frac{-m+p-2}{2}\right)$ is non-integer since $m\in\mathbb{Z}^-$ is even and p is odd. Since $-m+p>0$, we have

$$\left(\frac{-m+p-2}{2}\right)!=\Gamma\left(\frac{-m+p-2}{2}+1\right)=\Gamma\left(\frac{-m+p}{2}\right)=\frac{2^{m-p+1}(-m+p-1)!\sqrt{\pi}}{\left(\frac{-m+p-1}{2}\right)!}.$$

Hence, by substitution, we have

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=$$

$$\frac{(-1)^{\frac{p}{2}}k_1^{\frac{p}{2}}\left(\frac{-m+p-2}{2}\right)!}{\left(\frac{-m-2}{2}\right)!}\sqrt{\left(k_1x\pm k_2\right)^{m-p}}=\frac{(-1)^{\frac{p}{2}}k_1^{\frac{p}{2}}}{\left(\frac{-m-2}{2}\right)!}\left(\frac{2^{m-p+1}(-m+p-1)!\sqrt{\pi}}{\left(\frac{-m+p-1}{2}\right)!}\right)\sqrt{\left(k_1x\pm k_2\right)^{m-p}}=$$

$$\frac{(-1)^{\frac{p}{2}}k_1^{\frac{p}{2}}2^{m-p+1}(-m+p-1)!}{\left(\frac{-m+p-1}{2}\right)!\left(\frac{-m-2}{2}\right)!}\sqrt{\pi\left(k_1x\pm k_2\right)^{m-p}}=\frac{\left((-1)^{\frac{1}{2}}\right)^p k_1^{\frac{p}{2}}2^{m-p+1}(-m+p-1)!}{\left(\frac{-m+p-1}{2}\right)!\left(\frac{-m-2}{2}\right)!}\sqrt{\pi\left(k_1x\pm k_2\right)^{m-p}}=$$

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{i^p k_1^{\frac{p}{2}}2^{m-p+1}(-m+p-1)!}{\left(\frac{-m+p-1}{2}\right)!\left(\frac{-m-2}{2}\right)!}\sqrt{\pi\left(k_1x\pm k_2\right)^{m-p}}. \qquad \blacksquare$$

(ii) Suppose $f(x)=\left(k_1x\pm k_2\right)^{\frac{m}{2}}$, $m\in\mathbb{Z}^-$ is even, $m-p>0$, p is odd and $k_1,k_2,x\in\mathbb{R}$ with $k_1,k_2\neq 0$. By (i)

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{(-1)^{\frac{p}{2}}\left(k_1\right)^{\frac{p}{2}}\left(\frac{-m+p-2}{2}\right)!}{\left(\frac{-m-2}{2}\right)!}\left(k_1x\pm k_2\right)^{\frac{m-p}{2}}.$$

Since $m\in\mathbb{Z}^-$ is even, then for $-m+p>0$, we have

$$\left(\frac{-m+p-2}{2}\right)!=\Gamma\left(\frac{-m+p-2}{2}+1\right)=\Gamma\left(\frac{-m+p}{2}\right).$$

It follows that

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{(-1)^{\frac{p}{2}}k_1^{\frac{p}{2}}\left(\frac{-m+p-2}{2}\right)!}{\left(\frac{-m-2}{2}\right)!}\left(k_1x\pm k_2\right)^{\frac{m-p}{2}}=$$

$$\frac{(-1)^{\frac{p}{2}}k_1^{\frac{p}{2}}\Gamma\left(\frac{-m+p}{2}\right)}{\left(\frac{-m-2}{2}\right)!}\left(k_1x\pm k_2\right)^{\frac{m-p}{2}}.$$

Since $-m+p<0$ then, we get

$$\Gamma\left(\frac{-m+p}{2}\right)=\Gamma\left(-\left(\frac{m-p}{2}\right)\right)=\frac{(-1)^{\left(\frac{m-p+1}{2}\right)}2^{m-p}\left(\frac{m-p-1}{2}\right)!\sqrt{\pi}}{(m-p)!}.$$

Hence, by substitution, we have

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{(-1)^{\left(\frac{m+1}{2}\right)}2^{m-p}k_1^{\frac{p}{2}}\left(\frac{m-p-1}{2}\right)!}{(m-p)!\left(\frac{-m-2}{2}\right)!}\sqrt{\pi\left(k_1x\pm k_2\right)^{m-p}}$$

$$=\frac{(-1)^{\left(\frac{m}{2}\right)}(-1)^{\left(\frac{1}{2}\right)}2^{m-p}k_1^{\frac{p}{2}}\left(\frac{m-p-1}{2}\right)!}{(m-p)!\left(\frac{-m-2}{2}\right)!}\sqrt{\pi\left(k_1x\pm k_2\right)^{m-p}}=\frac{(-1)^{\left(\frac{m}{2}\right)}i2^{m-p}k_1^{\frac{p}{2}}\left(\frac{m-p-1}{2}\right)!}{(m-p)!\left(\frac{-m-2}{2}\right)!}\sqrt{\pi\left(k_1x\pm k_2\right)^{m-p}}.\ \blacksquare$$

(iii) Suppose $f(x)=\left(k_1x\pm k_2\right)^{\frac{m}{2}}$, $m\in\mathbb{Z}^-$ is odd, $m-p<0$, p is odd and $k_1,k_2,x\in\mathbb{R}$ with $k_1,k_2\neq 0$ .By (i) $D_x^n\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\left(\frac{k_1}{2}\right)^n(-1)^n\frac{(-m+2n-2)!}{(-m-2)!}\left(k_1x\pm k_2\right)^{\frac{m}{2}-n}$, if $m\in\mathbb{Z}^-$ is odd so we let $m=-2s+1$ for any $s\in\mathbb{Z}^+$ implies

$$(-m+2n-2)!!=(2s+2n-3)!!=\left(2\left(\frac{2s+2n-2}{2}\right)-1\right)!!=\frac{(2s+2n-2)!}{2^{\left(\frac{2s+2n-2}{2}\right)}\left(\frac{2s+2n-2}{2}\right)!}$$

and

$$(-m-2)!!=(2s-2-1)!!=\left(2(s-1)-1\right)!!=\frac{(2s-2)!}{2^{s-1}(s-1)!}.$$

By substitution, it follows that

$$D_x^n\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{\left(k_1\right)^n(-1)^n(2s+2n-2)!(s-1)!}{2^{2n}\left(\frac{2t+2n-2}{2}\right)!(2s-2)!}\left(k_1x\pm k_2\right)^{\frac{m}{2}-n}.$$

Consequently, substituting $s=\dfrac{-m+1}{2}$ and $n=\dfrac{p}{2}$ then we will get

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{\left(k_1\right)^{\frac{p}{2}}\left(-1\right)^{\frac{p}{2}}(-m+p-1)!\left(\frac{-m-1}{2}\right)!}{2^p\left(\frac{-m+p-1}{2}\right)!(-m-1)!}\sqrt{\left(k_1x\pm k_2\right)^{m-p}}=\frac{\left(k_1\right)^{\frac{p}{2}}i^p(-m+p-1)!\left(\frac{-m-1}{2}\right)!}{2^p\left(\frac{-m+p-1}{2}\right)!(-m-1)!}\sqrt{\left(k_1x\pm k_2\right)^{m-p}}.$$

Note that $\left(\frac{-m+p-1}{2}\right)$ is non-integer since $m\in\mathbb{Z}^-$ is odd and p is odd. Since $-m+p>0$ then from (ii) in Corollary 2,

$$\left(\frac{-m+p-1}{2}\right)!=\Gamma\left(\left(\frac{-m+p-1}{2}\right)+1\right)=\Gamma\left(\frac{-m+p+1}{2}\right)=\frac{2^{m-p}(-m+p)!\sqrt{\pi}}{\left(\frac{-m+p}{2}\right)!}.$$

Hence, by substitution, we get

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{k_1^{\frac{p}{2}}i^p\left(\frac{-m+p}{2}\right)!\left(\frac{-m-1}{2}\right)!}{2^m(-m+p)(-m-1)!}\sqrt{\frac{\left(k_1x\pm k_2\right)^{m-p}}{\pi}}.\qquad\blacksquare$$

Example 6. Let $f(x)=(x\pm1)^{-1}$, $m=-2$, $p=-1$, $k_1=1$ and $k_2=1$. Since $m\in\mathbb{Z}^-$ is even, $m-p>0$ and p is odd, we can apply (i) in Corollary 6 so we have

$$D_x^{-\frac{1}{2}}\left(x\pm1\right)^{-\frac{2}{2}}=D_x^{-\frac{1}{2}}\left(x\pm1\right)^{-1}=\frac{(i)^{-1}(1)^{-\frac{1}{2}}2^{-2+1+1}\left(-(-2)+(-1)-1\right)!}{\left(\frac{-(-2)+(-1)-1}{2}\right)!\left(\frac{-(-2)-2}{2}\right)!}\sqrt{\pi\left(x\pm1\right)^{-2-(-1)}}=-i\sqrt{\frac{\pi}{(x\pm1)}}.$$

Corollary 7. *Let $f(x)=\left(k_1x\pm k_2\right)^{\frac{m}{2}}$. Then the fractional derivative $D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}$ is zero if $m\in\mathbb{Z}^+$ and $m-p<0$ where p and m are odd.*

Proof: Let $m\in\mathbb{Z}^+$ is odd, $m-p<0$, and p is odd. By (iii) in Theorem 5, we have

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{\left(k_1\right)^{\frac{p}{2}}m!\left(\frac{m-p-1}{2}\right)!}{2^p(m-p)!\left(\frac{m-1}{2}\right)!}\left(k_1x\pm k_2\right)^{\left(\frac{m-p}{2}\right)},$$

we can express $\left(\frac{m-p-1}{2}\right)!=\Gamma\left(\frac{m-p-1}{2}+1\right)=\Gamma\left(\frac{m-p+1}{2}\right)$ and $(m-p)!=\Gamma(m-p+1)$ so we get

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{\left(k_1\right)^{\frac{p}{2}}m!\Gamma\left(\frac{m-p+1}{2}\right)}{2^p\Gamma(m-p+1)\left(\frac{m-1}{2}\right)!}\left(k_1x\pm k_2\right)^{\left(\frac{m-p}{2}\right)}\text{ for }m-p+1>0.$$

Since $m\in\mathbb{Z}^+$ is odd, $m-p<0$ and p is odd, we let $m=2s-1$ and $p=2r+1$ where $s,r\in\mathbb{Z}^+$. Furthermore,

$$m-p<0\Rightarrow 2s-1-(2r+1)<0\Rightarrow 2s-1-2r-1<0\Rightarrow 2s-2r-2<0\Rightarrow 2(s-r-1)<0\Rightarrow$$
$$s-r-1<0\Rightarrow s-r<1.$$

Substituting $s=\dfrac{m+1}{2}$ and $r=\dfrac{p-1}{2}$, it follows that

$$s-r<1 \Rightarrow \frac{m+1}{2}-\left(\frac{p-1}{2}\right)<1 \Rightarrow \frac{m-p+2}{2}<1 \Rightarrow m-p+2<2 \Rightarrow m-p+1<1.$$

Then by definition, $\Gamma(m-p+1)=\tilde{\infty}$. So it implies that

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{\left(k_1\right)^{\frac{p}{2}}m!\,\Gamma\left(\frac{m-p+1}{2}\right)}{2^p\,\tilde{\infty}\left(\frac{m-1}{2}\right)!}\left(k_1x\pm k_2\right)^{\left(\frac{m-p}{2}\right)}.$$

By arithmetical operations of complex infinity, $\dfrac{c}{\tilde{\infty}}=0$ where $c\in\mathbb{C}$, hence,

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{1}{\tilde{\infty}}\cdot\frac{\left(k_1\right)^{\frac{p}{2}}m!\,\Gamma\left(\frac{m-p+1}{2}\right)}{2^p\left(\frac{m-1}{2}\right)!}\left(k_1x\pm k_2\right)^{\left(\frac{m-p}{2}\right)}=0\cdot \quad \blacksquare$$

Example 7. Let $f(x)=(2x\pm4)^{\frac{5}{2}}$, $m=5$, $p=9$, $k_1=2$ and $k_2=4$. Then $D_x^{\frac{9}{2}}(2x\pm4)^{\frac{5}{2}}=0$ by Corollary 7, since $m-p=5-9=-4<0$.

Corollary 8. *Let* $f(x)=\left(k_1x\pm k_2\right)^{\frac{m}{2}}$. *Then the fractional derivative* $D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}$ *is a complex infinity* $\tilde{\infty}$ *if* $m\in\mathbb{Z}^-$ *and* $m-p>0$ *where* p *and* m *are odd.*

Proof: Let $m\in\mathbb{Z}^-$ is odd, $m-p>0$ and p is odd. By (iii) in Corollary 6, we have

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{i^p\left(k_1\right)^{\frac{p}{2}}(-m+p-1)!\left(\frac{-m-1}{2}\right)!}{2^p\left(\frac{-m+p-1}{2}\right)!(-m-1)!}\sqrt{\left(k_1x\pm k_2\right)^{m-p}},$$

we can express $(-m+p-1)!=\Gamma(-m+p)$ and $\left(\frac{-m+p-1}{2}\right)!=\Gamma\left(\frac{-m+p-1}{2}+1\right)=\Gamma\left(\frac{-m+p+1}{2}\right)$ so we get

$$D_x^{\frac{p}{2}}\left(k_1x\pm k_2\right)^{\frac{m}{2}}=\frac{i^p\left(k_1\right)^{\frac{p}{2}}\Gamma(-m+p)\left(\frac{-m-1}{2}\right)!}{2^p\,\Gamma\left(\frac{-m+p+1}{2}\right)(-m-1)!}\sqrt{\left(k_1x\pm k_2\right)^{m-p}} \text{ for } -m+p>0.$$

Since $m\in\mathbb{Z}^-$ is odd, $m-p>0$ and p is odd, $m=-2s+1$ and $p=-2r-1$ where $s,r\in\mathbb{Z}^+$. Furthermore,

$$m-p>0 \Rightarrow -2s+1-(-2r-1)>0 \Rightarrow -2s+1+2r+1>0 \Rightarrow -2s+2r+2>0 \Rightarrow$$
$$-2(s-r-1)>0 \Rightarrow s-r-1<0 \Rightarrow s-r<1.$$

Substituting $s=\dfrac{-m+1}{2}$ and $r=\dfrac{-p-1}{2}$, it follows that

$$s-r<1 \Rightarrow \frac{-m+1}{2}-\left(\frac{-p-1}{2}\right)<1 \Rightarrow \frac{-m+p+2}{2}<1 \Rightarrow -m+p+2<2 \Rightarrow -m+p<0 .$$

Then by definition, $\Gamma(-m+p)=\tilde{\infty}$. So it implies that

$$D_x^{\frac{p}{2}}\left(k_1 x \pm k_2\right)^{\frac{m}{2}}=\frac{i^p\left(k_1\right)^{\frac{p}{2}} \tilde{\infty}\left(\frac{-m-1}{2}\right)!}{2^p \Gamma\left(\frac{-m+p+1}{2}\right)(-m-1)!} \sqrt{\left(k_1 x \pm k_2\right)^{m-p}} .$$

By arithmetical operations of complex infinity, $c \cdot \tilde{\infty}=\tilde{\infty}$ where $c \in \mathbb{C}$, hence,

$$D_x^{\frac{p}{2}}\left(k_1 x \pm k_2\right)^{\frac{m}{2}}=\tilde{\infty} \cdot \frac{i^p\left(k_1\right)^{\frac{p}{2}}\left(\frac{-m-1}{2}\right)!}{2^p \Gamma\left(\frac{-m+p+1}{2}\right)(-m-1)!} \sqrt{\left(k_1 x \pm k_2\right)^{m-p}}=\tilde{\infty} . \blacksquare$$

Example 8. Let $f(x)=(x \pm 1)^{-\frac{1}{2}}$, $m=-1$, $p=-3$, $k_1=1$ and $k_2=1$. Then $D_x^{-\frac{3}{2}}(x \pm 1)^{-\frac{1}{2}}=\tilde{\infty}$ by Corollary 8, since $m-p=-1-(-3)=2>0$.

The graphs of $D_x^{\frac{p}{2}}\left(k x^{\frac{m}{2}}\right)$ and $D_x^{\frac{p}{2}}\left(k_1 x \pm k_2\right)^{\frac{m}{2}}$ where $k, k_1, k_2=1$ (nonzero real numbers), for some odd p and some $m \in \mathbb{Z}^{+} \cup\{0\}$ are shown in the figures below. Fig. 1 illustrates the $\frac{p}{2} th$ derivative of the unit function $f(x)=x^0=1$ where $p=\{\pm 1, \pm 3, \pm 5, \pm 7\}$. This graph also shows that the curves lie on the first and fourth quadrants.But $\frac{p}{2} th$ derivative of this function where p is positive has the values of x in the interval $(0, \infty)$ and the curves have horizontal asymptotes at $D_x^{\frac{p}{2}} x^0=0$. On the other hand, if p is negative, then we can see in the curves that their point of origin is at $(0,0)$ and is moving upward in the different directions.

Fig. 2 shows the $\frac{p}{2} th$ derivative of the function $f(x)=x^{\frac{1}{2}}$ where $p=\{\pm 1,-3,-5,-7\}$. Accordingly, since the value of $D_x^{\frac{1}{2}}\left(x^{\frac{1}{2}}\right)=\frac{\sqrt{\pi}}{2}$ then its curve is parallel to the x axis (a constant function). Also, since $D_x^{\frac{p}{2}}\left(x^{\frac{1}{2}}\right)$ is undefined when $p=\{3,5,7, \ldots .\}$, their graphs are not available. But take notice also that since $D_x^{-\frac{1}{2}}\left(x^{\frac{1}{2}}\right)=\frac{\sqrt{\pi}}{2} x$ then its curve is a straight line (a linear function), $D_x^{-\frac{3}{2}}\left(x^{\frac{1}{2}}\right)=\frac{\sqrt{\pi}}{4} x^2$ is a parabola (a quadratic function), $D_x^{-\frac{5}{2}}\left(x^{\frac{1}{2}}\right)=\frac{\sqrt{\pi}}{12} x^3$

which gives a curve of a cubic function so on and so forth. This means thatthe values for $D_x^{\frac{p}{2}}\left(x^{\frac{1}{2}}\right)$ where $p \in \mathbb{Z}^-$ (odd) are power functions.

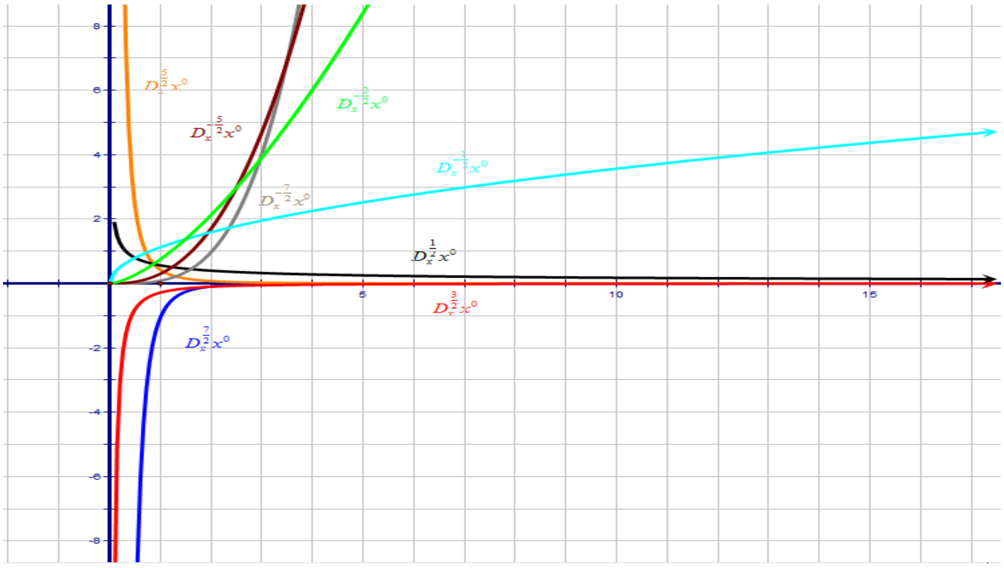

Fig. 1. $D_x^{\frac{p}{2}}\left(x^0\right)$ where $k=1$ and $p=\{\pm1,\pm3,\pm5,\pm7\}$

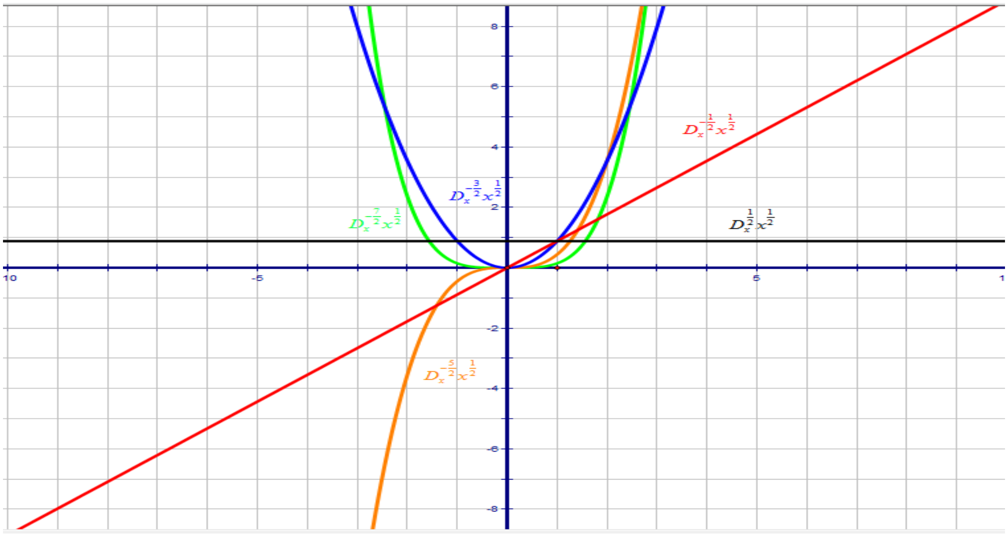

Fig. 2. $D_x^{\frac{p}{2}}\left(x^{\frac{1}{2}}\right)$ where $k=1$ and $p=\{\pm1,-3,-5,-7\}$

Fig. 3 shows the $\frac{p}{2}th$ derivative of the identity function $f(x)=x$ where $p=\{\pm1,\pm3,\pm5,\pm7\}$ .As we have observed, the curves lie on the first and fourth quadrants. But $\frac{p}{2}th$ derivative of this function where p is positive has the values of x in the interval $(0,\infty)$

and the curves have horizontal asymptotes at $D_x^{\frac{p}{2}} x = 0$ except at $p = 1$. On the other hand, if p is negative, then we can see in the curves that their point of origin is at $(0,0)$ and is moving upward in the different directions.

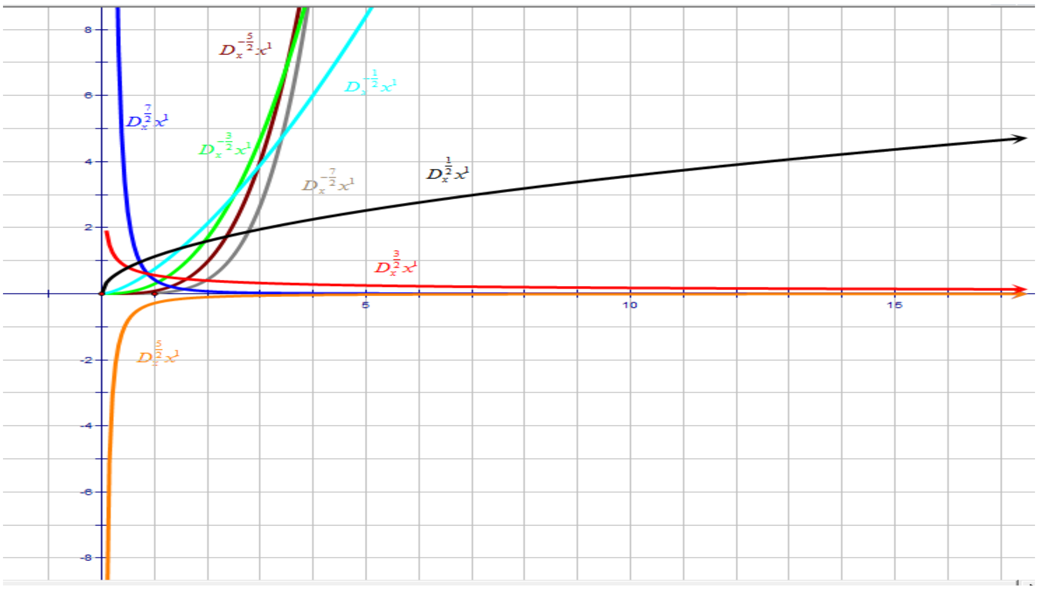

Fig. 3. $D_x^{\frac{p}{2}}(x)$ where $k = 1$ and $p = \{\pm1, \pm3, \pm5, \pm7\}$.

Fig. 4 shows the $\frac{p}{2}th$ derivative of the unit function $f(x) = (x+1)^0 = 1$ where $p = \{\pm1, \pm3, \pm5, \pm7\}$. As we have noticed, the curves lie on all the four quadrants. But $\frac{p}{2}th$ derivative of this function where p is positive hasthe values of x in the interval $(-1, \infty)$ and the curves have horizontal asymptotes at $D_x^{\frac{p}{2}}(x+1)^0 = 0$. On the other hand, if p is negative, then we can see in the curves that their point of origin is at $(-1,0)$ and is moving upward in the different directions.

Fig. 5 shows the $\frac{p}{2}th$ derivative of the function $f(x) = x+1$ where $p = \{\pm1, \pm3, \pm5, \pm7\}$.As we have observed, the curves lie on all the four quadrants. But $\frac{p}{2}th$ derivative of this function where p is positive hasthe values of x in the interval $(-1, \infty)$ and the curves have horizontal asymptotes at $D_x^{\frac{p}{2}}(x+1) = 0$ except at $p = 1$. On the other hand, if p is negative, then we can see in the curves that their point of origin is at $(-1,0)$ and is moving upward in the different directions.

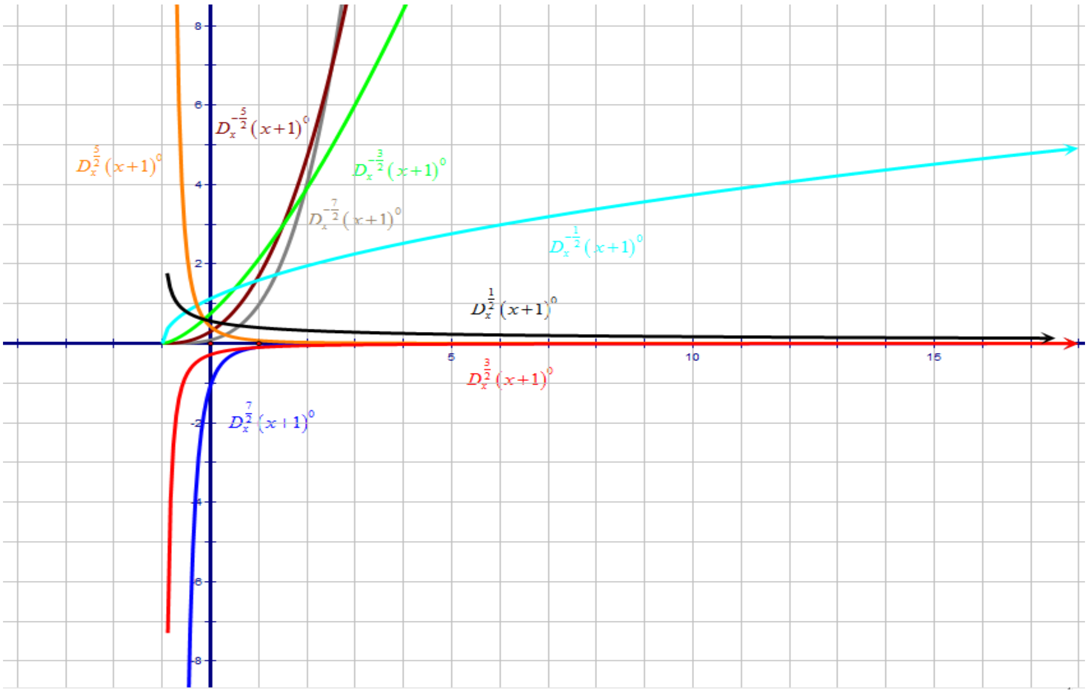

Fig. 4. $D_x^{\frac{p}{2}}(x+1)^0$ where $k=1$ and $p=\{\pm1,\pm3,\pm5,\pm7\}$.

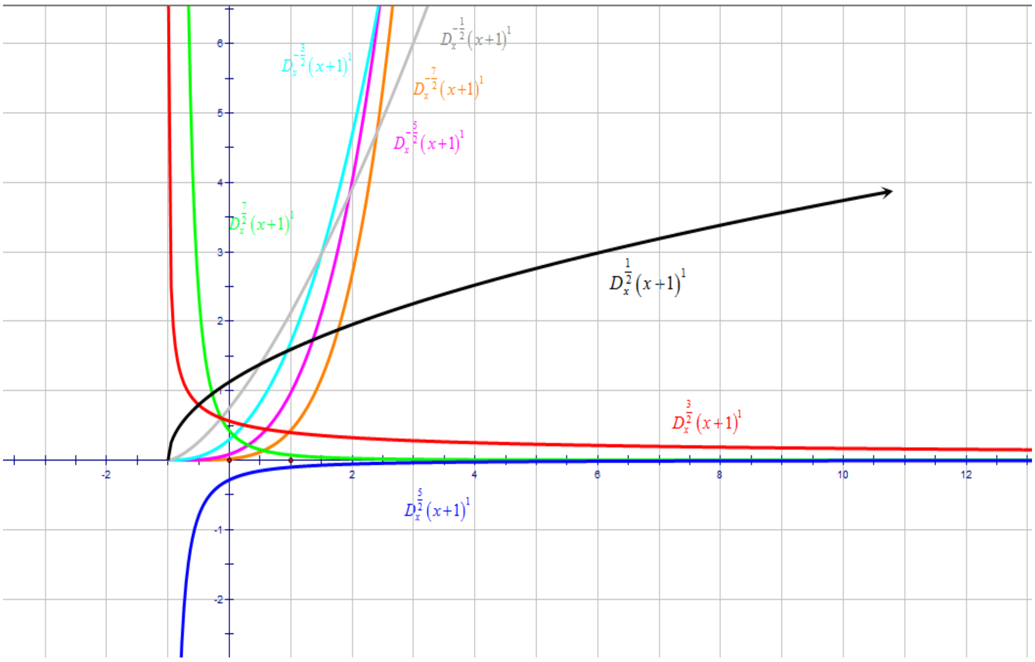

Fig. 5. $D_x^{\frac{p}{2}}(x+1)$ where $k=1$ and $p=\{\pm1,\pm3,\pm5,\pm7\}$.

III. CONCLUSIONS

The following are the conclusions of the study. Functions of the form $f(x)=kx^{\frac{m}{2}}$ and $f(x)=(k_1x\pm k_2)^{\frac{m}{2}}$ maybe a real number, a complex number, or a complex infinity with respect to fractional derivative. The graphical representation of a fractional derivative may consists of various graphs of common functions.

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