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# CONSTRUCTION OF FUNDAMENTAL THEOREMS OF FRACTIONAL CALCULUS

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***Abstract.*** *This paper discusses the theory of derivatives and integrals in the form of fractions with a particular order initiated by Lioville. Specifically, regarding the correlation between fractional derivatives and integrals, by examining definitions, determining the kernel function, and applying them to several examples, so a general formula will be obtained regarding the relationship between the two. This formula is the product of the fractional derivative of an order of a polynomial function of* $m$*-degree which is equal to the* $(n+1)^{th}$ *derivative of the related order fractional integral of a polynomial function of* $m$*-degree that the truth is proved by using Mathematical Induction.*
***Keywords:*** *fractional derivative; fractional integral; Fundamental Theorem of Calculus.*

## I. INTRODUCTION

One of the fundamental studies in mathematics, particularly in Calculus, is about derivatives and integrals. Lacroix developed a formula for the $n^{th}$ derivatives of $x^m$ in early 18th century [1], $\frac{d^n}{dx^n}x^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$. Remember that $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}$, $n > 0$ [2] and $\Gamma(n+1) = n\Gamma(n)$ where $\Gamma(1) = 1$ [3]. Lacroix initiated replacing $n$ with $\frac{1}{2}$ and $m = 1$ so it is obtained $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x = \frac{2\sqrt{x}}{\sqrt{\pi}}$ and note that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Therefore, since then, studies related to fractional order derivatives have emerged. Several mathematicians who studied Fractional Calculus included Joseph Fourier, who initiated integral notation to denote derivatives with non-integer orders. Joseph Liouville and Bernhard Riemann established a general notation for fractional integrals. Mittag-Leffler, Grünwald, and Letnikov who greatly contributed to develop the Fractional Calculus [4]

Several researchers discussed and established the Fractional Calculus, H. Vit Danon was one of those who developed the theory, namely the Fundamental Theorem of Fractional Calculus[5], $D^{\frac{1}{2}}f(x) = DF^{-\frac{1}{2}}(x)$, Podlubny [6] discusses fractional partial differential functions, Hadamard [7] also states that derivative of non integer order of an analytic function can be found using the Taylor series. Gunawan et al [8]explain a method to find a continuous and well-defined function of two real variables on a surface that minimizes the fractional integral energy.

The focus of this research is to show the relationship between fractional derivatives and integrals, and construct the Fundamental Theorem of Fractional Calculus of a particular order of polynomial function of $m$-degree, and prove using Mathematical Induction. This article is expected to provide new knowledge for students and serve as a reference source for other researchers to apply and develop in related fields.

## II. RIEMANN LIOUVILLE'S FRACTIONAL DERIVATIVES AND INTEGRALS

The Fundamental Theorem of Fractional Calculus is constructed using an analytical method. First, we will discuss the concept introduced by the French mathematician, Joseph Liouville, namely Liouville's Fractional Integral or Riemann-Liouville's Fractional Integral, which is a generalization of the Riemann-Stieltjes Integral for functions that have fractional derivatives [9]. Then, the corresponding kernel function is determined to obtain a generalized formula for fractional derivatives and integrals of order $n+\frac{k}{k+1}$and $n+\frac{k}{k+2}$ from polynomial functions of $m$-degree with some examples to be analyzed. However, in this article, we used the polynomial function[10] [11] $f(x)=x^m$. The final step is to create a general formula for the Fundamental Theorem of Fractional Calculus of order $n+\frac{k}{k+1}$ and $n+\frac{k}{k+2}$ that the truth is proved by using Mathematical Induction.

The fractional derivatives and integrals are obtained by extending the derivatives and integrals of integer order to the $q\in\mathbb{Q}$. In this article, Riemann Liouville's fractional derivatives and integrals can be identified by looking at the sign of each order, either positive or negative. If the sign is positive then it means fractional derivatives, whereas if the sign is negative then it means fractional integrals. First, Definition 1 and 2 will be given.

**Definition 1** [12]*Let $f$ and $g$ Lebesgue Integrable on the interval $(-\infty,\infty)$. The convolution of function $f$ and $g$ can be denoted as $h=f*g$, where $g$ is the kernel of convolution, and defined as*

$$h(x)=\int_{-\infty}^{\infty}f(t)g(x-t)\,dt.$$

**Definition 2** *Let $f\colon G\to G'$ is a group homomorphism. The kernel of $f$ can be denoted as Ker($f$) and defined as Ker($f$) $=\{x\in G|\ f(x)=e'\}$* [13][14]

Liouville defines fractional derivatives of order $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, and fractional integrals order $-\frac{1}{2}$, $-\frac{1}{3}$, and $-\frac{2}{3}$ of the function $f$ in [5][15]. We can apply the definitions to calculate the fractional derivatives of order $\frac{1}{2}$, $\frac{2}{3}$ and the integrals order $-\frac{1}{2}$ of $f(x)=x$, where $x\in[0,2]$, as follows

$$D^{\frac{1}{2}}x=\frac{1}{\Gamma\left(-\frac{1}{2}\right)}\int_0^x(x-t)^{-\frac{3}{2}}t\,dt=\frac{-1}{2\sqrt{\pi}}\left[2t(x-t)^{-\frac{1}{2}}+4(x-t)^{\frac{1}{2}}\right]_0^x=\frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} \tag{1}$$

$$D^{\frac{2}{3}}x=\frac{1}{\Gamma\left(\frac{-2}{3}\right)}\int_0^x(x-t)^{-\frac{5}{3}}t\,dt=\frac{-\Gamma\left(\frac{2}{3}\right)}{\pi\sqrt{3}}\left[\frac{3}{2}t(x-t)^{-\frac{2}{3}}+\frac{9}{2}(x-t)^{\frac{1}{3}}\right]_0^x=\frac{9\Gamma\left(\frac{2}{3}\right)x^{\frac{1}{3}}}{2\pi\sqrt{3}} \tag{2}$$

In other side, we obtain fractional integrals of order $-\frac{1}{2}$

$$F^{-\frac{1}{2}}(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x (x-t)^{-\frac{1}{2}} t\, dt = \frac{1}{\sqrt{\pi}} \left[-2t(x-t)^{\frac{1}{2}} - \frac{4}{3}(x-t)^{\frac{3}{2}}\right]_0^x = \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}} \quad (3)$$

The existence of the definition of Riemann-Liouville's Fractional Derivative, as the convolution of $f$ and kernel, inspired the creation of several kernels for fractional derivatives of a particular order, which can be seen in Table 1.

Table 1. Kernel of Fractional Derivatives of Particular Orders

| Form | Order | Kernel | Order | Kernel | Order | Kernel |
|---|---|---|---|---|---|---|
| I | $\frac{3}{4}$ | $\frac{x^{-\frac{7}{4}}}{\Gamma\left(-\frac{3}{4}\right)}$ | $\frac{7}{4}$ | $\frac{x^{-\frac{11}{4}}}{\Gamma\left(-\frac{7}{4}\right)}$ | $\frac{11}{4}$ | $\frac{x^{-\frac{15}{4}}}{\Gamma\left(-\frac{11}{4}\right)}$ |
| | $\frac{4}{5}$ | $\frac{x^{-\frac{9}{5}}}{\Gamma\left(-\frac{4}{5}\right)}$ | $\frac{9}{5}$ | $\frac{x^{-\frac{14}{5}}}{\Gamma\left(-\frac{9}{5}\right)}$ | $\frac{14}{5}$ | $\frac{x^{-\frac{19}{5}}}{\Gamma\left(-\frac{14}{5}\right)}$ |
| | $\frac{5}{6}$ | $\frac{x^{-\frac{11}{6}}}{\Gamma\left(-\frac{5}{6}\right)}$ | $\frac{11}{6}$ | $\frac{x^{-\frac{17}{6}}}{\Gamma\left(-\frac{11}{6}\right)}$ | $\frac{17}{6}$ | $\frac{x^{-\frac{23}{6}}}{\Gamma\left(-\frac{17}{6}\right)}$ |
| II | $\frac{2}{4}$ | $\frac{x^{-\frac{6}{4}}}{\Gamma\left(-\frac{2}{4}\right)}$ | $\frac{6}{4}$ | $\frac{x^{-\frac{10}{4}}}{\Gamma\left(-\frac{6}{4}\right)}$ | $\frac{10}{4}$ | $\frac{x^{-\frac{14}{4}}}{\Gamma\left(-\frac{10}{4}\right)}$ |
| | $\frac{3}{5}$ | $\frac{x^{-\frac{8}{5}}}{\Gamma\left(-\frac{3}{5}\right)}$ | $\frac{8}{5}$ | $\frac{x^{-\frac{13}{5}}}{\Gamma\left(-\frac{8}{5}\right)}$ | $\frac{13}{5}$ | $\frac{x^{-\frac{18}{5}}}{\Gamma\left(-\frac{13}{5}\right)}$ |
| | $\frac{4}{6}$ | $\frac{x^{-\frac{10}{6}}}{\Gamma\left(-\frac{4}{6}\right)}$ | $\frac{10}{6}$ | $\frac{x^{-\frac{16}{6}}}{\Gamma\left(-\frac{10}{6}\right)}$ | $\frac{16}{6}$ | $\frac{x^{-\frac{22}{6}}}{\Gamma\left(-\frac{16}{6}\right)}$ |

Based on Table 1, the formula can be formed for fractional derivatives of several orders, such as order $\frac{3}{4}$ and $\frac{4}{6}$ in Definitions 3 and 4.

**Definition 3** *Let $f$ integrable function over the interval $[a,b]$. There exists $\frac{x^{-\frac{7}{4}}}{\Gamma\left(-\frac{3}{4}\right)}$ the kernel of fractional derivatives of order $\frac{3}{4}$, based on Table 1 and Definition 1 then the fractional derivatives of order $\frac{3}{4}$ can be denoted as $D^{\frac{3}{4}}f(x)$, defined as the convolution of $f$ and kernel $\frac{x^{-\frac{7}{4}}}{\Gamma\left(-\frac{3}{4}\right)}$ over the interval $[a,x]$, such as*

$$D^{\frac{3}{4}}f(x) = \frac{1}{\Gamma\left(-\frac{3}{4}\right)} \int_a^x (x-t)^{-\frac{7}{4}} f(t)dt \quad (4)$$

**Definition 4** *Let f integrable function over the interval [a,b]. There exists* $\frac{x^{-\frac{10}{6}}}{\Gamma\left(-\frac{4}{6}\right)}$ *the kernel of fractional derivatives of order* $\frac{4}{6}$, *based on Table 1 and Definition 1 then the fractional derivatives of order* $\frac{4}{6}$ *can be denoted as* $D^{\frac{4}{6}}f(x)$, *defined as the convolution of f and kernel* $\frac{x^{-\frac{10}{6}}}{\Gamma\left(-\frac{4}{6}\right)}$ *over the interval* $[a, x]$, *such as*

$$D^{\frac{4}{6}}f(x) = \frac{1}{\Gamma\left(-\frac{4}{6}\right)} \int_a^x (x-t)^{-\frac{10}{6}} f(t)dt \tag{5}$$

We will use the definition to calculate fractional derivatives of $f(x) = x$.

**Example 1** Let $f(x) = x$, where $x \in [0,2]$. We will calculate the fractional derivatives of order $\frac{4}{6}$ and $\frac{3}{4}$, so we obtain

$$D^{\frac{4}{6}}f(x) = \frac{1}{\Gamma\left(\frac{-4}{6}\right)} \int_0^x (x-t)^{-\frac{10}{6}} udu = -\frac{\Gamma\left(\frac{4}{6}\right)}{\pi\sqrt{3}} \left[\frac{6}{4} t(x-t)^{-\frac{4}{6}} + \frac{18}{4}(x-t)^{\frac{2}{6}}\right]_0^x = \frac{18\Gamma\left(\frac{2}{3}\right) x^{\frac{2}{6}}}{4\pi\sqrt{3}}$$

$$D^{\frac{3}{4}}f(x) = \frac{1}{\Gamma\left(-\frac{3}{4}\right)} \int_0^x (x-t)^{-\frac{7}{4}} t\, dt = \frac{-3\Gamma\left(\frac{3}{4}\right)}{4\pi\sqrt{2}} \left[\frac{4}{3} t(x-t)^{-\frac{3}{4}} + \frac{16}{3}(x-t)^{\frac{1}{4}}\right]_0^x = \frac{16\Gamma\left(\frac{3}{4}\right) x^{\frac{1}{4}}}{4\pi\sqrt{2}}$$

Furthermore, the result in Example 1 (specifically order $\frac{4}{6}$) will be compared with order $\frac{2}{3}$ of $f(x) = x$ as we obtained previously. By (2), we have $\frac{9\Gamma\left(\frac{2}{3}\right)x^{\frac{1}{3}}}{2\pi\sqrt{3}}$ fractional derivatives of order $\frac{2}{3}$, while $\frac{18\Gamma\left(\frac{2}{3}\right)x^{\frac{2}{6}}}{4\pi\sqrt{3}}$ as the fractional derivatives of order $\frac{4}{6}$ and equal to $\frac{9\Gamma\left(\frac{2}{3}\right)x^{\frac{1}{3}}}{2\pi\sqrt{3}}$. Hence, we can state that the fractional derivatives of order $\frac{2}{3}$ and $\frac{4}{6}$ of $f(x) = x$ have the same result. So, this can be convincing that there is no contradiction between the definition in Table 1 and Definition 4 that has been made. Next, several examples of fractional derivatives with a particular order will be given based on the kernel search formed in Table 1.

**Example 2** Let $f(x) = x$, where $x \in [0,2]$. We will calculate the fractional derivatives of order $\frac{7}{4}$, we obtain

$$D^{\frac{7}{4}}f(x) = \frac{1}{\Gamma\left(\frac{-7}{4}\right)} \int_0^x (x-t)^{-\frac{11}{4}} t\, dt = \frac{1}{\Gamma\left(\frac{-7}{4}\right)} \left[\frac{4}{7} t(x-t)^{-\frac{7}{4}} + \frac{16}{21}(x-t)^{-\frac{3}{4}}\right]_0^x = \frac{\Gamma\left(\frac{3}{4}\right)\sqrt[4]{x^{-3}}}{\pi\sqrt{2}}.$$

In similar way, we can calculate other examples as in Table 2 and 3. This can be used to construct the Fundamental Theorem of Fractional Calculus of a particular order.

Table 2. Example of Fractional Derivatives of Order Form I of $f(x) = x$ and $f(x) = x^2$

| | Fractional Derivatives of $f(x)$ | | | | | |
|---|---|---|---|---|---|---|
| | Order | $x$ | $x^2$ | Order | $x$ | $x^2$ |
| 1 | $\frac{1}{2}$ | $\frac{2\sqrt{x}}{\sqrt{\pi}}$ | $\frac{8\sqrt{x^3}}{3\sqrt{\pi}}$ | $\frac{3}{2}$ | $\frac{\sqrt{x^{-1}}}{\sqrt{\pi}}$ | $\frac{4\sqrt{x}}{\sqrt{\pi}}$ |
| 2 | $\frac{2}{3}$ | $\frac{9\Gamma\left(\frac{2}{3}\right)\sqrt[3]{x}}{2\pi\sqrt{3}}$ | $\frac{27\Gamma\left(\frac{2}{3}\right)\sqrt[3]{x^4}}{4\pi\sqrt{3}}$ | $\frac{5}{3}$ | $\frac{3\Gamma\left(\frac{2}{3}\right)\sqrt[3]{x^{-2}}}{2\pi\sqrt{3}}$ | $\frac{9\Gamma\left(\frac{2}{3}\right)\sqrt[3]{x}}{\pi\sqrt{3}}$ |
| 3 | $\frac{3}{4}$ | $\frac{4\Gamma\left(\frac{3}{4}\right)\sqrt[4]{x}}{\pi\sqrt{2}}$ | $\frac{32\Gamma\left(\frac{3}{4}\right)\sqrt[4]{x^5}}{5\pi\sqrt{2}}$ | $\frac{7}{4}$ | $\frac{\Gamma\left(\frac{3}{4}\right)\sqrt[4]{x^{-3}}}{\pi\sqrt{2}}$ | $\frac{8\Gamma\left(\frac{3}{4}\right)\sqrt[4]{x}}{\pi\sqrt{2}}$ |
| 4 | $\frac{4}{5}$ | $\frac{5\Gamma\left(\frac{4}{5}\right)\sqrt[5]{x}}{\pi\csc\left(\frac{1}{5}\pi\right)}$ | $\frac{50\Gamma\left(\frac{4}{5}\right)\sqrt[5]{x^6}}{6\pi\csc\left(\frac{1}{5}\pi\right)}$ | $\frac{9}{5}$ | $\frac{\Gamma\left(\frac{4}{5}\right)\sqrt[5]{x^{-4}}}{\pi\csc\left(\frac{1}{5}\pi\right)}$ | $\frac{10\Gamma\left(\frac{4}{5}\right)\sqrt[5]{x}}{\pi\csc\left(\frac{1}{5}\pi\right)}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $k$ | $0+\frac{k}{k+1}$ | $D^{\left(0+\frac{k}{k+1}\right)}x$ | $D^{\left(0+\frac{k}{k+1}\right)}x^2$ | $1+\frac{k}{k+1}$ | $D^{\left(1+\frac{k}{k+1}\right)}x$ | $D^{\left(1+\frac{k}{k+1}\right)}x^2$ |

Table 3. Example of Fractional Derivatives of Order Form II of $f(x) = x$ and $f(x) = x^2$

| | Fractional Derivatives of $f(x)$ | | | | | |
|---|---|---|---|---|---|---|
| | Order | $x$ | $x^2$ | Orde | $x$ | $x^2$ |
| 1 | $\frac{1}{3}$ | $\frac{3x^{\frac{2}{3}}}{2\Gamma\left(\frac{2}{3}\right)}$ | $\frac{9x^{\frac{5}{3}}}{5\Gamma\left(\frac{2}{3}\right)}$ | $\frac{4}{3}$ | $\frac{x^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)}$ | $\frac{3x^{\frac{2}{3}}}{\Gamma\left(\frac{2}{3}\right)}$ |
| 2 | $\frac{2}{4}$ | $\frac{8x^{\frac{2}{4}}}{4\sqrt{\pi}}$ | $\frac{32x^{\frac{6}{4}}}{12\sqrt{\pi}}$ | $\frac{6}{4}$ | $\frac{x^{-\frac{1}{2}}}{\sqrt{\pi}}$ | $\frac{4x^{\frac{1}{2}}}{\sqrt{\pi}}$ |
| 3 | $\frac{3}{5}$ | $\frac{5\Gamma\left(\frac{3}{5}\right)x^{\frac{2}{5}}}{2\pi\csc\left(\frac{2}{5}\pi\right)}$ | $\frac{25\Gamma\left(\frac{3}{5}\right)x^{\frac{7}{5}}}{7\pi\csc\left(\frac{2}{5}\pi\right)}$ | $\frac{8}{5}$ | $\frac{\Gamma\left(\frac{3}{5}\right)x^{-\frac{3}{5}}}{\pi\csc\left(\frac{2}{5}\pi\right)}$ | $\frac{5\Gamma\left(\frac{3}{5}\right)x^{\frac{2}{5}}}{\pi\csc\left(\frac{2}{5}\pi\right)}$ |
| 4 | $\frac{4}{6}$ | $\frac{9\Gamma\left(\frac{2}{3}\right)x^{\frac{1}{3}}}{2\pi\sqrt{3}}$ | $\frac{27\Gamma\left(\frac{2}{3}\right)x^{\frac{4}{3}}}{4\pi\sqrt{3}}$ | $\frac{10}{6}$ | $\frac{3\Gamma\left(\frac{2}{3}\right)x^{-\frac{2}{3}}}{2\pi\sqrt{3}}$ | $\frac{9\Gamma\left(\frac{2}{3}\right)x^{\frac{1}{3}}}{\pi\sqrt{3}}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $k$ | $0+\frac{k}{k+2}$ | $D^{\left(0+\frac{k}{k+2}\right)}x$ | $D^{\left(0+\frac{k}{k+2}\right)}x^2$ | $1+\frac{k}{k+2}$ | $D^{\left(1+\frac{k}{k+2}\right)}x$ | $D^{\left(1+\frac{k}{k+2}\right)}x^2$ |

## III. CONSTRUCTION OF FUNDAMENTAL THEOREMS OF FRACTIONAL CALCULUS

The relationship between fractional derivatives and integrals is stated in the Fundamental Theorem of Fractional Calculus. The theorem states that the fractional derivative of order $\frac{1}{2}$ is equal to the derivative of fractional integrals of order $-\frac{1}{2}$. If Equations (3) and (1) are substituted into the Fundamental Theorem of Fractional Calculus of Order $\frac{1}{2}$, we will obtain

$$DF^{-\frac{1}{2}}(x) = \frac{d}{dx}\frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}} = \frac{4\left(\frac{3}{2}\right)x^{\frac{3}{2}-1}}{3\sqrt{\pi}} = \frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} = D^{\frac{1}{2}}x.$$

In particular, if the Fundamental Theorem of Fractional Calculus of Order $\frac{1}{2}$ is tried for higher order, we have

$$D^{\frac{3}{2}}f(x) = DD^{\frac{1}{2}}f(x) = D^2F^{-\frac{1}{2}}(x)$$
$$D^{\frac{5}{2}}f(x) = D^2D^{\frac{1}{2}}f(x) = D^3F^{-\frac{1}{2}}(x)$$
$$\vdots$$

More generally,

$$D^{n+\frac{1}{2}}f(x) = D^nD^{\frac{1}{2}}f(x) = D^{n+1}F^{-\frac{1}{2}}(x), n \geq 0.$$

In different order, we obtain

$$D^{n+\frac{1}{3}}f(x) = D^nD^{\frac{1}{3}}f(x) = D^{n+1}F^{-\frac{2}{3}}(x), n \geq 0$$
$$D^{n+\frac{2}{3}}f(x) = D^nD^{\frac{2}{3}}f(x) = D^{n+1}F^{-\frac{1}{3}}(x), n \geq 0.$$

If several examples of fractional derivatives in Table 2 and 3 are analyzed based on the Fundamental Theorem of Fractional Calculus of Order $\frac{1}{2}$, we have

$$D^{\frac{3}{4}}f(x) = D^{1-\frac{1}{4}}f(x) = DD^{-\frac{1}{4}}f(x) = DF^{-\frac{1}{4}}(x)$$
$$D^{\frac{4}{5}}f(x) = D^{1-\frac{1}{5}}f(x) = DD^{-\frac{1}{5}}f(x) = DF^{-\frac{1}{5}}(x)$$
$$D^{\frac{5}{6}}f(x) = D^{1-\frac{1}{6}}f(x) = DD^{-\frac{1}{6}}f(x) = DF^{-\frac{1}{6}}(x) \quad (6)$$

and

$$D^{\frac{2}{4}}f(x) = D^{1-\frac{2}{4}}f(x) = DD^{-\frac{2}{4}}f(x) = DF^{-\frac{2}{4}}(x)$$
$$D^{\frac{3}{5}}f(x) = D^{1-\frac{2}{5}}f(x) = DD^{-\frac{2}{5}}f(x) = DF^{-\frac{2}{5}}(x)$$
$$D^{\frac{4}{6}}f(x) = D^{1-\frac{2}{6}}f(x) = DD^{-\frac{2}{6}}f(x) = DF^{-\frac{2}{6}}(x) \quad (7)$$

Equation (6) and (7) initiate to establish the kernel of fractional integrals of particular order in Table 4.

By Table 4, we can calculate some examples in similar way as Example 2, which presented in Table 5.

Tabel 4. The Kernel of Fractional Integrals of Particular Orders

| Order | $-\frac{1}{4}$ | $-\frac{1}{5}$ | $-\frac{1}{6}$ | $-\frac{2}{4}$ | $-\frac{2}{5}$ | $-\frac{2}{6}$ |
|---|---|---|---|---|---|---|
| Kernel | $\frac{x^{-\frac{3}{4}}}{\Gamma\left(\frac{1}{4}\right)}$ | $\frac{x^{-\frac{4}{5}}}{\Gamma\left(\frac{1}{5}\right)}$ | $\frac{x^{-\frac{5}{6}}}{\Gamma\left(\frac{1}{6}\right)}$ | $\frac{x^{-\frac{2}{4}}}{\Gamma\left(\frac{2}{4}\right)}$ | $\frac{x^{-\frac{3}{5}}}{\Gamma\left(\frac{2}{5}\right)}$ | $\frac{x^{-\frac{4}{6}}}{\Gamma\left(\frac{2}{6}\right)}$ |

Tabel 5. Example of Fractional Integrals of Order Form I and II of $f(x) = x$ and $f(x) = x^2$

| No. | Fractional Integral $f(x)$ | | | | | |
|---|---|---|---|---|---|---|
| | Orde | $x$ | $x^2$ | Orde | $x$ | $x^2$ |
| 1 | $-\frac{1}{2}$ | $\frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}$ | $\frac{16x^{\frac{5}{2}}}{15\sqrt{\pi}}$ | $-\frac{2}{3}$ | $\frac{9x^{\frac{5}{3}}}{10\Gamma\left(\frac{2}{3}\right)}$ | $\frac{27x^{\frac{8}{3}}}{40\Gamma\left(\frac{2}{3}\right)}$ |
| 2 | $-\frac{1}{3}$ | $\frac{27\Gamma\left(\frac{2}{3}\right)}{8\pi\sqrt{3}}x^{\frac{4}{3}}$ | $\frac{81\Gamma\left(\frac{2}{3}\right)x^{\frac{7}{3}}}{28\pi\sqrt{3}}$ | $-\frac{2}{4}$ | $\frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}$ | $\frac{16x^{\frac{5}{2}}}{15\sqrt{\pi}}$ |
| 3 | $-\frac{1}{4}$ | $\frac{16\Gamma\left(\frac{3}{4}\right)x^{\frac{5}{4}}}{5\pi\sqrt{2}}$ | $\frac{128\Gamma\left(\frac{3}{4}\right)x^{\frac{9}{4}}}{45\pi\sqrt{2}}$ | $-\frac{2}{5}$ | $\frac{25\Gamma\left(\frac{3}{5}\right)x^{\frac{7}{5}}}{14\pi\csc\left(\frac{2}{5}\pi\right)}$ | $\frac{250\Gamma\left(\frac{3}{5}\right)x^{\frac{12}{5}}}{168\pi\csc\left(\frac{2}{5}\pi\right)}$ |
| 4 | $-\frac{1}{5}$ | $\frac{25\Gamma\left(\frac{4}{5}\right)x^{\frac{6}{5}}}{6\pi\csc\left(\frac{1}{5}\pi\right)}$ | $\frac{250\Gamma\left(\frac{4}{5}\right)x^{\frac{11}{5}}}{66\pi\csc\left(\frac{1}{5}\pi\right)}$ | $-\frac{2}{6}$ | $\frac{27\Gamma\left(\frac{2}{3}\right)}{8\pi\sqrt{3}}x^{\frac{4}{3}}$ | $\frac{81\Gamma\left(\frac{2}{3}\right)x^{\frac{7}{3}}}{28\pi\sqrt{3}}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $k$ | $-\frac{1}{k+1}$ | $F^{\left(-\frac{1}{k+1}\right)}(x)$ | $F^{\left(-\frac{1}{k+1}\right)}(x)$ | $-\frac{2}{k+2}$ | $F^{\left(-\frac{2}{k+2}\right)}(x)$ | $F^{\left(-\frac{2}{k+2}\right)}(x)$ |

By Table 5, Equation (6), and (7), we can generalize the Fundamental Theorem of Fractional Calculus of Order

Form I :

$$D^{\frac{k}{k+1}}f(x) = D^{1-\frac{1}{k+1}}f(x) = DD^{-\frac{1}{k+1}}f(x) = DF^{-\frac{1}{k+1}}(x),\ k \in \mathbb{N} \tag{8}$$

Form II :

$$D^{\frac{k}{k+2}}f(x) = D^{1-\frac{2}{k+2}}f(x) = DD^{-\frac{2}{k+2}}f(x) = DF^{-\frac{2}{k+2}}(x),\ k \in \mathbb{N} \tag{9}$$

Furthermore, if the product of fractional integrals of order $-\frac{1}{4}$ and $-\frac{2}{3}$ of $x$ is differentiated twice, then

$$DF^{-\frac{1}{4}}(x) = \frac{d}{dx}\frac{16\Gamma\left(\frac{3}{4}\right)x^{\frac{5}{4}}}{5\pi\sqrt{2}} = \frac{4\Gamma\left(\frac{3}{4}\right)x^{\frac{1}{4}}}{\pi\sqrt{2}}$$

$$D^2F^{-\frac{1}{4}}(x) = DDF^{-\frac{1}{4}}(x) = \frac{d}{dx}\frac{4\Gamma\left(\frac{3}{4}\right)x^{\frac{1}{4}}}{\pi\sqrt{2}} = \frac{\Gamma\left(\frac{3}{4}\right)x^{-\frac{3}{4}}}{\pi\sqrt{2}}.$$

The fact, $\frac{4\Gamma\left(\frac{3}{4}\right)x^{\frac{1}{4}}}{\pi\sqrt{2}}$ and $\frac{\Gamma\left(\frac{3}{4}\right)x^{-\frac{3}{4}}}{\pi\sqrt{2}}$ are the product of fractional derivatives of order $\frac{3}{4}$ and $\frac{7}{4}$ of $x$. In different order we get

$$DF^{-\frac{2}{3}}(x) = \frac{d}{dx}\frac{9x^{\frac{5}{3}}}{10\Gamma\left(\frac{2}{3}\right)} = \frac{3x^{\frac{2}{3}}}{2\Gamma\left(\frac{2}{3}\right)}$$

$$D^2F^{-\frac{2}{3}}(x) = DDF^{-\frac{2}{3}}(x) = \frac{d}{dx}\frac{3x^{\frac{2}{3}}}{2\Gamma\left(\frac{2}{3}\right)} = \frac{x^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)},$$

From this result were obtained $\frac{3x^{\frac{2}{3}}}{2\Gamma\left(\frac{2}{3}\right)}$ and $\frac{x^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)}$ as fractional derivatives of order $\frac{1}{3}$ and $\frac{4}{3}$ of $f(x) = x$. This needs to be done to examine that (8) and (9) have been formulated and constructed correctly. Accordingly, we can state that the product of the fractional derivative of order of $n + \frac{k}{k+1}$ of polynomial function of $m$-degree is equal to the $(n+1)^{th}$ derivative of fractional integral of $-\frac{1}{k+1}$ of polynomial function of $m$-degree. Likewise, the product of the fractional derivative of order of $n + \frac{k}{k+2}$ of polynomial function of $m$-degree is equal to the $(n+1)^{th}$ derivative of fractional integral of $-\frac{1}{k+2}$ of polynomial function of $m$-degree, for arbitrary $n \geq 0, k \in \mathbb{N}$. Thus, it can be formulated as Propositions 1 and 2. Moreover, we can prove that validity by using Mathematical Induction [16].

**Propositions 1** *if* $f(x) = x^m$, *then* $D^{n+\frac{k}{k+1}}f(x) = D^{n+1}F^{\left(-\frac{1}{k+1}\right)}(x)\ \forall n \geq 0,\ k \in \mathbb{N}$.

**Proof:**

Let $P(k)$: $D^{n+\frac{k}{k+1}}f(x) = D^{n+1}F^{\left(-\frac{1}{k+1}\right)}(x)$ for arbitrary $k \in \mathbb{N}$ and fix positive integer $n$.

a. Initial Step.
   If $k = 1$, then
   $P(1)$: $D^{n+\frac{1}{1+1}}f(x) = D^{n+\frac{1}{2}}f(x) = D^nD^{\frac{1}{2}}f(x) = D^{n+1}D^{-\frac{1}{2}}f(x) = D^{n+1}F^{-\frac{1}{2}}(x)$
   $\therefore P(1)$ is true
b. Inductive Step.
   Assume that $P(i)$: $D^{n+\frac{i}{i+1}}f(x) = D^{n+1}F^{\left(-\frac{1}{i+1}\right)}(x)$ is true.
   We want to prove that $P(i+1)$ is true, such that

$$\begin{aligned} P(i+1): D^{n+\frac{i+1}{(i+1)+1}}f(x) &= D^{n+\frac{i+1}{i+2}}f(x) \\ &= D^nD^{\frac{i+1}{i+2}}f(x) \\ &= D^nD^{\frac{i+2}{i+2}-\frac{1}{i+2}}f(x) \\ &= D^nD^{1-\frac{1}{i+2}}f(x) \\ &= D^{n+1}D^{-\frac{1}{i+2}}f(x) \\ &= D^{n+1}F^{\left(-\frac{1}{i+2}\right)}(x) \end{aligned}$$

   $\therefore P(i+1)$ is true

Hence a and b, $P(k)$: $D^{n+\frac{k}{k+1}}f(x) = D^{n+1}F^{\left(-\frac{1}{k+1}\right)}(x)$ is true for all $k \in \mathbb{N}, n \geq 0$. ■

**Proposition 2.** *If* $f(x) = x^m$, *then* $D^{n+\frac{k}{k+2}}f(x) = D^{n+1}F^{\left(-\frac{2}{k+2}\right)}(x)\ \forall n \geq 0,\ k \in \mathbb{N}$.

**Proof:**

Suppose $P(k)$: $D^{n+\frac{k}{k+2}}f(x) = D^{n+1}F^{\left(-\frac{2}{k+2}\right)}(x)$ for arbitrary $k \in \mathbb{N}$ and fix positive integer $n$.

a. Initial Step.
If $k = 1$, then
$P(1)$: $D^{n+\frac{1}{1+2}}f(x) = D^{n+\frac{1}{3}}f(x) = D^n D^{1-\frac{2}{3}}f(x) = D^{n+1}D^{-\frac{2}{3}}f(x) = D^{n+1}F^{-\frac{2}{3}}(x)$
$\therefore P(1)$ is true.

b. Inductive Step.
Assume that $P(i)$: $D^{n+\frac{i}{i+2}}f(x) = D^{n+1}F^{\left(-\frac{2}{i+2}\right)}(x)$ is true.
We want to prove that $P(i+1)$ is true, such that

$$
\begin{aligned}
P(i+1): D^{n+\frac{i+1}{(i+1)+2}}f(x) &= D^{n+\frac{i+1}{i+3}}f(x) \\
&= D^n D^{\frac{i+3}{i+3}-\frac{2}{i+3}}f(x) \\
&= D^n D^{1-\frac{2}{i+3}}f(x) \\
&= D^{n+1}D^{-\frac{2}{i+3}}f(x) \\
&= D^{n+1}F^{\left(-\frac{2}{i+3}\right)}(x)
\end{aligned}
$$

$\therefore P(i+1)$ is true
Hence a and b, $P(k)$: $D^{n+\frac{k}{k+2}}f(x) = D^{n+1}F^{\left(-\frac{2}{k+2}\right)}(x)$ is true for all $k \in \mathbb{N},\ n \geq 0$. ■

## IV. CONCLUSIONS

Based on the constructed Propositions 1 and 2, it can be concluded that there is a relationship between fractional derivatives and integrals. In particular, the product of the fractional derivative of order of $n + \frac{k}{k+1}$ and $n + \frac{k}{k+2}$ of polynomial function of $m$-degree which is equal to the $(n+1)^{th}$ derivative of fractional integral of $-\frac{1}{k+1}$ and $-\frac{1}{k+2}$ of polynomial function of $m$-degree, namely the Fundamental Theorem of Fractional Calculus of Order $n + \frac{k}{k+1}$ and $n + \frac{k}{k+2}, \forall n \geq 0, k \in \mathbb{N}$.

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