

# TOPOLOGY OF QUASI-PSEUDOMETRIC SPACES AND CONTINUOUS LINEAR OPERATOR ON ASYMMETRIC NORMED SPACES

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**Abstract.** In this paper, we will discuss about topological properties of quasi-pseudometric spaces and properties of linear operators in asymmetric normed spaces. The topological properties of quasi-pseudometric spaces will be given consisting of open and closed set properties in quasi-pseudometric spaces. The discussion about properties of linear operator on asymmetric normed spaces is focus on the uniform boundedness principle. The uniform boundedness theorem is proved by utilizing completeness properties and characteristic of closed sets on quasi-pseudometric spaces.

**Keywords:** quasi-pseudometric, asymmetric normed spaces, topological properties, uniform boundedness principle.

## I. INTRODUCTION

Quasi pseudometric is a generalization of a metric. This generalization is achieved by weakening two properties that metrics possess [9]. Furthermore, a quasi pseudometric is called a quasi metric if it satisfies a specific axiom. Based on the properties of quasi pseudometrics, Kelly [9] established the conjugate of quasi pseudometrics and the topology on quasi pseudometric spaces.

Based on the research conducted by Kelly [9] regarding quasi pseudometrics, Duffin et al [1] introduced the term asymmetric norm and established the relationship between asymmetric norm and quasi pseudometrics. Duffin et al [1] stated that a space with an asymmetric norm  $(X, p)$ , where  $X$  is a vector space equipped with the asymmetric norm  $p$ , can be considered as a quasi pseudometric space  $(X, d_p)$ , where  $d_p : X \times X \rightarrow [0, \infty)$ , and  $d_p(x, y) = p(y - x)$ , for every  $x, y \in X$ . The topology, properties of convergent sequences, and completeness properties of quasi pseudometric spaces have been studied by Reilly et al [12], Romaguera [13], Wilson [15], and Cobzas [6, 7]. In these studies, further investigation into the relationship between quasi pseudometrics and quasi metrics has not been explored. Additionally, there hasn't been extensive research on the properties of open and closed sets in quasi pseudometric spaces. Therefore, the authors were motivated to conduct research in this area and provide some supporting examples.

The research related to normed asymmetric spaces continues to progress, and one of its developments involves studying the properties of linear operators in asymmetric normed spaces as discussed by Cobzas [6, 2]. Cobzas' research builds upon the findings of previous studies conducted by Raffi and others [4]. One of the properties of linear operators in asymmetric normed spaces described by Cobzas [6] is the uniform boundedness property exhibited by collections of pointwise bounded continuous linear operators in asymmetric normed spaces. Furthermore, Cobzas [2] and Bouadjila et al [3] revisited the uniform boundedness property of collections

of continuous linear operators in asymmetric normed spaces by modifying the conditions required for such collections. In their work, Cobzas provided a proof of the uniform boundedness property of collections of continuous linear operators in asymmetric normed spaces using the Zabrejko Lemma introduced by Romaguera [13], while Bouadjila and colleagues utilized the properties of collections of lower semicontinuous linear operators. Due to the complexity of the proofs provided, the author has the idea to establish the theorem using an approach that leverages the properties of completeness and closedness of sets within asymmetric normed spaces.

Based on the elaboration above, in this paper, the author will establish the relationship between quasi-pseudometric and quasi-metric, provide topological properties in quasi-pseudometric spaces not covered by Cobzas [6], and offer supporting examples. Additionally, the author will present a proof of the uniform boundedness theorem for collections of continuous linear operators in asymmetric normed spaces using a simpler approach than the one employed by Cobzas [6, 2].

## II. PRELIMINARIES

Based on the research conducted by previous scholars, the following are the research findings that are essential for this study.

**Definition 2.1** [6, 2, 11, 17] *Let  $X$  be a non-empty set. A Function  $d : X \times X \rightarrow [0, \infty)$  is called quasi-pseudometric on  $X$  if*

1.  $d(x, y) \geq 0$  and  $d(x, x) = 0$ , for all  $x, y \in X$ ;
2.  $d(x, y) \leq d(x, z) + d(z, y)$ , for all  $x, y, z \in X$ .

*Furthermore, a quasi-pseudometric is called quasi-metric, if for every  $x, y \in X$  it satisfies*

3.  $d(x, y) = d(y, x) = 0 \implies x = y$ .

A set  $X$  equipped with a quasi-pseudometric  $d$  is called a quasi-pseudometric space and is usually denoted as  $(X, d)$ . Furthermore, if  $d$  is a quasi-metric, then  $(X, d)$  is referred to as a quasi-metric space. Moreover, because quasi-pseudometrics is not symmetric, Kelly [9] introduced the definition of quasi-pseudometric conjugate as follows.

**Definition 2.2** [9, 2, 11] *Given quasi-pseudometric  $(X, d)$ . The conjugate of a quasi-pseudometric  $d$  is  $\bar{d} : X \times X \rightarrow [0, \infty)$  defined as follow*

$$\bar{d}(x, y) = d(y, x), \text{ for every } x, y \in X.$$

Based on Definition 2.1 and Definition 2.2, Cobzas [2] wrote that the conjugate of a quasi-pseudometric is a quasi-pseudometric.

**Lemma 2.3** [2] *Given quasi-pseudometric  $(X, d)$ . The conjugate of a quasi-pseudometric  $d$ , denoted as  $\bar{d}$ , is a quasi-pseudometric on  $X$ . Furthermore, if  $d$  is a quasi-metric, then  $\bar{d}$  is a quasi-metric.*

Based on the definition of quasi-pseudometric and quasi-metric, Madula [11] wrote about another quasi-pseudometric that can be defined within the space of quasi-pseudometrics.

**Lemma 2.4** [11] *Let  $X$  be a non-empty set equipped with a quasi-pseudometric  $d : X \times X \rightarrow [0, \infty)$ . The function  $d^s(x, y) = \max\{d(x, y), \bar{d}(x, y)\}$ ,  $x, y \in X$ , is a pseudometric on  $X$ . Furthermore,  $d^s$  is a metric on  $X$  if and only if  $d$  is a quasi-metric on  $X$ .*

Given a quasi-pseudometric space  $(X, d)$ , Romaguera [13] defines that for every  $x \in X$  and  $r > 0$ , an open ball with center  $x$  and radius  $r$  is defined as

$$B_d(x, r) = \{y \in X : d(x, y) < r\}, \quad (1)$$

meanwhile, a closed ball with center  $x$  and radius  $r$  is defined as

$$B_d[x, r] = \{y \in X : d(x, y) \leq r\}. \quad (2)$$

Furthermore, based on the definition of open balls, Cobzas [6] provides the definition of open sets in quasi-pseudometric spaces as given in Definition 2.5.

**Definition 2.5** [6] *Given a quasi-pseudometric space  $(X, d)$  and subset  $U \subseteq X$ . The set  $U \subseteq X$  is called an open set in the quasi-pseudometric space  $(X, d)$ , if for every  $a \in U$  there exist  $r > 0$  such that*

$$B_d(a, r) \subseteq U.$$

A set  $U$  that satisfies these conditions is called an open set with respect to  $d$ . Conversely, a subset  $U \subseteq X$  is called a closed set with respect to  $d$  in the quasi-pseudometric space  $(X, d)$ , if  $U^c$  is an open set with respect to  $d$  in the quasi-pseudometric space  $(X, d)$ .

In metric spaces, it is true that closed balls in the metric space are closed sets. This does not hold in quasi-pseudometric spaces. The following lemma states that closed balls in quasi-pseudometric spaces are closed sets with respect to the conjugate of the quasi-pseudometric.

**Lemma 2.6** [6] *Given quasi-pseudometric space  $(X, d)$ , for any  $x \in X$  and  $r > 0$ ,  $B_d[x, r] = \{y \in X : d(x, y) \leq r\}$  is a closed set with respect to  $\bar{d}$ , where  $\bar{d}$  is the conjugate of the quasi-pseudometric  $d$ .*

By generalizing the definition of convergent sequences in metric spaces, the following is the definition of convergence for sequences in quasi-pseudometric spaces.

**Definition 2.7** [2] *Given a quasi-pseudometric space  $(X, d)$ . A sequence  $(x_n) \subseteq X$  is said to converge- $d$  to  $x \in X$ , if the sequence  $(d(x, x_n))$  in  $\mathbb{R}$  converges to 0.*

The sequence  $(x_n) \subseteq X$  that converges- $d$  to  $x \in X$  is commonly denoted as  $x_n \xrightarrow{d} x$ . Based on Definition 2.7, it is obtained that every sequence  $(x_n)$  in a quasi-pseudometric space  $X$  is said to converge- $d$  to  $x \in X$  if and only if the sequence  $(d(x, x_n))$  in  $\mathbb{R}$  converges to 0. In other words,

$$x_n \xrightarrow{d} x \iff d(x, x_n) \rightarrow 0.$$

Furthermore, based on Definition 2.7 and Definition 2.2, a similar definition of convergence with respect to  $\bar{d}$  can be derived as follows:

$$x_n \xrightarrow{\bar{d}} x \iff \bar{d}(x, x_n) \rightarrow 0 \iff d(x_n, x) \rightarrow 0.$$

To provide an understanding of the convergence of sequences in quasi-pseudometric spaces, the following are examples of convergent sequences in quasi-pseudometric spaces.

**Example 2.8** 1. Given a quasi-pseudometric space  $(X, d)$  where  $X = \mathbb{R}$  and  $d : X \times X \rightarrow [0, \infty)$ ,  $d(x, y) = \max\{0, x - y\}$ , for every  $x, y \in X$ . A sequence  $(x_n)$  in  $X$  where  $x_n = \frac{1}{n}$ , for every  $n \in \mathbb{N}$ . The sequence  $(x_n)$  converges- $d$  to  $0 \in X$  and converges- $\bar{d}$  to  $0 \in X$ .

2. Given a quasi-pseudometric space  $(X, p)$  where  $X = [0, 1]$  and  $p : X \times X \rightarrow [0, \infty)$ ,

$$p(x, y) = \begin{cases} 0, & \text{if } x \leq y \\ 1, & \text{if } x > y. \end{cases}$$

Given the sequence  $(x_n)$  in  $X$ , where

$$x_n = \frac{1}{n+1}, n \in \mathbb{N}.$$

The sequence  $(x_n)$  converges- $p$  to 0 and converges- $\bar{p}$  to 1.

Based on Example 2.8, it can be concluded that sequences in quasi-pseudometric spaces can converge- $d$  and converge- $\bar{d}$  to same or different point. According to the research findings by Cobzas [6], it is obtained that a sequence in a quasi-pseudometric space  $(X, d)$  can converge- $d$  and converge- $\bar{d}$  to the same point, if the sequence converges- $d^s$ . Sequences in metric spaces have unique convergence values, whereas sequences in quasi-pseudometric spaces can have non-unique convergence values. The proof of this statement can be found in the work of Cobzas [6].

Because quasi-pseudometrics are asymmetric, the definition of Cauchy sequences is differentiated into several definitions.

**Definition 2.9** [6] Given a quasi-pseudometric space  $(X, d)$ . A sequence  $(x_n)$  in  $(X, d)$  is called

a. *Left K-Cauchy*, if for every  $\epsilon > 0$ , there exist  $n_0 \in \mathbb{N}$  such that for every  $n, m \in \mathbb{N}$ ,  $n_0 \leq n < m$ ,

$$d(x_n, x_m) < \epsilon.$$

b. *Right K-Cauchy*, if for every  $\epsilon > 0$ , there exist  $n_0 \in \mathbb{N}$  such that for every  $n, m \in \mathbb{N}$ ,  $n_0 \leq n < m$ ,

$$d(x_m, x_n) < \epsilon.$$

Furthermore, Cobzas [2, 7] provides definitions of completeness within the pseudometric space in two definitions.

**Definition 2.10** [2, 7] *Given a quasi-pseudometric space  $(X, d)$ . The quasi-pseudometric space  $(X, d)$  is called*

1. *Right  $K$ -complete with respect to  $d$ , if for every right  $K$ -Cauchy sequences in  $X$  converges- $d$  in  $X$ .*
2. *Left  $K$ -complete with respect to  $d$ , if for every left  $K$ -Cauchy sequences in  $X$  converges- $d$  in  $X$ .*

An asymmetric norm is a generalization of normed spaces by weakening the axioms held by norms. Definition 2.11 provides the definition of an asymmetric norm.

**Definition 2.11** [4, 17] *Given a vector space  $X$  and a function  $p : X \rightarrow [0, \infty)$ . The function  $p$  is called an asymmetric normed if the following conditions are satisfied.*

1. *for every  $x \in X$ ,  $p(x) \geq 0$  and  $(p(x) = p(-x) = 0 \implies x = 0)$ ;*
2.  *$p(\alpha x) = \alpha p(x)$ , for all  $x \in X$  and  $\alpha \geq 0$ ;*
3.  *$p(x + y) \leq p(x) + p(y)$ , for every  $x, y \in X$ .*

The set  $X$  equipped with an asymmetric norm  $p$  is called an asymmetric normed space and is denoted by  $(X, p)$ .

Based on Definition 2.11, it is derived that every norm is an asymmetric norm. The following example provides an example of an asymmetric norm in  $\mathbb{R}$ .

**Example 2.12** *Given a vector space  $\mathbb{R}$  and a function  $u : \mathbb{R} \rightarrow [0, \infty)$ , where  $u(x) = \max\{x, 0\}$ , for every  $x \in \mathbb{R}$ . The function  $p$  is asymmetric norm on  $\mathbb{R}$ .*

Furthermore, based on the definition of quasi-pseudometric and asymmetric norm, Kelly [9] states that within an asymmetric normed space, a quasi-pseudometric can be defined.

**Lemma 2.13** *Given asymmetric normed space  $(X, p)$ . The function  $d_p : X \times X \rightarrow [0, \infty)$  where*

$$d_p(x, y) = p(y - x), x, y \in X,$$

*is a quasi-metric on  $X$ .*

The conjugate function of the asymmetric norm  $p$  o vector space  $X$  is defined as the function  $\bar{p} : X \rightarrow [0, \infty)$ , where

$$\bar{p}(x) = p(-x), \tag{3}$$

for every  $x \in X$ .

### III. TOPOLOGY IN QUASI-PSEUDOMETRIC SPACES

In this section, the properties of open and closed sets in quasi-pseudometric spaces will be discussed. Previously, based on Definition 2.1 outlined in the Introduction section, the following consequences were obtained.

**Corollary 3.1** Every quasi-metric is a quasi-pseudometric.

Based on the quasi-pseudometric examples provided by previous researchers, it cannot be concluded that every quasi-pseudometric is a quasi-metric. Here is an example of a quasi-pseudometric that is not a quasi-metric.

**Example 3.2** Let  $R[0, 1]$  be the set of all Riemann integrable functions on  $[0, 1]$ . Given a function  $d : R[0, 1] \times R[0, 1] \rightarrow [0, \infty)$ , where

$$d(f, g) = \max\{0, \int_0^1 (f(x) - g(x))dx\},$$

for every  $f, g \in R[0, 1]$ . The function  $d$  is a quasi-pseudometric on  $R[0, 1]$ , but not a quasi-metric.

*Proof.* Based on the properties of Riemann integrable functions, it is obtained that for every  $f, g \in R[0, 1]$ ,  $f - g$  is Riemann integrable. As a result, for every  $f, g \in R[0, 1]$ ,  $d(f, g)$  is well defined.

Next, it will be shown that  $d$  is a quasi-pseudometric on  $R[0, 1]$ . Let  $f, g, h \in R[0, 1]$  be arbitrary. Based on the definition of  $d$ , it is obtained that  $d(f, g) \geq 0$  and  $d(f, f) = 0$ . Based on the properties of integrals, it follows that

$$\begin{aligned} d(f, g) &= \max\{0, \int_0^1 (f(x) - g(x))dx\} \\ &= \max\{0, \int_0^1 (f(x) - h(x) + h(x) - g(x))dx\} \\ &\leq \max\{0, \int_0^1 (f(x) - h(x))dx\} + \max\{0, \int_0^1 (h(x) - g(x))dx\} \\ &= d(f, h) + d(h, g). \end{aligned}$$

Based on these conditions, it is obtained that  $d$  is a quasi-pseudometric on  $R[0, 1]$ .

Furthermore, it will be shown that  $d$  is not a quasi-metric. Consider the functions  $f$  and  $g$  below

$$f(x) = \begin{cases} 2 & \text{if } x = 0; \\ 0 & \text{if } 0 < x \leq 1; \end{cases} \text{ and } g(x) = \begin{cases} 1 & \text{if } x = 0; \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$$

The function  $f$  and  $g$  are Riemann integrable functions on  $[0, 1]$ , this means that  $f, g \in R[0, 1]$ . Note that

$$f(x) - g(x) = \begin{cases} 1 & \text{if } x = 0; \\ 0 & \text{if } 0 < x \leq 1; \end{cases}$$

and

$$g(x) - f(x) = \begin{cases} -1 & \text{if } x = 0; \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$$

Therefore, it is obtained that

$$\int_0^1 (f(x) - g(x))dx = 0 \text{ and } \int_0^1 (g(x) - f(x))dx = 0.$$



According to the definition of  $d$ , it is obtained that  $d(f, g) = d(g, f) = 0$ . On the other hand,  $f \neq g$ . Consequently, it is concluded that  $d$  is not a quasi-metric.  $\square$

Based on the definitions of open and closed sets provided in the Preliminaries, Example 3.3 gives an example of an open set in a quasi-pseudometric space.

**Example 3.3** Consider  $(X, p)$ , a quasi-pseudometric space defined in Example 2.8, a set  $A = (\frac{1}{2}, 1]$  is an open set in quasi-pseudometric space  $X$ .

*Proof.* Take any  $a \in A$ , this means that  $\frac{1}{2} < a \leq 1$ . Let  $r = \frac{1}{4}$ . Note that

$$\begin{aligned} B_p(a, \frac{1}{4}) &= \{x \in X : p(a, x) < \frac{1}{4}\} \\ &= \{x \in X : p(a, x) = 0\} \\ &= \{x \in X : a \leq x\}. \end{aligned}$$

Since  $a > \frac{1}{2}$  and for every  $x \in X$ , it holds that  $0 \leq x \leq 1$ , then for every  $y \in B_p(a, \frac{1}{4})$ , it is obtained that  $\frac{1}{2} < a \leq y \leq 1$ . This means that  $y \in A$ . In other words, it is obtained that  $B_p(a, \frac{1}{4}) \subseteq A$ . Thus, it is proven that  $A$  is an open set in  $X$ .  $\square$

Based on the definitions of open sets and neighborhoods provided in the Preliminaries, a generalization of the definition of the interior and closure of a set in quasi-pseudometric spaces can be made, as given in Definition 3.4.

**Definition 3.4** Given quasi-pseudometric space  $(X, d)$  and not empty set  $U \subseteq X$ .

1. A point  $a \in U$  is called an interior point of the set  $U$  if there exists  $r > 0$ , such that

$$B_d(a, r) \subseteq U.$$

The collection of all interior points of the set  $U$  is called the interior  $U$  and is denoted by  $int_d(U)$ .

2. A point  $a \in U$  is called an closure point of the set  $U$  if for every  $r > 0$ , it holds that

$$B_d(a, r) \cap U \neq \emptyset.$$

The collection of all closure point of the set  $U$  is called closure  $U$  and is denoted by  $cl_d(U)$ .

Furthermore, in Example 3.5, an example of the interior and closure of a set in a quasi-pseudometric space is provided.

**Example 3.5** Given quasi-pseudometric space  $(\mathbb{R}, d)$  as defined in Example 2.8. Let  $A = \{x \in \mathbb{R} : 5 < x < 20\}$ . It is obtained that  $int_d(A) = \emptyset$  and  $cl_d(A) = \{x \in \mathbb{R} : x \leq 20\}$ .

*Proof.* Assumed that  $int_d(A) \neq \emptyset$ , it means there exist  $a \in A$  and  $r > 0$  such that

$$B_d(a, r) \subseteq A.$$

Note that

$$\begin{aligned}
 B_d(a, r) &= \{x \in \mathbb{R} : d(a, x) < r\} \\
 &= \{x \in \mathbb{R} : d(a, x) = 0 \text{ or } 0 < d(a, x) < r\} \\
 &= \{x \in \mathbb{R} : a \leq x \text{ or } 0 < a - x < r\} \\
 &= \{x \in \mathbb{R} : x > a - r\}.
 \end{aligned} \tag{4}$$

Since  $a \in A$  and for any  $x \in \mathbb{R}$  with  $x > 20$ , it holds that  $x > 20 > 20 - r > a - r$ , it follows that  $x \in B_d(a, r)$ . This means that for every  $x \in \mathbb{R}$  with  $x > 20$ , it is contained within  $B_d(a, r)$ . As a result, it is obtained that  $B_d(a, r) \not\subseteq A$ . A contradiction occurs. Therefore, the assumption must be denied. Thus, it is proven that  $\text{int}_d(A) = \emptyset$ .

Next, it will be shown that  $\text{cl}_d(A) = \{x \in \mathbb{R} : x \leq 20\}$ .

Take any  $y \in \mathbb{R}$  such that  $y \leq 20$  and any  $r > 0$ . Based on Equation (4), it is obtained that

$$B_d(y, r) = \{x \in \mathbb{R} : x > y - r\}.$$

Note that for the given  $r > 0$  there are two possibilities: either  $r \geq 20$  or  $0 < r < 20$ . For  $r \geq 20$ , let  $z = 10$ . Consequently,  $z \in A$  and

$$z = 10 > 0 = 20 - 20 \geq y - r.$$

This means that  $z \in B_d(y, r) \cap A$ .

For  $0 < r < 20$ , we choose  $z = 20 - \min\{r, 20 - r\} < 0$ . Note that

$$z = 20 - \min\{r, 20 - r\} > y - r,$$

which mean  $z \in B_d(y, r)$  and it is also noted that

$$10 = \frac{1}{2}(20 - r) + \frac{1}{2}(20 - (20 - r)) < 2\left(\frac{1}{2}(20 - \min\{r, 20 - r\})\right) = z.$$

Therefore, it is obtained that  $10 < z < 20$ . In other words, it is obtained that  $z \in A$ . Consequently, it is obtained that  $z \in B_d(y, r) \cap A$ .

Based on the results from both of these possibilities, it is obtained that for every  $r > 0$ ,  $B_d(y, r) \cap A \neq \emptyset$ . That means  $y \in \text{cl}_d(A)$ .

In other words, it is obtained that  $\{x \in \mathbb{R} : x \leq 20\} \subseteq \text{cl}_d(A)$ .

For the converse, take any  $y \in \text{cl}_d(A)$ , it will be shown that  $y \in x \in \mathbb{R} : x \leq 20$

Assumed that  $y \notin \{x \in \mathbb{R} : x \leq 20\}$ , it means  $y > 20$ .

Defined  $r = \frac{1}{2}(y - 20) > 0$ . Moreover, it is noted that

$$\begin{aligned}
 B_d(y, r) &= \{x \in \mathbb{R} : x > y - r\} \\
 &= \{x \in \mathbb{R} : x > y - \frac{1}{2}(y - 20)\} \\
 &= \{x \in \mathbb{R} : x > \frac{1}{2}y + 10\}.
 \end{aligned}$$



Since  $y > 20$ , then for any  $z \in B_d(y, r)$ , it is obtained that

$$z > \frac{1}{2}y + 10 > \frac{1}{2} \cdot 20 + 10 = 20.$$

This means that  $z \notin A$ . In other words, it is obtained that  $B_d(y, r) \subseteq A^c$ , consequently  $B_d(y, r) \cap A = \emptyset$ . A contradiction arises with the fact that  $y \in cl_d(A)$ . Therefore, the assumption must be rejected.

In other words, it is proven that  $y \in \{x \in \mathbb{R} : x \leq 20\}$ . Thus,  $cl_d(A) \subseteq \{x \in \mathbb{R} : x \leq 20\}$ .

Since  $\{x \in \mathbb{R} : x \leq 20\} \subseteq cl_d(A)$  and  $cl_d(A) \subseteq \{x \in \mathbb{R} : x \leq 20\}$ , it is obtained that  $cl_d(A) = \{x \in \mathbb{R} : x \leq 20\}$ .  $\square$

In metric spaces, a set is closed if it contains all its limit points. This property still holds for sets in quasi-pseudometric spaces, as stated in Theorem 3.6.

**Theorem 3.6** Given quasi-pseudometric space  $(X, d)$  and  $U \subseteq X$ . The set  $U$  is a closed set with respect to  $d$  if and only if  $cl_d(U) = U$ .

*Proof.* ( $\implies$ ). Let  $U$  is a closed set with respect to  $d$ , it means  $U^c$  is an open set with respect to  $d$ . It will be shown that  $cl_d(U) = U$ . Since  $U \subseteq cl_d(U)$ , to prove it, it is sufficient to show that  $cl_d(U) \subseteq U$ . Assumed that  $cl_d(U) \not\subseteq U$ . This means that there exists  $x \in cl_d(U)$  but  $x \notin U$ . Since  $x \in U^c$  and  $U^c$  is an open set with respect to  $d$ , there exist  $r > 0$ , such that

$$B_d(x, r) \subseteq U^c.$$

Consequently, it is obtained that  $B_d(x, r) \cap U = \emptyset$ . This contradicts with the fact that  $x \in cl_d(U)$ . The assumption must be rejected. In other words, it is obtained that  $cl_d(U) \subseteq U$ .

So, it has been proven that  $cl_d(U) = U$ .

( $\impliedby$ ). It is known that  $cl_d(U) = U$ . It will be shown that  $U$  is a closed set with respect to  $d$ . To prove this, it is equivalent to show that  $U^c$  is an open set with respect to  $d$ . Take any  $x \in U^c$ , meaning  $x \notin U$ . Therefore, based on what is known, it follows that there exists  $r > 0$ , such that  $B_d(x, r) \cap U = \emptyset$ . As a result, it is obtained that  $B_d(x, r) \subseteq U^c$ . In other words, it is proven that  $U^c$  is an open set with respect to  $d$ . Therefore, it is established that  $U$  is a closed set with respect to  $d$ .  $\square$

**Example 3.7** Let  $(\mathbb{R}, d)$  as a quasi-pseudometric space defined in Example 2.8. Given not empty set  $W = \{x \in \mathbb{R} : x \leq 6\}$ ,  $W$  is a closed set with respect to  $d$ .

*Proof.* Based on the definition of the quasi-pseudometric  $d$ , it is obtained that

$$cl_d(W) = \{x \in \mathbb{R} : x \leq 6\}.$$

As a result, according to Theorem 3.6, it is concluded that  $W$  is a closed set with respect to  $d$ .  $\square$

The property of the interior of a set being an open set still holds in quasi-pseudometric spaces. This is provided in Theorem 3.8.

**Theorem 3.8** Given a quasi-pseudometric space  $(X, d)$  and a set  $A \subseteq X$ ,  $\text{int}_d(A)$  is an open set with respect to  $d$ .

*Proof.* Take any  $a \in \text{int}_d(A)$ . Therefore, there exists a number  $r > 0$  such that  $B_d(a, r) \subseteq A$ . Furthermore, take any  $y \in B_d(a, r)$ , then it follows that  $d(a, y) < r$ .

Let  $r_1 = r - d(a, y) > 0$  be defined. Take any  $x \in B_d(y, r_1)$ . Consequently, it is obtained that

$$d(a, x) \leq d(a, y) + d(y, x) \leq d(a, y) + r_1 = d(a, y) + r - d(a, y) = r.$$

This means that  $x \in B_d(a, r) \subseteq A$ . Because this applies to any  $x \in B_d(y, r_1)$ , it follows that  $B_d(y, r_1) \subseteq A$ . In other words, it is obtained that  $y \in \text{int}_d(A)$ . Furthermore, since  $y$  is any element of  $B_d(a, r)$ , it follows that  $B_d(a, r) \subseteq \text{int}_d(A)$ . Therefore, it is established that  $\text{int}_d(A)$  is an open set with respect to  $d$ .  $\square$

In Theorem 3.6, the relationship between closed sets and closures has been discussed. Furthermore, in Theorem 3.9, the relationship between open sets and interiors is presented.

**Theorem 3.9** Given a quasi-pseudometric space  $(X, d)$  and  $A \subseteq X$ . The set  $A$  is an open set with respect to  $d$  if and only if  $\text{int}_d(A) = A$ .

*Proof.* ( $\implies$ ). Based on the definition of interior points, it is obtained that  $\text{int}_d(A) \subseteq A$ . Furthermore, take any  $a \in A$ . Since  $A$  is an open set, there exists  $r > 0$  such that  $B_d(a, r) \subseteq A$ . This means that  $a \in \text{int}_d(A)$ . Therefore, it is obtained that  $A \subseteq \text{int}_d(A)$ . Thus, it is proven that  $\text{int}_d(A) = A$ .

( $\impliedby$ ). Conversely, it is known that  $\text{int}_d(A) = A$ . Therefore, based on Theorem 3.8, it is concluded that  $A$  is an open set with respect to  $d$ .  $\square$

**Example 3.10** Let  $(\mathbb{R}, d)$ , a quasi-pseudometric space as defined in Example 2.8.

Let  $C = \{x \in \mathbb{R} : x > 5\}$ ,  $C$  be an open set with respect to  $d$ .

*Proof.* Based on the definition of quasi-pseudometric  $d$ , it is obtained that

$$\text{int}_d(C) = \{x \in \mathbb{R} : x > 5\}.$$

As a result, according to Theorem 3.9, it is obtained that  $C$  is an open set with respect to  $d$ .  $\square$

Furthermore, the following theorem presents the relationship between the convergence of sequences in quasi-pseudometric spaces and closed sets.

**Theorem 3.11** Given quasi-pseudometric space  $(X, d)$  and not empty set  $S \subseteq X$ .

1. For every  $(x_n) \subseteq S$  with the property that  $(x_n)$  converges- $d$  to  $x$ , it follows that  $x \in \text{cl}_d(S)$ .
2. The set  $S$  is a closed set with respect to  $d$  if and only if for every sequences  $(x_n) \subseteq S$  where  $(x_n)$  converges- $d$  to  $x \in X$ , it implies  $x \in S$ .

*Proof.* 1. Take any sequence  $(x_n)$  in  $S$  with the property that  $(x_n)$  converges- $d$  to  $x \in X$ . As a result, for any  $r > 0$ , there exists  $n_0 \in \mathbb{N}$  such that for every  $n \in \mathbb{N}$ ,  $n \geq n_0$ , it holds that

$$d(x, x_n) < r.$$

This means that for every  $n \geq n_0$ ,  $x_n \in B_d(x, r)$ . Therefore, it is obtained that  $B_d(x, r) \cap S \neq \emptyset$ . In other words, it is obtained that  $x \in cl_d(S)$ .

2. ( $\implies$ ). Given that  $S$  is a closed set, it means  $S = cl_d(S)$ . Take any sequence  $(x_n)$  in  $S$  where  $(x_n)$  converges- $d$  to  $x \in X$ . According to Point 1, it is obtained that  $x \in cl_d(S)$ . Since  $S = cl_d(S)$ , it follows that  $x \in S$ .

( $\impliedby$ ). Take any  $x \in cl_d(S)$ . Based on the definition of closure, it is obtained that for every  $n \in \mathbb{N}$ , the following holds

$$B_d(x, \frac{1}{n}) \cap S \neq \emptyset. \quad (5)$$

This means that for every  $n \in \mathbb{N}$ , there exist  $x_n \in S$  such that

$$d(x, x_n) < \frac{1}{n}. \quad (6)$$

Define the sequence  $(x_n)$  inside  $S$  that satisfies Condition (6). Based on the definition of the sequence  $(x_n)$ , it is obtained that  $(x_n)$  is a sequence inside  $S$  that converges- $d$  to  $x$ . According to the given condition, it is established that  $x \in S$ . Therefore, it is deduced that  $cl_d(S) \subseteq S$ . In other words, it is concluded that  $S$  is a closed set. □

**Example 3.12** By using Theorem 3.11, it can be shown that the set  $W$  given in Example 3.7 is a closed set with respect to  $d$ .

*Proof.* Take any sequence  $(x_n)$  inside  $W$  with  $(x_n)$  converges- $d$  to  $x \in \mathbb{R}$ . It will be shown that  $x \in W$ . It is known that  $(x_n)$  is a sequence inside  $W$ . Consequently, for every  $n \in \mathbb{N}$ ,  $x_n \leq 6$ . Take  $\epsilon > 0$ , since  $(x_n)$  converges- $d$  to  $x$ , there exists  $n_0 \in \mathbb{N}$  such that for every  $n \in \mathbb{N}$ ,  $n \geq n_0$ , it holds that

$$d(x, x_n) < \epsilon \iff \max\{0, x - x_n\} < \epsilon.$$

As a result, for every  $n \geq n_0$ , it follows that

$$x - x_n < \epsilon \iff x < x_n + \epsilon.$$

Because this holds for any  $\epsilon > 0$ , it is obtained that  $x \leq x_n \leq 6$ . Thus, it is concluded that  $x \in W$ . Consequently, based on Theorem 3.11, it is obtained that  $W$  is a closed set with respect to  $d$ . □

#### IV. LINEAR OPERATOR IN ASYMMETRIC NORMED SPACES

In this section, the properties of linear operators in asymmetric normed spaces, which have not been extensively explored by previous researchers, will be presented. Additionally, this section will provide an alternative approach to prove the uniform boundedness property

of continuous linear operators in asymmetric normed spaces by utilizing the completeness and closedness properties of subsets within the asymmetric normed space.

Based on Lemma 2.13, it is obtained that every asymmetric normed space is a quasi-pseudometric space. Therefore, the definition of open balls and open sets in quasi-pseudometric spaces can be applied to asymmetric normed spaces. Furthermore, by replacing the norm with the asymmetric norm, the following is the definition of continuous linear operators in asymmetric normed spaces.

**Definition 4.1** *Given the asymmetric normed spaces  $(X, p)$  and  $(Y, q)$ , and also given the linear operator  $A : X \rightarrow Y$ . The linear operator  $A$  is said to be continuous at the point  $x_0 \in X$ , if for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that for every  $x \in X$ , with  $p(x - x_0) < \delta$ , the following holds*

$$q(Ax - Ax_0) < \epsilon.$$

The operator  $A$  is said to be a  $(p, q)$ -continuous operator on  $X$ , if the linear operator  $A$  is continuous at every  $x \in X$ .

In normed spaces, a linear operator is continuous if and only if it is bounded. The properties of continuous linear operators hold for continuous linear operators in asymmetric normed spaces, as stated in [4], [6], and [2]. Furthermore, based on the definition of convergent sequences and continuous linear operators, Theorem 4.2 presents the relationship between the continuity of continuous linear operators in asymmetric normed spaces and the convergence of sequences.

**Theorem 4.2** *Given the asymmetric normed spaces  $(X, p)$  and  $(Y, q)$ . The linear operator  $T : (X, p) \rightarrow (Y, q)$  is a  $(p, q)$ -continuous linear operator if and only if for every sequence  $(x_n)$  in  $(X, p)$  with  $(x_n)$  converges- $p$  to  $x \in X$ , it implies that  $(Tx_n)$  converges- $q$  to  $Tx$ .*

*Proof.* ( $\implies$ ). Take any sequence  $(x_n)$  in  $X$  with the property that  $(x_n)$  converges- $p$  to  $x \in X$ . Take any  $\epsilon > 0$ . Since  $T$  is continuous at  $x$ , there exists  $r > 0$  such that for every  $a \in X$  with  $p(a - x) < r$ , it follows

$$q(Ta - Tx) < \epsilon. \tag{7}$$

Furthermore, since  $(x_n)$  converges- $p$  to  $x$ , there exists  $n_0 \in \mathbb{N}$  such that for every  $n \in \mathbb{N}$ ,  $n \geq n_0$ , it is obtained that

$$p(x_n - x) < r. \tag{8}$$

Therefore, based on Condition (7), it is obtained that

$$q(Tx_n - Tx) < \epsilon.$$

This means that  $(Tx_n)$  converges- $q$  to  $Tx$ .

( $\impliedby$ ). Assume  $T$  is not continuous at  $x_0 \in X$ . Therefore, there exists  $\epsilon_0 > 0$  such that for every  $r > 0$ , there exists  $x_r \in X$  with  $p(x_r - x_0) < r$ , but

$$q(Tx_r - Tx_0) \geq \epsilon_0. \tag{9}$$

As a result, for every  $n \in \mathbb{N}$ , there exists  $x_n \in X$  such that  $p(x_n - x_0) < \frac{1}{n}$ , but

$$q(Tx_n - Tx_0) \geq \epsilon_0. \quad (10)$$

Furthermore, since for every  $n \in \mathbb{N}$ , it holds that  $p(x_n - x_0) < \frac{1}{n}$ , it follows that  $(x_n)$  is a sequence in  $X$  that converges- $p$  to  $x_0$ . Based on the given conditions, it is obtained that  $(Tx_n)$  converges- $q$  to  $Tx$ . This contradicts Condition (10). Therefore, the assumption must be rejected. In other words, it is concluded that  $T$  is a  $(p, q)$ -continuous operator.  $\square$

**Example 4.3** Given an asymmetric normed space  $(\mathbb{R}, u)$ , with  $u(x) = \max\{0, x\}$ , for every  $x \in \mathbb{R}$ . Also given a linear operator  $T : \mathbb{R} \rightarrow \mathbb{R}$ , defined as

$$T(x) = 2x,$$

for every  $x \in \mathbb{R}$ . Using Theorem 4.2, it can be shown that  $T$  is a  $(u, u)$ -continuous linear operator.

*Proof.* Take any sequence  $(x_n)$  in  $(\mathbb{R}, u)$  with  $(x_n)$  converges- $u$  to  $x$ . It will be shown that  $(Tx_n)$  converges- $u$  to  $Tx = 2x$ . Take any  $\epsilon > 0$ . Since  $(x_n)$  converges- $u$  to  $x$ , there exists  $n_0 \in \mathbb{N}$  such that for every  $n \in \mathbb{N}$ ,  $n \geq n_0$ , it holds that

$$u(x_n - x) = \max\{0, x_n - x\} < \frac{\epsilon}{2}.$$

As a result, for every  $n \in \mathbb{N}$  with  $n \geq n_0$ , it follows that

$$u(Tx_n - Tx) = \max\{0, 2x_n - 2x\} = 2 \max\{0, x_n - x\} < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

So, it is obtained that  $(Tx_n)$  converges- $u$  to  $Tx$ . Based on Theorem 4.2, it is concluded that  $T$  is a  $(u, u)$ -continuous linear operator.  $\square$

The collection of continuous linear operators from the asymmetric normed space  $(X, p)$  to the asymmetric normed space  $(Y, q)$  is denoted by  $\mathcal{L}_{p,q}(X, Y)$ . The collection of continuous linear operators in asymmetric normed spaces forms a set structure that is closed under addition and non-negative scalar multiplication. Such a set structure satisfying these properties is called a cone. The definition of a cone can be studied in [8], [14], and [18].

Furthermore, by generalizing the definition of the collection of continuous linear operators on normed spaces pointwise, in [2] and [3] the definition for a collection of continuous linear operators in normed asymmetric spaces is written.

In [2] and [3], the proof of the uniform boundedness theorem for collections of continuous linear operators in asymmetric normed spaces has been presented using two different approaches. In this paper, the proof of the uniform boundedness theorem will be provided by utilizing the properties of closed sets within asymmetric normed spaces.

**Theorem 4.4** Given asymmetric normed spaces  $(X, p)$  and  $(Y, q)$ . Also, given  $A \subseteq \mathcal{L}_{p,q}(X, Y)$ , a collection of continuous linear operators that are pointwise bounded with respect to both  $q$

and  $\bar{q}$ . If  $(X, p)$  is a right  $K$ -complete space with respect to  $\bar{p}$ , then

$$\sup_{T \in \mathcal{A}} \sup_{x \in B_p} \bar{q}(Tx) < \infty,$$

where  $B_p = \{y \in X : p(y) \leq 1\}$ .

*Proof.* If  $X = \{\theta\}$ , the proof is trivial. It is assumed that  $X \neq \{\theta\}$ . Let  $\mathcal{A} \subseteq \mathcal{L}_{p,q}(X, Y)$  be a collection of continuous linear operators that are pointwise bounded with respect to both  $q$  and  $\bar{q}$ . This means that for every  $x \in X$ , there exists  $C_x > 0$  and  $H_x > 0$  such that

$$\sup_{T \in \mathcal{A}} q(Tx) < C_x \text{ and } \sup_{T \in \mathcal{A}} \bar{q}(Tx) < H_x. \quad (11)$$

First, it will be shown that there exist  $x_0 \in X, \rho > 0$ , and  $C' > 0$  such that

$$\bar{q}(Tx) \leq C',$$

for every  $x \in B_p(x_0, \rho)$  and for every  $T \in \mathcal{A}$ .

Assume there are no  $x_0 \in X, \rho > 0$ , and  $C' > 0$  such that

$$\bar{q}(Tx) \leq C',$$

for every  $x \in B_p(x_0, \rho)$  and for every  $T \in \mathcal{A}$ . This means that, for every  $x_0 \in X$  and  $\rho > 0$ , the set

$$\{\bar{q}(Tx) : x \in B_p(x_0, \rho), T \in \mathcal{A}\} \quad (12)$$

is unbounded. Specifically, for  $x_0 = 0$  and  $\rho = 1$ , it follows that

$$\{\bar{q}(Tx) : x \in B_p(0, 1), T \in \mathcal{A}\}$$

is unbounded. This means there exist  $x_1 \in B_p(0, 1)$  and  $T_1 \in \mathcal{A}$  such that

$$\bar{q}(T_1x_1) > 1.$$

Since  $T_1$  is continuous, so  $T_1$  is continuous at  $x_1$ . Therefore, there exists  $h_1 > 0$  such that for every  $x \in X$  with  $p(x - x_1) < h_1$ , it implies

$$\begin{aligned} q(T_1x - T_1x_1) &< \bar{q}(T_1x_1) - 1 \\ \implies \bar{q}(T_1x_1 - T_1x) &< \bar{q}(T_1x_1) - 1 \\ \implies \bar{q}(T_1x_1) - \bar{q}(T_1x) &< \bar{q}(T_1x_1) - 1 \\ \implies \bar{q}(T_1x) &> 1. \end{aligned}$$

Choose  $\rho_1 \in \mathbb{R}$  with  $0 < \rho_1 < \min\{h_1, \frac{1}{2}, 1 - p_1(x)\}$ , then  $\rho_1 \in (0, \frac{1}{2})$  such that  $B_p[x_1, \rho_1] \subseteq B_p(0, 1)$  and

$$\bar{q}(T_1x) > 1,$$

for every  $x \in B_p[x_1, \rho_1]$ .

Based on Condition (??), it is obtained that the set

$$\{\bar{q}(Tx) : x \in B_p(x_1, \rho_1), T \in \mathcal{A}\}$$

is unbounded. Hence, there exist  $x_2 \in B_p(x_1, \rho_1)$  and  $T_2 \in \mathcal{A}$  such that

$$\bar{q}(T_2x_2) > 2.$$

Since  $T_2$  is continuous, so  $T_2$  is continuous at  $x_2$ . Therefore, there exists  $h_2 > 0$  such that for every  $x \in X$  with  $p(x - x_2) < h_2$ , it implies

$$\begin{aligned} q(T_2x - T_2x_2) &< \bar{q}(T_2x_2) - 2 \\ \implies \bar{q}(T_2x_2 - T_2x) &< \bar{q}(T_2x_2) - 2 \\ \implies \bar{q}(T_2x_2) - \bar{q}(T_2x) &< \bar{q}(T_2x_2) - 2 \\ \implies \bar{q}(T_2x) &> 2. \end{aligned}$$

Choose  $\rho_2 \in \mathbb{R}$  with  $0 < \rho_2 < \min\{h_2, \frac{1}{2^2}, \rho_1 - p_1(x_2 - x_1)\}$ , then  $\rho_2 \in (0, \frac{1}{2^2})$  such that  $B_p[x_2, \rho_2] \subseteq B_p(x_1, \rho_1)$  and

$$\bar{q}(T_2x) > 2,$$

for every  $x \in B_p[x_2, \rho_2]$ .

If the process continues inductively, then it will be obtained that

- a. the sequence  $(x_n)$  in  $X$ ,
- b. the sequence  $(T_n)$  in  $\mathcal{A}$ , and
- c. a sequence of real numbers  $(\rho_n)$  satisfying the condition that for every  $n \in \mathbb{N}$ ,  $0 < \rho_n < \frac{1}{2^n}$ , such that  $B_p[x_{n+1}, \rho_{n+1}] \subseteq B_p(x_n, \rho_n)$  and

$$\bar{q}(T_{n+1}x) > n + 1,$$

for every  $x \in B_p(x_{n+1}, \rho_{n+1})$ .

Note that  $(\rho_n)$  is a sequence of real numbers that converges to 0. Therefore, for  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$ , such that for every natural number  $n \geq n_0$ , it holds that

$$\rho_n < \epsilon.$$

Furthermore, since  $B_p[x_{n+1}, \rho_{n+1}] \subseteq B_p(x_n, \rho_n)$ , it follows that for every  $m, n \in \mathbb{N}$ , with  $n_0 \leq n < m$ ,  $x_m \in B(x_n, \rho_n)$ . In other words, it is obtained that

$$\bar{p}(x_n - x_m) = p(x_m - x_n) < \rho_n < \epsilon.$$

This means that  $(x_n) \subseteq X$  is a right  $K$ -Cauchy sequence with respect to  $\bar{p}$ . Since  $X$  is a right  $K$ -complete space with respect to  $\bar{p}$ , then there exists  $x \in X$ , such that the sequence  $(x_n)$  converges- $\bar{p}$  to  $x$ . Note that for every natural numbers  $n$  and  $m$  with  $m \geq n$ , it holds that  $x_m \in B_p(x_n, \rho_n)$ . This means that for any natural number  $n$ , the sequence  $(x_n, x_{n+1}, x_{n+2}, \dots)$ , is a



sequence inside  $B_p(x_n, \rho_n) \subseteq B_p[x_n, \rho_n]$ . Since the sequence converges- $\bar{p}$  to  $x$  and  $B_p[x_n, \rho_n]$  is a closed set with respect to  $\bar{p}$ , it follows that  $x \in B_p[x_n, \rho_n]$ . As a result,

$$\bar{q}(T_n x) > n.$$

This contradicts with the fact that for every  $x \in X$ , there exists  $H_x > 0$  such that

$$\sup_{T \in \mathcal{A}} \bar{q}(Tx) < H_x.$$

Therefore, the assumption must be rejected. In other words, it is proven that there exist  $x_0 \in X$ ,  $\rho > 0$ , and  $C' > 0$  such that

$$\bar{q}(Tx) \leq C', \quad (13)$$

for every  $x \in B_p(x_0, \rho)$  and for every  $T \in \mathcal{A}$ .

Next, take any  $a \in B_p[0, \frac{1}{2}\rho]$  and any  $T \in \mathcal{A}$ . Note that

$$p(x_0 + a - x_0) = p(a) \leq \frac{1}{2}\rho < \rho.$$

This means that  $x_0 + a \in B_p(x_0, \rho)$ . Consequently, based on Condition (13), it follows that  $\bar{q}(T(x_0 + a)) \leq C'$ , thus satisfying

$$\begin{aligned} \bar{q}(Ta) &= \bar{q}(T(x_0 + a - x_0)) \\ &= \bar{q}(T(x_0 + a) - Tx_0) \\ &\leq \bar{q}(T(x_0 + a)) + \bar{q}(-Tx_0) \\ &= \bar{q}(T(x_0 + a)) + q(Tx_0) \\ &\leq C' + C_{x_0}. \end{aligned} \quad (14)$$

Let  $C = 2\rho^{-1}(C' + C_{x_0})$ . For any  $a \in B_p$ , it follows that  $\frac{1}{2}\rho a \in B_p[0, \frac{1}{2}\rho]$ . Consequently, based on Condition (14), it can be concluded that

$$\bar{q}(Ta) \leq 2\rho^{-1}(C' + C_{x_0}) = C,$$

for all  $T \in \mathcal{A}$ . Since this condition holds for any  $a \in B_p$  and for any  $T \in \mathcal{A}$ , it follows that

$$\sup_{T \in \mathcal{A}} \sup_{x \in B_p} \bar{q}(Tx) \leq C < \infty.$$

□

## V. CONCLUSION

Based on Theorem 3.9, Theorem 3.8, and Lemma 2.6, it is obtained that the topological properties in normed spaces still hold within quasi-pseudometric spaces. Furthermore, based on Example 2.8, it is found that a sequence in a quasi-pseudometric space  $(X, d)$  can have the same or different  $d$ -convergence values compared to its  $\bar{d}$ -convergence values. However, if the sequence  $(x_n) \subseteq X$  converges- $d^s$  to  $x \in X$ , then the sequence will also converges- $d$  and converges- $\bar{d}$  to  $x$ . Furthermore, based on Corollary 3.1, it is concluded that every quasi-metric

is a quasi-pseudometric, but the reverse may not necessarily be true. This fact is emphasized by Example 3.2. Furthermore, based on Theorem 3.6, Theorem 3.8, and Theorem 3.9, it is obtained that the properties of open sets and closed sets in metric spaces still hold in quasi-pseudometric spaces. Furthermore, based on Theorem 4.2, it is concluded that the continuity property of linear operators can preserve the convergence properties of convergent sequences in asymmetric normed spaces. Based on Theorem 4.4, it is obtained that by utilizing the closedness property and completeness in asymmetric normed spaces, the uniform boundedness theorem can be proven without relying on the Zabrejko Lemma or lower semicontinuity properties of functions.

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