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THE L(2,1)-LABELING OF MONGOLIAN TENT, LOBSTER, TRIANGULAR SNAKE, AND KAYAK PADDLE GRAPH

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Abstract. Let $G = (V, E)$ be a simple graph. $L(2,1)-$labeling defined as a function $f : V(G) \to \mathbf{N_0}$ such that, x and y are two adjacent vertices in V. If x and y are two adjacent vertices in $V(G)$, then $|f(y) - f(x)| \geq 2$ and if the distance between x and y is 2, then $|f(y) - f(x)| \geq 1$. The $L(2,1)$-labeling number of G, called $\lambda_{2,1}(G)$, is the smallest number m of G. In this paper, we will further discuss the $L(2,1)$-labeling of mongolian tent, lobster, triangular snake, and kayak paddle. We establish a lower and upper bound and then calculate the precise value of $\lambda_{2,1}$ of mongolian tent, lobster, triangular snake, and kayak paddle.
Keywords: L(2,1)-Labeling, mongolian tent, lobster, triangular snake, kayak paddle.

I. INTRODUCTION

Graph theory is widely applied in everyday life. One of them was to solve the problem of assigning FM station frequencies in the early 20th century. This problem occurs because more and more stations are requesting a frequency so it becomes difficult to determine a frequency without having a new station interrupting the broadcasts from other stations nearby. Problematic channel assignment is the technical problem of assigning channels (nonnegative integers) to each FM radio station within a given station group such that there is no inter-station interference and the tuned channel range is minimized. The level of interference between two FM radio stations correlates with the geographic location of the stations. The closer the stations, the stronger the interference, so the difference between the assigned channels should be greater.

Graphs defined here are finite, undirected, simple, and connected. The notion of $L(2,1)$-labeling is described by Griggs and Yeh [1]. Griggs and Yeh define the $L(2,1)$-labeling of G as a function f that maps every $x, y \in V$ to the whole number such that $|f(y) - f(x)| \geq 2$ if x and y are adjacent to each other and $|f(y) - f(x)| \geq 1$ if x and y have the distance 2 [1]. Moreover, they examine $\lambda_{2,1}$ on graphs of paths, cycles, cubes, wheels, trees, stars. In this research, we determine the minimum span value of $\lambda_{2,1}$ of mongolian tent, lobster, triangular snake, and kayak paddle graph.

As for previous studies related to this $L(2,1)$ labeling are Improved upper bound on the $L(2,1)-$labeling of Cartesian sum of graphs [2], $L(2,1)-$Labeling In The Context Of Some Graph Operations [3], $L(2,1)-$labeling of direct product of paths and cycles [4], $L(2,1)-$labeling of interval graphs [5], $L(2,1)$ Labeling of Lollipop and Pendulum Graphs [6], $L(2,1)-$Labeling of the Strong Product of Paths and Cycles [7], The $L(2,1)-$Labeling Problem on Graphs [8],$L(2,1)-$labellings for direct products of a triangle and a cycle [9], On the $L(2,1)-$labelling of block graphs [10], The $L(2,1)-$Labeling and Operations of Graphs [11], The $L(2,1)-$labeling of $K_{1,n}-$free graphs and its applications [12]. In order to do so, we use the following definition and known results.

Definition 1 *Let $G=(V,E)$ be a simple graph. $L(2,1)-$labeling defined as $f:V(G)\longrightarrow \mathbf{N_0}$ such that, whenever x and y are two adjacent vertices in $V(G)$, then $|f(x)-f(y)|\geq 2$, and whenever the distance between x and y is 2, then $|f(x)-f(y)|\geq 1$.*

Proposition 1 *[1] Let P_n be a path with n vertices. Then (i) $\lambda_{2,1}(P_2)=2,$ (ii) $\lambda_{2,1}(P_3)=\lambda_{2,1}(P_4)=3,$ and (iii) $\lambda_{2,1}(P_n)=4,$ for $n\geq 5$.*

Proposition 2 *[1] Let C_n be a cycle with $n\geq 3$ vertices. Then $\lambda_{2,1}(C_n)=4$.*

Proposition 3 *[1] Let $K_{1,n}$ be a star with $n+1$ vertices. Then $\lambda_{2,1}(K_{1,n})=n+1$.*

Lemma 1 *[13] If H is a subgraph of G, then $\lambda_{2,1}(H)\leq\lambda_{2,1}(G)$.*

II. RESULTS AND DISCUSSION

In this section, the L(2,1)-labeling of four types of graphs are discussed, i. e., mongolian tent, lobster, triangular snake, and kayak paddle graph.

2.1. The $L(2,1)$ Labeling of Mongolian Tent

Definition 1 *The mongolian tent $M_{m,n}$ is the graph obtained from the grid graph of path with m vertices and path with odd n vertices by adding an extra vertex above the graph and joining every other vertex of the top row to the additional vertex.*

Theorem 1 *Let $m,n\geq 2$ be positive integers and n is odd. If $M_{m,n}$ be the mongolian tent graph, then*

$$\lambda_{2,1}(M_{m,n})=\begin{cases}6, & n=3,\\ 7, & n\geq 5.\end{cases}$$

Proof. Let $V(M_{m,n})=\{u_0\}\cup\{u_{11},u_{12},u_{13},...,u_{1m}\}\cup\{u_{21},u_{22},u_{23},...,u_{2m}\}\cup ...\cup\{u_{n1},u_{n2},u_{n3},...,u_{nm}\}$ and $E(M_{m,n})=\{u_0u_{ij}\mid i\in[1,n],$ i is odd, $j\in[1,m]\}\cup\{u_{ij}u_{ij+1}|\, i\in[1,n],j\in[1,m]\}\cup\{u_{ij}u_{i+1j}|\, i\in[1,n],j\in[1,m]\}$.

- Case I: for $n=3$
  Suppose that $\lambda_{2,1}(M_{m,n})\leq 5$. Since $M_{m,n}$ for $n=3$ contains $M_{2,3}$ where $M_{2,3}$ contains three cycles that are connected to each other in figure 1. The first and second cycle can be labeled by 0, 1, 2, 3, 4, 5, but on the third cycle there is a vertex with no label, by pigeonhole principle. So that, $\lambda_{2,1}(M_{m,n})\geq\lambda_{2,1}(M_{m,n})=6$. *Contradiction*. Next, from figure 1 shows that $M_{m,n}$ for $n=3$ can be labeled with no more label more than 6. Therefore, $\lambda_{2,1}(M_{m,n})\leq 6$, for $n=3$.

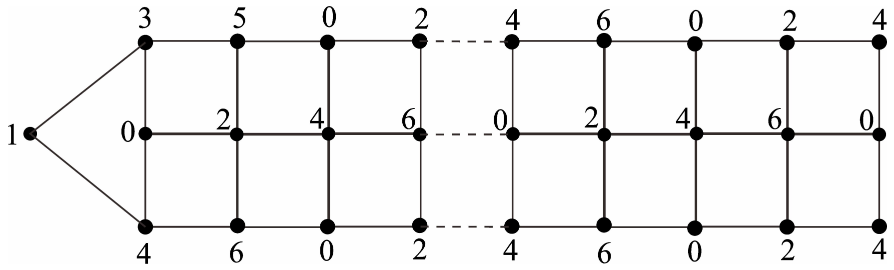

Figure 1. Mongolian Tent graph for $n = 3$

- Case II: for $n \geq 5$
  Since $M_{m,n}$ for $n \geq 5$ contains $M_{2,5}$ where $M_{2,5}$ contains six cycles that are connected to each other in figure 2. Five cycles can be labeled by 0, 1, 2, 3, 4, 5, 6 but on the other cycle there is a vertex with no label, by pigeonhole principle. So that, $\lambda_{2,1}(M_{m,n}) \geq \lambda_{2,1}(M_{m,n}) = 7$. *Contradiction*. Next, from figure 2 shows that $M_{m,n}$ for $n \geq 5$ can be labeled with no more label more than 7. Thus, $\lambda_{2,1}(M_{m,n}) \leq 7$.

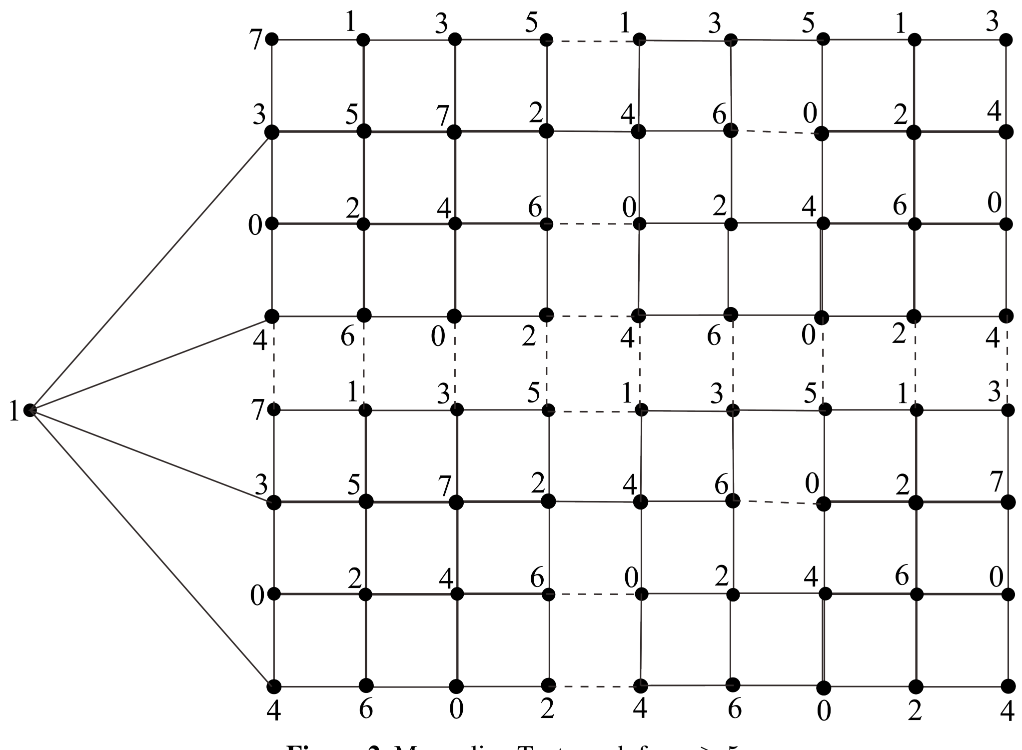

Figure 2. Mongolian Tent graph for $n \geq 5$

□

We can see that from the figure 2 that the fix label for the leftmost vertex is 1, and for the vertices on every rows on the figure 2 have the pattern such as the first row (bottom) has the labels 4, 6, 0, 2, ..., 4, 6, 0, 2.

2.2. The $L(2,1)$ Labeling of Lobster Graph

Definition 2 *The lobster graph $L(n,q,r)$ is a tree graph contains the property that the removal of leaf nodes leaves a caterpillar graph.*

Theorem 2 *Let $n,q,r \geq 2$ be positive integers. If $L(n,q,r)$ be the lobster graph, then*

$$\lambda_{2,1}(L(n,q,r)) = \begin{cases} r+3, & n=2,\ q=2,3,\ r\geq 2 \\ & \textit{or } n=2,\ q\geq 4,\ r\geq q, \\ & \textit{or } n\geq 3,\ q=2,\ r\geq 2, \\ & \textit{or } n=3,\ q\geq 2,\ r\geq q+1, \\ q+2, & n=2,\ q\geq 4,\ r\in[2,q-1], \\ n+3, & n\geq 3,\ q\geq 3,\ r\in[2,q]. \end{cases}$$

Proof. Let $V(L(n,q,r)) = \{v_1,v_2,...,v_n\} \cup \{v_1^1,v_1^2,...,v_1^q\} \cup \{v_n^1,v_n^2,...,v_n^q\} ... \cup \{v_1^{11},v_1^{12},...,v_1^{1r}\} \cup ... \cup \{v_n^{q1},v_n^{q2},...,v_n^{qr}\}$ and $E(L(n,q,r)) = \{v_i v_i^j | i\in[1,n],\ j\in[1,q]\} \cup \{v_i^j v_i^{jk} | i\in[1,n],\ j\in[1,q],\ k\in[1,r]\}$.

- Case I: for $n=2,\ q=2,\ r\geq 2$,
  Suppose that $\lambda_{2,1}(L(n,q,r)) \leq r+2$. Since $L(n,q,r)$ contains P_5 with $r-1$ vertices that adjacent to the leaf of path so that the possible labels are 0, 1, 2, 3, ..., r+2. But there will be at least a pair of vertices that has no label by pigeonhole principle. *Contradiction.* Thus, $\lambda_{2,1}(L(n,q,r)) \geq r+3$. Claim if $n=2,\ q=2,\ r\geq 2$, then $\lambda_{2,1}(L(n,q,r)) = r+3$. Next, defined a labelling function f of $L(n,q,r)$ as follows.

$$f(v_1)=0,\ f(v_2)=4,$$

$$f(v_i^j) = \begin{cases} j+1, & i=1,\ j\in[1,2], \\ j, & i=2,\ j\in[1,2]. \end{cases} \quad f(v_i^{jk}) = \begin{cases} k+3, & i=1,\ j=1,\ k\in[1,r], \\ & \text{or } i=1,\ j=2,\ k\in[2,r], \\ & \text{or } i=2,\ j=1,\ k\in[2,r], \\ & \text{or } i=2,\ j=2,\ k\in[2,r], \\ 1, & i=1,\ j=2,\ k=1, \\ 3, & i=2,\ j=1,\ k=1, \\ 0, & i=2,\ j=2,\ k=1. \end{cases}$$

- Case II: for $n=2,\ q=3,\ r\geq 2$,
  Since $L(n,q,r)$ for $n=2,\ q=3,\ r\geq 2$ contains $L(n,q,r)$ for $n=2,\ q=2,\ r\geq 2$. Thus, $\lambda_{2,1}L(n,q,r)$ for $n=2,\ q=2,\ r\geq 2 \geq \lambda_{2,1}L(n,q,r)$ for $n=2,\ q=3,\ r\geq 2 = r+3$. Claim if $n=2,\ q=3,\ r\geq 2$, then $\lambda_{2,1}(L(n,q,r)) = r+3$. Next, defined a labelling function f of $L(n,q,r)$ as follows.

$$f(v_i) = \begin{cases} 0, & i=1, \\ 5, & i=2, \end{cases} \quad f(v_i^j) = \begin{cases} j+1, & i=1,\ j\in[1,3], \\ j, & i=2,\ j\in[1,3]. \end{cases}$$

$$f(v_i^{jk}) = \begin{cases} k+3, & i=1,\ j=1,\ k\in[1,r], \\ & \text{or } i=1,\ j=2,\ k\in[2,r], \\ & \text{or } i=1,\ j=3,\ k\in[3,r], \\ k, & i=1,\ j=2,\ k=1, \\ & \text{or } i=1,\ j=3,\ k\in[1,2], \\ k+2, & i=2,\ j=1,\ k\in[1,r], \\ & \text{or } i=2,\ j=2,\ k\in[2,r], \\ & \text{or } i=2,\ j=3,\ k\in[3,r], \\ k-1, & i=2,\ j=2,\ k=1, \\ & \text{or } i=2,\ j=3,\ k\in[1,2]. \end{cases}$$

- Case III: for $n=2,\ q\geq 4,\ r\geq q$,
  Since $L(n,q,r)$ for $n=2,\ q\geq 4,\ r\geq q$ contains $L(n,q,r)$ for $n=2,\ q=2,\ r\geq 2$. Thus, $\lambda_{2,1}L(n,q,r)$ for $n=2,\ q=2,\ r\geq 2 \geq \lambda_{2,1}L(n,q,r)$ for $n=2,\ q\geq 4,\ r\geq q=r+3$. Claim if $n=2,\ q\geq 4,\ r\geq q$, then $\lambda_{2,1}(L(n,q,r))=r+3$. Next, defined a labelling function f of $L(n,q,r)$ as follows.

$$f(v_i) = \begin{cases} 0, & i=1, \\ r+3, & i=2. \end{cases} \quad f(v_i^j) = \begin{cases} j+1, & i=1,\ j\in[1,q], \\ j, & i=2,\ j\in[1,q]. \end{cases}$$

$$f(v_i^{jk}) = \begin{cases} k+3, & i=1,\ j=1,\ k\in[1,r], \\ & \text{or } i=1,\ j=2,\ k\in[2,r], \\ & \text{or } i=1,\ j\geq 3,\ k\in[j,r], \\ j, & i=1,\ j=2,\ k=1, \\ & \text{or } i=1,\ j\geq 3,\ k\in[1,j-1], \\ k+2, & i=2,\ j=1,\ k\in[1,r], \\ & \text{or } i=2,\ j=2,\ k\in[2,r], \\ & \text{or } i=2,\ j\geq 3,\ k\in[j,r], \\ j-1, & i=2,\ j=2,\ k=1, \\ & \text{or } i=2,\ j\geq 3,\ k\in[1,j-1]. \end{cases}$$

- Case IV: for $n\geq 3,\ q=2,\ r\geq 2$,
  Since $L(n,q,r)$ for $n\geq 3,\ q=2,\ r\geq 2$ contains $L(n,q,r)$ for $n=2,\ q=2,\ r\geq 2$. Thus, $\lambda_{2,1}L(n,q,r)$ for $n=2,\ q=2,\ r\geq 2\geq \lambda_{2,1}L(n,q,r)$ for $n\geq 3,\ q=2,\ r\geq 2=r+3$. Claim if $n\geq 3,\ q=2,\ r\geq 2$, then $\lambda_{2,1}(L(n,q,r))=r+3$. Next, defined a

labelling function f of $L(n,q,r)$ as follows.

$$f(v_i)=\begin{cases}2, & i\equiv 1 \mod 5,\\ 0, & i\equiv 2 \mod 5,\\ 3, & i\equiv 3 \mod 5,\\ 1, & i\equiv 4 \mod 5,\\ 4, & i\equiv 0 \mod 5.\end{cases} f(v_i^j)=\begin{cases} j+3, & i\equiv 1,2,3,4 \mod 5,\ j\in[1,2],\\ & \text{or } i\equiv 0 \mod 5,\ j=2,\\ 0, & i\equiv 0 \mod 5,\ j=1.\end{cases}$$

$$f(v_i^{jk})=\begin{cases} k-1, & i\equiv 1 \mod 5,\ j\in[1,2],\ k\in[1,2],\\ & \text{or } i\equiv 2 \mod 5,\ j=1,\ k\in[1,3],\\ & \text{or } i\equiv 3 \mod 5,\ j\in[1,2],\ k\in[1,3],\\ & \text{or } i\equiv 4 \mod 5,\ j\in[1,2],\ k=1,\\ & \text{or } i\equiv 0 \mod 5,\ j=2,\ k\in[1,3],\\ k, & i\equiv 1 \mod 5,\ j=1,\ k=3,\\ & \text{or } i\equiv 1 \mod 5,\ j=2,\ k\in[3,4],\\ & \text{or } i\equiv 2 \mod 5,\ j=1,\ k\in[1,3],\\ & \text{or } i\equiv 2 \mod 5,\ j=2,\ k\in[1,4],\\ & \text{or } i\equiv 3 \mod 5,\ j=2,\ k=4,\\ & \text{or } i\equiv 4 \mod 5,\ j=1,\ k\in[2,3],\\ & \text{or } i\equiv 4 \mod 5,\ j=2,\ k\in[2,4],\\ k+3, & i\equiv 1,2,3,4 \mod 5,\ j=1,\ k\geq 4,\\ & \text{or } i\equiv 1,2,3,4 \mod 5,\ j=2,\ k\geq 5,\\ k+1, & i\equiv 0 \mod 5,\ j=1,\ k\in[1,2],\\ k+2, & i\equiv 0 \mod 5,\ j=1,\ k\geq 3,\\ k+4, & i\equiv 0 \mod 5,\ j=2,\ k\geq 4.\end{cases}$$

Therefore, $\lambda_{2,1}(L(n,q,r))\leq r+3$.

- Case V: for $n=3,\ q\geq 2,\ r\geq q+1$,
  Since $L(n,q,r)$ for $n=3,\ q\geq 2,\ r\geq q+1$ contains $L(n,q,r)$ for $n=2,\ q=2,\ r\geq 2$. Thus, $\lambda_{2,1}L(n,q,r)$ for $n=2,\ q=2,\ r\geq 2\geq \lambda_{2,1}L(n,q,r)$ for $n=3,\ q\geq 2,\ r\geq q+1=r+3$. Claim if $n=3,\ q\geq 2,\ r\geq q+1$, then $\lambda_{2,1}(L(n,q,r))=r+3$. Next, defined a labelling function f of $L(n,q,r)$ as follows.

$$f(v_i)=\begin{cases}2, & i=1,\\ 0, & i=2,\\ q+3, & i=3.\end{cases} f(v_i^j)=\begin{cases} q+3, & i=1,\ j\in[1,q],\\ q+2, & i=2,\ j\in[1,q],\\ q, & i=3,\ j\in[1,q].\end{cases}$$

$$f(v_i^{jk}) = \begin{cases} k-1, & i=1,\ j \in [1,q],\ k \in \left[1, \frac{q+3}{2}\right], \\ k+3, & i=1,\ j \in [1,q],\ k \in \left[\frac{q+3}{2}+1, r\right] \end{cases}$$

- Case VI: for $n = 2,\ q \geq 4,\ r \in [2, q-1]$,

$$f(v_i) = \begin{cases} 0, & i=1, \\ r+3, & i=2. \end{cases} \quad f(v_i^j) = \begin{cases} j+1, & i=1,\ j \in [1,q], \\ j, & i=2,\ j \in [1,q]. \end{cases}$$

$$f(v_i^{jk}) = \begin{cases} k+3, & i=1,\ j=1,\ k \in [1,r], \\ & \text{or } i=1,\ j=2,\ k \in [2,r], \\ & \text{or } i=1,\ j \geq 3,\ k \in [j,r], \\ j, & i=1,\ j=2,\ k=1, \\ & \text{or } i=1,\ j \geq 3,\ k \in [1, j-1], \\ k+2, & i=2,\ j=1,\ k \in [1,r], \\ & \text{or } i=2,\ j=2,\ k \in [2,r], \\ & \text{or } i=2,\ j \geq 3,\ k \in [j,r], \\ j-1, & i=2,\ j=2,\ k=1, \\ & \text{or } i=2,\ j \geq 3,\ k \in [1, j-1]. \end{cases}$$

Therefore, $\lambda_{2,1}(L(n,q,r)) \leq q+2$.

- Case VII: for $n \geq 3,\ q \geq 3,\ r \in [2,q]$,
Suppose that $\lambda_{2,1}(L(n,q,r)) \leq n+2$. Since $L(n,q,r)$ contains $K_{1,q+2}$ where $q+2 > n+2$, then there will be at least a vertex of $L(n,q,r)$ has no label by pigeonhole principle. Thus, $\lambda_{2,1}(L(n,q,r)) \geq n+3$. Claim if $n \geq 3,\ q \geq 3,\ r \in [2,q]$, then $\lambda_{2,1}(L(n,q,r)) = n+3$. Next, defined a labelling function f of $L(n,q,r)$ as follows.

$$f(v_i) = \begin{cases} 2, & i \equiv 1 \mod 5, \\ 0, & i \equiv 2 \mod 5, \\ 3, & i \equiv 3 \mod 5, \\ 1, & i \equiv 4 \mod 5, \\ 4, & i \equiv 0 \mod 5. \end{cases} \quad f(v_i^j) = \begin{cases} j+3, & i \equiv 1,2,3,4 \mod 5, j \in [1,2], \\ & \text{or } i \equiv 0 \mod 5,\ j=2, \\ 0, & i \equiv 0 \mod 5,\ j=1. \end{cases}$$

$$f(v_i^{jk}) = \begin{cases} k-1, & i \equiv 1 \mod 5,\ j \in [1,q],\ k \in [1,q+2], \\ & \text{or } i \equiv 4 \mod 5,\ j \in [1,q],\ k = 1, \\ & \text{or } i \equiv 3 \mod 5,\ j \in [1,q],\ k \in [1,q+2], \\ & \text{or } i \equiv 0 \mod 5,\ j \in [2,q],\ k \in [1,q+2], \\ k+3, & i \equiv 1,2,3,4 \mod 5,\ j \in [1,q],\ k \in [q+3,r], \\ & \text{or } i \equiv 0 \mod 5,\ j = 1,\ k \geq 4, \\ & \text{or } i \equiv 0 \mod 5,\ j \in [2,q],\ k \in [q+3,r], \\ k, & i \equiv 2 \mod 5,\ j \in [1,q],\ k \in [1,q+2], \\ & \text{or } i \equiv 4 \mod 5,\ j \in [1,q],\ k \in [2,q+2], \\ k+1, & i \equiv 0 \mod 5,\ j = 1,\ k \in [1,2], \\ k+2, & i \equiv 0 \mod 5,\ j = 1,\ k = 3, \end{cases}$$

From the proof of Theorem 2, we can imply that the more vertex that adjacent to more vertices so the more upper bound limit increases of $\lambda_{2,1}$ of lobster graph. □

2.3. The $L(2,1)$ Labeling of Triangular Snake Graph

The triangular snake graph B_m is the graph on m vertices with m odd defined by starting with the P_{m-1} and adding edges $(2s-1, 2s+1)$ for $s \in [1, m-1]$.

Theorem 3 *Let $m \geq 5$ be positive integers and m is odd. If TS_m be the triangular snake graph, then*

$$\lambda_{2,1}(TS_m) = \begin{cases} 5, & m = 5, \\ 6, & m \geq 7. \end{cases}$$

Proof. Let $V(TS_m) = \{v_1, v_2, ..., v_m\} \cup \{v^{12}, v^{23}, ..., v^{m-1m}\}$ and $E(TS_m) = \{v_1 v_{i+1} | i \in [1,m]\} \cup \{v_i v^{ii+1} | i \in [1,m]\}$.

- Case I: for $m = 5$,
  Since TS_m for $m = 5$ contains $K_{1,4}$, then by Lemma 1.1, $\lambda_{2,1}(TS_m) \leq \lambda_{2,1}(K_{1,4}) = 5$. Claim if $m = 5$, then $\lambda_{2,1}(TS_m) = 5$. Next, defined a labeling function f of TS_m as follows.

$$f(v_1) = 4,\ f(v_2) = 0,\ f(v_3) = 3.$$
$$f(v^{12}) = 2,\ f(v^{23}) = 5.$$

- Case II: for $m \geq 7$,
  Suppose that $\lambda_{2,1}(TS_m) \leq 5$. Since TS_m for $m \geq 7$ contains three cycles, then the first and second cycle can be labeled by 0, 2, 3, 4, 5. So that, there will be at least a vertex on the third cycle has no label by the definition of $L(2,1)$. Thus, $\lambda_{2,1}(TS_m) \geq 6$. Claim if

$m \geq 7$, then $\lambda_{2,1}(TS_m) = 6$. Next, defined a labeling function f of TS_m as follows.

$$f(v_i) = \begin{cases} 4, & i = 1, \\ 0, & i \equiv 2 \mod 3, \\ 3, & i \equiv 0 \mod 3, \\ 6, & i \equiv 1 \mod 3, i \geq 4. \end{cases} \qquad f(v^{jk}) = \begin{cases} 2, & i \equiv 1 \mod 3,\ j \equiv 2 \mod 3, \\ 5, & i \equiv 2 \mod 3,\ j \equiv 0 \mod 3, \\ 1, & i \equiv 0 \mod 3,\ j \equiv 1 \mod 3. \end{cases}$$

Therefore, $\lambda_{2,1}(TS_m) \leq 6$, for $m \geq 7$.

From the following proof, it is clear that the triangular snake graph for $m = 5$ can be labeled by the labels of star graph with five vertices. Otherwise, the label of one of vertices of triangular snake graph need to be increase by 1, such that the labels of triangular snake graph fit the definition of $L(2,1)$. □

2.4. The $L(2,1)$ Labeling of Kayak Paddle Graph

Definition 3 *The kayak paddle $KP(p,q,l)$ is the graph obtained by joining two cycles with p vertices and q vertices by a path of length l.*

Theorem 4 *Let $p, q \geq 3$ and $l \geq 2$ be positive integers. If $KP(p,q,l)$ be the kayak paddle graph, then*

$$\lambda_{2,1}(KP(p,q,l)) = \begin{cases} 4, & \textit{for } p,\ q \equiv 0 \mod 3,\ l \equiv 4 \mod 5, \\ & \textit{or } p \equiv 0 \mod 3,\ q,\ l \equiv 1 \mod 3, \\ & \textit{or } p,\ l \equiv 1 \mod 3,\ q \equiv 0 \mod 3, \\ & \textit{or } p,\ q \equiv 1 \mod 3,\ l \equiv 3 \mod 5, \\ 5, & \textit{for } p,\ q \equiv 0 \mod 3,\ l \equiv 1,2,3,0 \mod 5, \\ & \textit{or } p \equiv 0 \mod 3,\ q \equiv 1 \mod 3,\ l \equiv 2,3,4,0 \mod 5, \\ & \textit{or } p \equiv 0,1 \mod 3,\ q \equiv 2 \mod 3,\ l \equiv 1,2,3,4,0 \mod 5, \\ & \textit{or } p \equiv 1 \mod 3,\ q \equiv 0 \mod 3,\ l \equiv 2,3,4,0 \mod 5, \\ & \textit{or } p,\ q \equiv 1 \mod 3,\ l \equiv 1,2,4,0, \mod 5, \\ & \textit{or } p \equiv 2 \mod 3,\ q \equiv 1,2,0 \mod 3,\ l \equiv 1,2,3,4,0 \mod 5. \end{cases}$$

Proof. Let $V(KP(p,q,l)) = \{u_1, u_2, ..., u_p\} \cup \{v_1, v_2, ..., v_l\} \cup \{w_1, w_2, ..., w_q\}$, where $u_1 = v_1,\ w_1 = v_q$ and $E(KP_{p,q,l}) = \{u_z u_{z+1} \mid z \in [1, p-1]\} \cup \{u_p u_1\} \cup \{v_z v_{z+1} \mid z \in [1, l-1]\} \cup \{w_z w_{z+1} | z \in [1, q-1]\} \cup \{w_q w_1\}$.

- Case I: for $\lambda_{2,1}(KP(p,q,l)) = 4$
  Suppose that $\lambda_{2,1}(KP(p,q,l)) = 3$. Since $KP(p,q,l)$ contains C_p then by Lemma 1.1, we wil get $\lambda_{2,1}(KP(p,q,l)) \geq \lambda_{2,1}(C_p) = 4$. *Contradiction*. Thus, $\lambda_{2,1}(KP(p,q,l)) \geq 4$. To prove that $\lambda_{2,1}(KP(p,q,l)) = 4$, we define the labeling function f of $KP(p,q,l)$ and we divided into four subcases as follows.
  - For $p,\ q \equiv 0 \mod 3,\ l \equiv 4 \mod 5$
    For the first cycle, we give the labels 0, 2, 4, ..., 0, 2, 4. Then we obtain the function

for the first cycle as $f(u_z) = 0$ if $z \equiv 1 \mod 3$, $f(u_z) = 2$ if $z \equiv 2 \mod 3$, and $f(u_z) = 4$ if $z \equiv 0 \mod 3$. Next, for the path, we give the labels 0, 3, 1, 4, 2, ..., 0, 3, 1, 4, 2 that we obtain the function for the path as $f(v_z) = 0$, if $z \equiv 1 \mod 5$, $f(v_z) = 3$, if $z \equiv 2 \mod 5$, $f(v_z) = 1$, if $z \equiv 3 \mod 5$, $f(v_z) = 4$, if $z \equiv 4 \mod 5$, and $f(v_z) = 2$, if $z \equiv 0 \mod 5$. Finally, for the second cycle, we give the labels 4, 2, 0, ..., 4, 2, 0 . Then we obtain the function for the second cycle as $f(w_z) = 4$ if $z \equiv 1 \mod 3$, $f(w_z) = 2$ if $z \equiv 2 \mod 3$, and $f(w_z) = 0$ if $z \equiv 0 \mod 3$.

- For $p \equiv 0 \mod 3,\ q,\ l \equiv 1 \mod 3$
  For the first cycle, we give the labels 0, 2, 4, ..., 0, 2, 4. Then we obtain the function for the first cycle as $f(u_z) = 0$ if $z \equiv 1 \mod 3$, $f(u_z) = 2$ if $z \equiv 2 \mod 3$, and $f(u_z) = 4$ if $z \equiv 0 \mod 3$. Next, for the path, we give the labels 0, 3, 1, 4, 2, ..., 0, 3, 1, 4, 2 that we obtain the function for the path as $f(v_z) = 0$, if $z \equiv 1 \mod 5$, $f(v_z) = 3$, if $z \equiv 2 \mod 5$, $f(v_z) = 1$, if $z \equiv 3 \mod 5$, $f(v_z) = 4$, if $z \equiv 4 \mod 5$, and $f(v_z) = 2$, if $z \equiv 0 \mod 5$. Finally, for the second cycle, we give the labels 0, 4, 1, 3, 0, 4, 2, ..., 0, 4, 2. Then we obtain the function for the second cycle as $f(w_z) = 0$ if $z = 1$ or $z \equiv 2 \mod 3$, $f(w_z) = 4$ if $z = 2$ or $z \equiv 1 \mod 3$, $f(w_z) = 1$ if $z = 3$, $f(w_z) = 3$ if $z = 4$, and $f(w_z) = 2$ if $z \equiv 0 \mod 3$.

- For $p,\ l \equiv 1 \mod 3,\ q \equiv 0 \mod 3$
  For the first cycle, we give the labels 0, 4, 1, 3, 0, 4, 2, ..., 0, 4, 2. Then we obtain the function for the first cycle as $f(u_z) = 0$ if $z = 1$ or $z \equiv 2 \mod 3$, $f(u_z) = 4$ if $z = 2$ or $z \equiv 1 \mod 3$, $f(u_z) = 1$ if $z = 3$, $f(u_z) = 3$ if $z = 4$, and $f(u_z) = 2$ if $z \equiv 0 \mod 3$. Next, for the path, we give the labels 0, 3, 1, 4, 2, ..., 0, 3, 1, 4, 2 that we obtain the function for the path as $f(v_z) = 0$, if $z \equiv 1 \mod 5$, $f(v_z) = 3$, if $z \equiv 2 \mod 5$, $f(v_z) = 1$, if $z \equiv 3 \mod 5$, $f(v_z) = 4$, if $z \equiv 4 \mod 5$, and $f(v_z) = 2$, if $z \equiv 0 \mod 5$. Finally, for the second cycle, we give the labels 0, 2, 4, ..., 0, 2, 4. Then we obtain the function for the second cycle as $f(w_z) = 0$ if $z \equiv 1 \mod 3$, $f(w_z) = 2$ if $z \equiv 2 \mod 3$, and $f(w_z) = 4$ if $z \equiv 0 \mod 3$.

- For $p,\ q \equiv 1 \mod 3,\ l \equiv 3 \mod 5$
  For the first cycle, we give the labels 0, 3, 1, 4, 0, 2, ..., 4, 0, 2. Then we obtain the function for the first cycle as $f(u_z) = 0$ if $z = 1$ or $z \equiv 2 \mod 3$, $f(u_z) = 3$ if $z = 2$, $f(u_z) = 1$ if $z = 3$, $f(u_z) = 4$ if $z = 4$ or $z \equiv 1 \mod 3$, and $f(u_z) = 2$ if $z \equiv 0 \mod 3$. Next, for the path, we give the labels 0, 2, 4, 1, 3, ..., 0, 2, 4, 1, 3 that we obtain the function for the path as $f(v_z) = 0$, if $z \equiv 1 \mod 5$, $f(v_z) = 2$, if $z \equiv 2 \mod 5$, $f(v_z) = 4$, if $z \equiv 3 \mod 5$, $f(v_z) = 1$, if $z \equiv 4 \mod 5$, and $f(v_z) = 3$, if $z \equiv 0 \mod 5$. Finally, for the second cycle, we give the labels 4, 1, 3, 0, 2, 4, ..., 0, 2, 4. Then we obtain the function for the second cycle as $f(w_z) = 4$ if $z = 1$ or $z \equiv 0 \mod 3$, $f(w_z) = 1$ if $z = 2$, $f(w_z) = 3$ if $z = 3$, $f(w_z) = 0$ if $z = 4$ or $z \equiv 1 \mod 3$, and $f(w_z) = 2$ if $z \equiv 2 \mod 3$.

f in the definition above is $L(2,1)$-labeling of $KP(p,q,l)$ for $p,\ q \equiv 0 \mod 3,\ l \equiv 4 \mod 3,\ p \equiv 0 \mod 3,\ q,\ l \equiv 1 \mod 3,\ p,\ l \equiv 1 \mod 3,\ q \equiv 0 \mod 3$, or $p,\ q \equiv 1 \mod 3,\ l \equiv 3 \mod 3$. Therefore, $\lambda_{2,1}(KP(p,q,l)) \leq 4$.

- Case II: for $\lambda_{2,1}(KP(p,q,l)) = 5$
  Suppose that $\lambda_{2,1}(KP(p,q,l)) = 4$, then the possible labels are 0, 1, 2, 3, and 4. Since $KP(p,q,l)$ contains a path of l length so we can label the path as 0, 3, 1, 4, 2, 0, 3, 1, 4, 2, ... For $p, q \equiv 0 \mod 3$, so the label of endvertex of path is 0, 3, 1, or 2. Since each endvertices of path are adjacent with two cycle, then there will be at least a pair of vertices on $KP(p,q,l)$ has the same label by the pigeonhole principle. So it does not satisfy the definition of $L(2,1)$-labeling. The similar proof holds for $p \equiv 0 \mod 3,\ q \equiv 1 \mod 3,\ l \equiv 2,3,4,0 \mod 5,\ p \equiv 0 \mod 3,\ q \equiv 2 \mod 3,\ l \equiv 1,2,3,4,0 \mod 5,\ p \equiv 1 \mod 3,\ q \equiv 0 \mod 3,\ l \equiv 2,3,4,0 \mod 5,\ p,\ q \equiv 1 \mod 3,\ l \equiv 1,2,4,0 \mod 5,\ p \equiv 1 \mod 3,\ q \equiv 2 \mod 3,\ l \equiv 1,2,3,4,0 \mod 5,\ p \equiv 2 \mod 3,\ q \equiv 0 \mod 3,\ l \equiv 1,2,3,4,0 \mod 5,\ p \equiv 2 \mod 3,\ q \equiv 1 \mod 3,\ l \equiv 1,2,3,4,0 \mod 5,\ p,\ q \equiv 2 \mod 3,\ l \equiv 1,2,3,4,0 \mod 5$, *contradiction.* Thus, $\lambda_{2,1}(KP(p,q,l)) \geq 5$. To show that $\lambda_{2,1}(KP(p,q,l)) = 5$, we define the labeling function f of $KP(p,q,l)$ as follows.

  - For $p,\ q \equiv 0 \mod 3,\ l \equiv 1,2,3,0 \mod 5$
    For the first cycle, we give the labels 0, 2, 4, ..., 0, 2, 4. Then we obtain the function for the first cycle as $f(u_z) = 0$ if $z \equiv 1 \mod 3$, $f(u_z) = 2$ if $z \equiv 2 \mod 3$, and $f(u_z) = 4$ if $z \equiv 0 \mod 3$. Next, for the path, we define the labeling function $f(v_z) = 0$, if $z \equiv 1 \mod 5$, or $z = l$ for $l \equiv 1 \mod 5$, $f(v_z) = 3$ if $z \equiv 2 \mod 5$ or $z = l - 1$ for $l \equiv 1 \mod 5$, $f(v_z) = 1$ if $z \equiv 3 \mod 5$, $f(v_z) = 4$ if $z \equiv 4 \mod 5$, $f(v_z) = 2$ if $z \equiv 0 \mod 5$, and $f(v_z) = 5$ if $z = l - 2$ for $l \equiv 1 \mod 5$ or $z = l - 1$ for $l \equiv 0 \mod 5$. Finally, for the second cycle, we have four cases based on the length l of the path. For $l \equiv 1 \mod 5$, we give the labels 0, 2, 4, ..., 0, 2, 4. For $l \equiv 2 \mod 5$, we give the labels 3, 5, 1, ..., 3, 5, 1. For $l \equiv 3 \mod 5$, we give the labels 5, 2, 0, ..., 5, 2, 0. For $l \equiv 0 \mod 5$, we give the labels 2, 4, 0, ..., 2, 4, 0.

  - For $p \equiv 0 \mod 3,\ q \equiv 1 \mod 3,\ l \equiv 2,3,4,0 \mod 5$
    For the first cycle, we give the labels 0, 2, 4, ..., 0, 2, 4. Then we obtain the function for the first cycle as $f(u_z) = 0$ if $z \equiv 1 \mod 3$, $f(u_z) = 2$ if $z \equiv 2 \mod 3$, and $f(u_z) = 4$ if $z \equiv 0 \mod 3$. Next, for the path, we define the labeling function $f(v_z) = 0$ if $z \equiv 1 \mod 5$ or $z = l$ for $l \equiv 0 \mod 5$, $f(v_z) = 3$ if $z \equiv 2 \mod 5$ or $z = l - 1$ for $l \equiv 4 \mod 5$, $f(v_z) = 1$ if $z \equiv 3 \mod 5$, $f(v_z) = 4$ if $z \equiv 4 \mod 5$, $f(v_z) = 2$ if $z \equiv 0 \mod 5$, and $f(v_z) = 5$ if $z = l - 1$ for $l \equiv 2,0 \mod 5$ or $z = l - 2$ for $l \equiv 4 \mod 5$. Finally, for the second cycle, we have four cases based on the length l of the path. For $l \equiv 2 \mod 5$, we give the labels 3, 1, 4, 0, 2, 4, ..., 0, 2, 4. For $l \equiv 3 \mod 5$, we give the labels 1, 4, 0, 5, 2, 0, 4, ..., 2, 0, 4. For $l \equiv 4 \mod 5$, we give the labels 4, 1, 3, 0, 2, 4, 0, ..., 2, 4, 0. For $l \equiv 0 \mod 5$, we give the labels 0, 4, 1, 3, 0, 2, 4, ..., 0, 2, 4.

  - For $p \equiv 0 \mod 3,\ q \equiv 2 \mod 3,\ l \equiv 1,2,3,4,0 \mod 5$
    For the first cycle, we give the labels 0, 2, 4, ..., 0, 2, 4. Then we obtain the function for the first cycle as $f(u_z) = 0$ if $z \equiv 1 \mod 3$, $f(u_z) = 2$ if $z \equiv 2 \mod 3$, and $f(u_z) = 4$ if $z \equiv 0 \mod 3$. Next, for the path, we define the labeling function

$f(v_z) = 0$ if $z \equiv 1 \mod 5$ or $z = l-1$ for $l \equiv 4 \mod 5$, $f(v_z) = 3$ if $z \equiv 2 \mod 5$ or $z = l-1$ for $l \equiv 3 \mod 5$ or $z = l-2$ for $l \equiv 1 mod 5$, $f(v_z) = 1$ if $z \equiv 3 \mod 5$, $f(v_z) = 4$ if $z \equiv 4 \mod 5$, $f(v_z) = 2$ if $z \equiv 0 \mod 5$, and $f(v_z) = 5$ if $z = l-3$ for $l \equiv 4 \mod 5$ or $z = l-1$ for $l \equiv 0, 1, 2 \mod 5$. Finally, for the second cycle, we have four cases based on the length l of the path. For $l \equiv 1 \mod 5$, we give the labels 0, 3, 1, 4, 2, 0, 4, 2, ..., 0, 4, 2. For $l \equiv 2 \mod 5$, we give the labels 3, 0, 2, 4, 0, 2, 4, ..., 0, 2, 4, 1. For $l \equiv 3 \mod 5$, we give the labels 1, 4, 2, 0, 4, 2, 0, ..., 4, 2, 0, 3. For $l \equiv 4 \mod 5$, we give the labels 4, 1, 3, 0, 2, 4, 0, 2, ..., 4, 0, 2. For $l \equiv 0 \mod 5$, we give the labels 2, 0, 3, 1, 4, 2, 0, 4, ..., 2, 0, 4.

- For $p \equiv 1 \mod 3,\ q \equiv 0 \mod 3,\ l \equiv 2, 3, 4, 0 \mod 5$ The labeling function f for $p \equiv 1 \mod 3,\ q \equiv 0 \mod 3,\ l \equiv 2, 3, 4, 0 \mod 5$ is isomorphic with the labeling function f for $p \equiv 0 \mod 3,\ q \equiv 1 \mod 3,\ l \equiv 2, 3, 4, 0 \mod 5$

- For $p,\ q \equiv 1 \mod 3,\ l \equiv 1, 2, 4, 0 \mod 5$
  For the first cycle, we give the labels 0, 3, 1, 4, 0, 2, 4, ..., 0, 2, 4. Next, for the path, we define the labeling function $f(v_z) = 0$ if $z \equiv 1 \mod 5$ or $z = l$ for $l \equiv 2 \mod 5$, $f(v_z) = 2$ if $z \equiv 2 \mod 5$, $f(v_z) = 4$ if $z \equiv 3 \mod 5$, $f(v_z) = 1$ if $z \equiv 4 \mod 5$ or $z = l-1$ for $l \equiv 2 \mod 5$, $f(v_z) = 5$ if $z = l-1$ for $l \equiv 4 \mod 5$ or $f(v_z) = l-2$ for $l \equiv 2 \mod 5$. Finally, for the second cycle, we have five cases based on the length l of the path. For $l \equiv 1 \mod 5$, we give the label 0, 4, 1, 3, 0, 2, 4, ..., 0, 2, 4. For $l \equiv 2 \mod 5$, we give the labels 3, 0, 4, 2, ..., 0, 4, 2, 1. For $l \equiv 3 \mod 5$, we give the labels 4, 1, 3, 0, 2, 4, ..., 0, 2, 4. For $l \equiv 4 \mod 5$, we give the labels 1, 4, 0, 2, 4, 0, 2, ..., 4, 0, 2. For $l \equiv 0 \mod 5$, we give the labels 3, 0, 2, 4, 0, 2, ..., 4, 0, 2.

- For $p \equiv 2 \mod 3,\ q \equiv 0 \mod 3,\ l \equiv 1, 2, 3, 4, 0 \mod 5$ The labeling function f for $p \equiv 2 \mod 3,\ q \equiv 0 \mod 3,\ l \equiv 1, 2, 3, 4, 0 \mod 5$ is isomorphic with the labeling function f for $p \equiv 0 \mod 3,\ q \equiv 2 \mod 3,\ l \equiv 1, 2, 3, 4, 0 \mod 5$

- For $p \equiv 2 \mod 3,\ q \equiv 1 \mod 3,\ l \equiv 1, 2, 3, 4, 0 \mod 5$ The labeling function f for $p \equiv 2 \mod 3,\ q \equiv 1 \mod 3,\ l \equiv 1, 2, 3, 4, 0 \mod 5$ is isomorphic with the labeling function f for $p \equiv 1 \mod 3,\ q \equiv 2 \mod 3,\ l \equiv 1, 2, 3, 4, 0 \mod 5$

- For $p \equiv 2 \mod 3,\ q \equiv 2 \mod 3,\ l \equiv 1, 2, 3, 4, 0 \mod 5$ For the first cycle, we give the labels 2, 0, 3, 1, 4, 2, 0, 4, 2, ..., 0, 4, 2. Next, for the path, we give the labels 2, 5, 3, 1, 4, ..., 2, 5, 3, 1, 4. Finally, for the second cycle, we have five cases based on the length l of the path. For $l \equiv 1 \mod 5$, we give the labels 2, 0, 3, 1, 4, 2, 0, 4, ..., 2, 0, 4. For $l \equiv 2 \mod 5$, we give the labels 5, 2, 0, ..., 5, 2, 0, 3, 1. For $l \equiv 3 \mod 5$, we give the labels 3, 0, 2, 4, ..., 0, 2, 4, 1. For $l \equiv 4 \mod 5$, we give the labels 1, 4, 2, 0, ..., 4, 2, 0, 3. For $l \equiv 0 \mod 5$, we give the labels 4, 2, 0, ..., 4, 2, 0, 3, 1.

From the proof, we can imply that we can use the labeling of subgraph, such as, path subgraph of kayak paddle graph can be labeled as same as the label of path with same vertices, the same way holds for the cycle subgraph. If we found the same label that does not fit with the definition of $L(2,1)$, then add up the labels by 1. On the proof, we found that with some cases need add up the labels, so for some cases $\lambda_{2,1}$ of kayak paddle is 5. □

III. CONCLUSIONS

In this paper, we examined the exact value of the L(2,1)-labeling of mongolian tent, lobster, triangular snake, and kayak paddle graph. For some graph, we can get the $L(2,1)-$labeling number of G from the already precise value of biggest connected subgraph of G.

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