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ON THE RELATIVE-INVERSE OF AN EXCLUSIVE-SUBMATRIX: AN INVERSE OF A NON-SQUARE MATRIX

Louie Resti S. Rellon[1*], James Alfred B. Lozada[2], Genevieve T. Guieb[3] and Afoljen S. Mejorada[4]

[1,2,3,4]*Mathematics Department, University of Mindanao, Philippines*
Email : [1]louie_rellon@umindanao.edu.ph, [2]j8mlozada93@yahoo.com, [3]genevieveguieb@gmail.com, [4]afoljenmejorada@gmail.com
[*]Corresponding author

Abstract. A system of linear equations $AX = B$ with non-square coefficient matrix A may not appear to be solvable using matrix inversion. In this paper, we established that if matrix A of order $m \times n$ is an exclusive-submatrix of an invertible matrix then there exists a non-square matrix $A^{\sim}$ of order $n \times m$ called the relative-inverse of A such that either $AA^{\sim} = I_m$ or $A^{\sim}A = I_n$. This result enables us to solve the first statement provided that it is a consistent system. In particular, if $\operatorname{rank}(A) = n$ and $m > n$, then the system has a unique solution of the form $X = A^{\sim}B$; and if $\operatorname{rank}(A) < n$ and $m < n$, then the system has infinitely many solutions of the form $X = A^{\sim}B$ and each one of it is unique with respect to the chosen invertible matrix. The results support and confirm the existence of the inverse of a non-square matrix. The findings of this research will influence future advances in cryptography and cryptanalysis, and will serve as a bridge linking developments in coding theory. This research can be used as a guide for technological developments in programming language and especially for video game developers, when performing the inverse transformation of an object..
Keywords: one-sided inverse; non-square matrix; exclusive-submatrix; relative-inverse

I. INTRODUCTION

Have you ever heard of an inverse of a non-square matrix? If not, then consider this interesting discussion. To understand this idea, it is necessary to talk about what matrix is. We will define it in several ways so that we can see different perspectives. First, according to Anton [1], a matrix is a rectangular array of expressions called entries from the field of numbers. While this definition is a good starting point, let us see to it that the term "expression" mentioned here should also include numbers alone. Second, according to Beauregard and Fraleigh [3], it is a collection of abstract quantities arranged in rows and columns. It reminds of the assumption that one can count any group of objects, whether it is perceivable or not. And finally, according to Lang [11], it is an indexed family of elements defined under addition and multiplication.

Linear algebra is a core area in the field of mathematics that centers on the notion of linear equations. It has an extensive use of matrix that is commonly known in the context of compactly representing simultaneous linear equations. A classical question in this area is to determine a solution, assuming it exists, to a given system of linear equations. One way to determine its solution is to incorporate the concept of determinant in the process called matrix inversion. It was in 1801 when Gauss introduced the concept of determinant during his study on quadratic forms [2]. Moreover, according to Campbell and Meyer [6], the system of equations in the form of $AX = B$ takes place in various aspects of mathematics. They assert that, through matrix inversion, the prior equation can be easily solved and has a unique solution of $X = A^{-1}B$ provided that the coefficient matrix A is invertible. However, the existence of a determinant was known to be possible only for square matrices. That is, solving a system of linear equations will be a difficult task if it has a non-square coefficient matrix. This motivates the pure beginning in considering the study of matrix inversion involving non-square matrix.

Several mathematicians, in earlier times, have studied matrix inversion involving non-square matrices who independently developed the concept of pseudo-inverse, widely known as Moore-Penrose inverse, which was generally accepted as an inverse of a non-square matrix [4]. In the paper of F. Toutounian, F. & F. Soleymani [14], they proposed a new high-order iterative method computational algorithm for finding an approximate inverse of a square matrix and extended the method to find the pseudo-inverse (also known as the Moore–Penrose inverse) of a singular or rectangular matrix. More recently, Cordero, Manayon, and Sagpang [7], in the paper entitled On the Quasi-inverse of a Non-square Matrix: An Infinite Solution, have established a different way of finding the inverse of a non-square matrix called the "quasi-inverse" matrix.

When talking about the inverse of a non-square matrix, we are referring to the notion of the one-sided inverse: which is often used to mean the left or right inverse. To justify that a non-square matrix has a one-sided inverse, consider an $m \times n$ matrix A with $m \neq n$. Then we can see that A^{-1} has to be $n \times m$ because that is the only way for $AA^{-1} = I_m$ and also for $A^{-1}A = I_n$ to be satisfied and from this notice that instead of having I_m, we obtain I_n; and since the definition of inverse requires to operate on both sides of A, then it is enough to assume that the left and the right inverse of a non-square matrix are not the same. Thus, it has a one-sided inverse.

In this study, we produce a different one-sided inverse by integrating the idea of a submatrix in the process called deletion of arbitrary rows or columns. This study aimed to establish the inverse of a non-square matrix and its properties and extend its notion in finding a solution to the linear system $AX = B$ with a non-square coefficient matrix A.

II. RESULTS

The following definitions and theorems are all results of this study.

Exclusive-submatrix of a Matrix

Definition 1. *Let A be a matrix of order $m \times n$. A non-square matrix B of order $v \times n$ (or $m \times v$) is an exclusive-submatrix of A provided that it can be obtained from A by deleting rows or columns of A, but not both, where $1 \leq v < m$ (or $1 \leq v < n$).*

The following illustration demonstrates this definition.

Illustration 1. Let

$$A=\begin{bmatrix}1 & 3 & 0 & 2 & -1\\ 2 & 1 & 5 & 1 & 3\\ 4 & 3 & 0 & 1 & 2\end{bmatrix}.$$

If we delete the second row, we obtain the exclusive-submatrix

$$B=\begin{bmatrix}1 & 3 & 0 & 2 & -1\\ 4 & 3 & 0 & 1 & 2\end{bmatrix}.$$

We could also delete the second and fifth columns of matrix A and obtain

$$C=\begin{bmatrix}1 & 0 & 2\\ 2 & 5 & 1\\ 4 & 0 & 1\end{bmatrix},$$

which gives another exclusive-submatrix of matrix A.

Remark 1. All matrices under consideration have real entries.

Lemma 1. *Let* $A=\left[a_{ij}\right]$ *be a matrix of order* $m\times q$ *and let* $B=\left[b_{ij}\right]$ *be a matrix of order* $q\times n$ *such that* $AB=C$, *where* $C=\left[c_{ij}\right]$ *is a matrix of order* $m\times n$. *Then* k *corresponding rows (or* h *corresponding columns) of* C *are deleted if* k *rows of* A *(or* h *columns of* B*) are deleted, where* $0\le k<m$ *and* $0\le h<n$.

PROOF: Let $A=\left[a_{ij}\right]$ be a matrix of order $m\times q$ and let $B=\left[b_{ij}\right]$ be a matrix of order $q\times n$ such that $AB=C$, where $C=\left[c_{ij}\right]$ is a matrix of order $m\times n$. Then,

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{iq}b_{qj}=\sum_{k=1}^{q}a_{ik}b_{kj}.$$

For instance, when $i=1$ and $j=2$, then this means that the entry

$$c_{12}=a_{11}b_{12}+a_{12}b_{22}+\ldots+a_{1q}b_{q2}=\sum_{s=1}^{q}a_{1s}b_{s2}$$

is composed of the entries in first row $(i=1)$ of matrix A and second column $(j=2)$ of matrix B. In general, the subscript i of the entry c_{ij} corresponds to the subscript i in a_{ij} and the subscript j of the entry c_{ij} corresponds to the subscript j in b_{ij}; and clearly, all entries c_{ij} in the ith row and jth column of C is composed of the entries in the ith row of A and jth column of B, respectively. So, the deletion of k rows of A (or h columns of B), where $0\le k<m$ and $0\le h<n$, would result to the corresponding deletion of k rows (or q columns) of C. Therefore, k corresponding rows (or q corresponding columns) of C are deleted.

Relative-inverse of an Exclusive-submatrix

Definition 2. *Let B be an $m \times n$ matrix and let A be a square matrix of order m or n such that B is an exclusive-submatrix of A. The one-sided inverse of B denoted by $B^{\sim}$ different from the pseudo-inverse A^{+} or the quasi-inverse A^{*} which is an $n \times m$ exclusive-submatrix of A^{-1} is called the relative-inverse of B if $BB^{\sim} = I_m$ or $B^{\sim}B = I_n$. We say B is relatively-invertible if the relative-inverse $B^{\sim}$ exists.*

Theorem 1. *Let B be an $m \times n$ matrix. Then we can find a square matrix A where B is an exclusive-submatrix that can produce a relative-inverse $B^{\sim}$ of B if the following conditions are satisfied:*

i. *A is of order n if $m < n$, otherwise A is of order m;*
ii. *A has k number of rows added below the mth row of B if $m < n$ and k number of columns added to the right of nth column of B if $m > n$;*
iii. *A is invertible;*
iv. *Deletion of k columns of A^{-1} corresponds when A has k more rows than B and deletion of k rows of A^{-1} corresponds when A has k more columns than B.*

PROOF: Let B be an $m \times n$ matrix. Here we prove the case for $m < n$ since $m > n$ can be proven similarly. Suppose that $m < n$. Then, A must be of order n, otherwise A cannot contain B. Since B is of order $m < n$ and is contained in A, it follows that A has k number of rows added below the mth row of B. This tells us that B is an exclusive-submatrix of A, by Definition 2.1, and is of order $m = (n-k) \times n$. At this point, conditions (i) and (ii) are satisfied. Now, by condition (iii),let us assume that A is invertible. Then A^{-1} exists. Consider the equation $AA^{-1} = I_n$. Recall that A has k rows added below the mth row of B, that is, A has k more rows than B. Applying Lemma 2.1 in the prior equation, we obtain another equation $BA^{-1} = I_{(m \times n)}$. Observe that the right-hand side of the latter equation is no longer an identity matrix. Then, k corresponding columns must be deleted in $I_{(m \times n)}$ to obtain another identity matrix. To make this possible, we first delete such columns of A^{-1}, that is, k corresponding columns of A^{-1} must be deleted first so that the resulting matrix is of order $n \times (n-k) = m$; and then, the deletion in $I_{(m \times n)}$ follows immediately by Lemma 2.1. Thus, the equation $BA^{-1} = I_{(m \times n)}$ reduces to the equation $BB^{\sim} = I_{(n-k)} = I_m$. Since we found that

$B^{\sim}$ is of order $n\times(n-k)=m$ exclusive-submatrix of A^{-1}, then Definition 2.2 tells us that $B^{\sim}$ is the relative-inverse of B. Therefore, the relative-inverse $B^{\sim}$ of B exists. Hence, the theorem is proved.

Corollary 1. *If B is an $m\times n$ exclusive-submatrix of an invertible matrix A and $B^{\sim}$ denotes the relative-inverse of B, then* $B^{\sim}=\begin{bmatrix}\frac{\langle A_{11}\rangle}{\det(A)} & \frac{\langle A_{21}\rangle}{\det(A)} & \cdots & \frac{\langle A_{n1}\rangle}{\det(A)}\\ \frac{\langle A_{12}\rangle}{\det(A)} & \frac{\langle A_{22}\rangle}{\det(A)} & \cdots & \frac{\langle A_{n2}\rangle}{\det(A)}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\langle A_{1(n-k)}\rangle}{\det(A)} & \frac{\langle A_{2(n-k)}\rangle}{\det(A)} & \cdots & \frac{\langle A_{n(n-k)}\rangle}{\det(A)}\end{bmatrix}$ *which is of order* $(n-k)\times n$ *if* $m<n$ *and* $B^{\sim}=\begin{bmatrix}\frac{\langle A_{11}\rangle}{\det(A)} & \frac{\langle A_{21}\rangle}{\det(A)} & \cdots & \frac{\langle A_{(m-k)1}\rangle}{\det(A)}\\ \frac{\langle A_{12}\rangle}{\det(A)} & \frac{\langle A_{22}\rangle}{\det(A)} & \cdots & \frac{\langle A_{(m-k)2}\rangle}{\det(A)}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\langle A_{1m}\rangle}{\det(A)} & \frac{\langle A_{2m}\rangle}{\det(A)} & \cdots & \frac{\langle A_{(m-k)m}\rangle}{\det(A)}\end{bmatrix}$ *which is of order* $m\times(m-k)$ *if* $m>n$*, where A has k more columns or rows than B, respectively.*

PROOF: From Theorem 2.1, k rows of A exceed to B if $m<n$, so the corresponding deletion of k columns in A^{-1} follows. Hence, $B^{\sim}$ is of order $(n-k)\times n$ if $m<n$. Similarly, k columns of A exceed to B if $m>n$, so the corresponding deletion of k rows in A^{-1} follows. Hence, $B^{\sim}$ is of order $m\times(m-k)$ if $m>n$.

Illustration 2. To illustrate Theorem 2.1 and Corollary 2.1, consider the non-square matrix below. Let

$$B=\begin{bmatrix}1 & 4\\ 7 & 2\\ 0 & 5\end{bmatrix}.$$

Since B has more rows than columns ($m>n$), we choose $A=\begin{bmatrix}1 & 4 & 0\\ 7 & 2 & 0\\ 0 & 5 & 1\end{bmatrix}$.Notice that conditions (i) and (ii) are satisfied, then B is an exclusive-submatrix of A. It turns out that matrix A is invertible (otherwise, we find another appropriate matrix), by direct computations of determinant of A we obtain

$$\det(A) = 0\begin{vmatrix} 7 & 2 \\ 0 & 5 \end{vmatrix} - 0\begin{vmatrix} 1 & 4 \\ 0 & 5 \end{vmatrix} + (1)\begin{vmatrix} 1 & 4 \\ 7 & 2 \end{vmatrix} = -26.$$

Since A has is invertible, A^{-1} exists. This means that condition (iii) is satisfied. By straightforward computations of A^{-1} we get

$$A^{-1} = \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & 0 \\ \frac{7}{26} & -\frac{1}{26} & 0 \\ -\frac{35}{26} & \frac{5}{26} & 1 \end{bmatrix}.$$

Since A has 1 column added to the right of the second column of B, then A has 1 more column than B, so 1 corresponding row (which is the third row in this case) of A^{-1} must be deleted. Hence, the resulting matrix is of order 2×3 and also observe that Corollary 2.1 is satisfied. Thus, by condition (iv), we obtain

$$B^{\sim} = \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & 0 \\ \frac{7}{26} & \frac{-1}{26} & 0 \end{bmatrix}$$

and we have that

$$B^{\sim}B = \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & 0 \\ \frac{7}{26} & \frac{-1}{26} & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 7 & 2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

All succeeding results represent the properties and characteristics of a relative-inverse matrix. Observe from the last illustration that A can have different values on its deleted column. That is, matrix B can be an exclusive-submatrix of another invertible matrix. Intuitively, this tells us that there can be an infinite number of invertible matrices containing A as an exclusive-submatrix. The following theorem verifies this claim.

Theorem 2. *Let B be an $m\times n$ matrix with m linearly independent rows if $m<n$ and n linearly independent columns if $m>n$. Then there are infinitely many invertible matrices that contain B as an exclusive-submatrix.*

PROOF: Let B be an $m\times n$ matrix with m linearly independent rows if $m<n$ and n linearly independent columns if $m>n$. Here we prove the case for $m<n$ since $m>n$ can be proven similarly. Suppose that $m<n$. Let us consider a square matrix A of order n that

contain B as an exclusive-submatrix. Since B has m linearly independent rows, it follows that A must also have at least m linearly independent rows, otherwise B is not an exclusive-submatrix of A. Since B is an exclusive-submatrix of A, then k rows of A exceed to B, where $1 \le k < n$. Suppose, further, that all k rows of A are linearly independent. Then A must have $m+k=n$ linearly independent rows, so A is guaranteed to be invertible, otherwise A has a row of zeros. Then Theorem 2.1 implies that B is relatively-invertible. For the sake of illustration, we let sth and rth rows of A be one of the deleted rows and remaining rows of A, respectively. Let $C=\left[c_{ij}\right]$ be a matrix obtained from A by adding a non-zero multiple of rth row to the sth row. Then, we have $c_{sj}=a_{sj}+ta_{rj}$ for the deleted sth row; and $c_{ij}=a_{ij}$ for the remaining rows excluding the sth row, where $t \in \mathbb{R}^*$. Since t is a non-zero real number, then C is row equivalent to A. Then C is invertible. Since the added non-zero multiple is only on the sth row and sth row is one of the deleted rows, it follows that C contains B as an exclusive-submatrix and $C \neq A$, otherwise the sth row of C and A are equal. That is, we have $c_{sj}=a_{sj}$, but $c_{sj}=a_{sj}+ta_{rj}$ so $a_{sj}+ta_{rj}=a_{sj}$ implies $ta_{rj}=0$ contradicting the hypothesis that the added multiple is non-zero. Furthermore, since t is a non-zero real number and every choice of t yields an invertible matrix that contains B as an exclusive-submatrix, then there are infinitely many invertible matrices that contains B as an exclusive-submatrix.

Illustration 3. To illustrate this theorem, we produce three distinct relative-inverses of matrix

$$B=\begin{bmatrix}1 & 4\\ 7 & 2\\ 0 & 5\end{bmatrix}.$$

SOLUTION: Choose $A=\begin{bmatrix}1 & 4 & 0\\ 7 & 2 & 0\\ 0 & 5 & 1\end{bmatrix}$.Then to obtain the other two matrices of the form described in Theorem 2.2, we use the column operation since B is obtained from A by deleting the third column and then we proceed as follows. Apply column operation $2\mathrm{c}_1+\mathrm{c}_3 \to \mathrm{c}_3$ in A to get

$$C=\begin{bmatrix}1 & 4 & 2\\ 7 & 2 & 14\\ 0 & 5 & 1\end{bmatrix}.$$

And apply column operation $5c_1+c_3 \to c_3$ in A to get

$$D=\begin{bmatrix}1 & 4 & 5\\ 7 & 2 & 35\\ 0 & 5 & 1\end{bmatrix}.$$

From Illustration 2.2, we know that B is invertible. Also, both C and D are column-equivalent to B which implies that these matrices are invertible. Hence, matrices C and D the first three conditions of Theorem 2.1.By straightforward computations of the inverse of each of these matrices, we have

$$A^{-1}=\begin{bmatrix}\frac{-1}{13} & \frac{2}{13} & 0\\ \frac{7}{26} & \frac{-1}{26} & 0\\ \frac{-35}{26} & \frac{5}{26} & 1\end{bmatrix},\ A_1^{-1}=\begin{bmatrix}\frac{34}{13} & \frac{-3}{13} & -2\\ \frac{7}{26} & \frac{-1}{26} & 0\\ \frac{-35}{26} & \frac{5}{26} & 1\end{bmatrix},\ A_2^{-1}=\begin{bmatrix}\frac{173}{26} & \frac{-21}{26} & -5\\ \frac{7}{26} & \frac{-1}{26} & 0\\ \frac{-35}{26} & \frac{5}{26} & 1\end{bmatrix}.$$

By condition (iv) of Theorem 2.1, the third row must be deleted on each of these matrices to obtain the relative-inverse of B. Thus, we obtain

$$B^{\sim}=\begin{bmatrix}\frac{-1}{13} & \frac{2}{13} & 0\\ \frac{7}{26} & \frac{-1}{26} & 0\end{bmatrix}, B_1^{\sim}=\begin{bmatrix}\frac{27}{104} & \frac{11}{104} & \frac{-1}{4}\\ \frac{7}{26} & \frac{-1}{26} & 0\end{bmatrix}, B_2^{\sim}=\begin{bmatrix}\frac{307}{104} & \frac{-29}{104} & \frac{-9}{4}\\ \frac{7}{26} & \frac{-1}{26} & 0\end{bmatrix},$$

respectively. We see that

$$B^{\sim}B=\begin{bmatrix}\frac{-1}{13} & \frac{2}{13} & 0\\ \frac{7}{26} & \frac{-1}{26} & 0\end{bmatrix}\begin{bmatrix}1 & 4\\ 7 & 2\\ 0 & 5\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$$B_1^{\sim}B=\begin{bmatrix}\frac{27}{104} & \frac{11}{104} & \frac{-1}{4}\\ \frac{7}{26} & \frac{-1}{26} & 0\end{bmatrix}\begin{bmatrix}1 & 4\\ 7 & 2\\ 0 & 5\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$$B_2^{\sim}B=\begin{bmatrix}\frac{307}{104} & \frac{-29}{104} & \frac{-9}{4}\\ \frac{7}{26} & \frac{-1}{26} & 0\end{bmatrix}\begin{bmatrix}1 & 4\\ 7 & 2\\ 0 & 5\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Corollary 2. *Let B be an $m\times n$ matrix and let $B^{\sim}$ be the relative-inverse of B. If B is right (or left) relatively-invertible, then B^T is left (or right) relatively-invertible.*

PROOF: Let B be an $m \times n$ matrix and let $B^{\sim}$ be the relative-inverse of B. Suppose that B is right relatively-invertible. Then the relative-inverse $B^{\sim}$ of B exists such that $BB^{\sim} = I$. So, we have $\left(BB^{\sim}\right)^T = I^T = I$ implies $\left(BB^{\sim}\right)^T = B^{\sim T}B^T = I$. This tells us that B^T is left relatively-invertible. Similarly, suppose that B is left relatively-invertible. Then the relative inverse $B^{\sim}$ of B exists such that $B^{\sim}B = I$. So, we have $\left(B^{\sim}B\right)^T = I^T = I$ implies $\left(B^{\sim}B\right)^T = B^T B^{\sim T} = I$ which tells us that B^T is right relatively-invertible.

Illustration 4. Using the result in Illustration 2.2, we have

$$B^{\sim}B = \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & 0 \\ \frac{7}{26} & \frac{-1}{26} & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 7 & 2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\left(B^{\sim}B\right)^T = B^T B^{\sim T} = \begin{bmatrix} 1 & 7 & 0 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{-1}{13} & \frac{7}{26} \\ \frac{2}{13} & \frac{-1}{26} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, B^T is right relatively-invertible.

Corollary 3. *Let A be an $m \times n$ matrix. If $m > n$ (or $m < n$)and A has linearly dependent column (or row), then A is not relatively-invertible.*

PROOF: Let A bean $m \times n$ matrix. Suppose that $m > n$ and A has linearly dependent column. Then there exists a non-trivial linear combination of rows, which is equal to 0. So, any square matrix containing A as an exclusive-submatrix must also contain the same property and so such square matrix has 0 determinant. Thus, A is not relatively-invertible. Similarly, A is not relatively invertible if $m < n$ and has linearly dependent row.

Illustration 5. Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & -3 & 12 \end{bmatrix}$. Notice that matrix A is of order 2×3 ($m < n$). The last row of this matrix is linearly dependent. In particular, the second row is equal to thrice the first row. So, any square matrix containing A as an exclusive-submatrix must have determinant of zero. Hence, A is not relatively-invertible.

Illustration 6. Let A be a square matrix of order n. By Definition 2.1, the exclusive-submatrix of A has an order of $v \times n$ or $n \times v$, where $1 \le v < n$. Let $E(n)$ denote the number of exclusive-submatrices in A of order is n. Let us observe the following matrices of order 1×1, 2×2, and 3×3.

$$\begin{bmatrix} a_{11} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Clearly, the Definition 2.1 tells us that the order must be at least 1, so a matrix of order 1×1 contains no exclusive-submatrix. Hence $E(1) = 0$. The following are the only exclusive-submatrices of a 2×2 matrix:

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix}, \quad \begin{bmatrix} a_{21} & a_{22} \end{bmatrix}, \quad \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \quad \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

Hence, we have $E(2) = 4$.

For a 3×3 matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}, \quad \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{33} & a_{33} \end{bmatrix}$$

Hence, we get $E(3) = 12$.

As we continue to observe in this manner, we end up with the pattern shown below:

$$0, \quad 4, \quad 12, \quad 28, \quad 60, \quad \ldots$$

Equivalently,

$$2^{1+1} - 4, \quad 2^{2+1} - 4, \quad 2^{3+1} - 4, \quad 2^{4+1} - 4, \quad 2^{5+1} - 4, \quad \ldots$$

Thus, we claim that the number of exclusive-submatrices in A of order n is $2^{n+1} - 4$. That is, we have $E(n) = 2^{n+1} - 4$. It follows from this claim that there are $2^{n+1} - 4$ relatively-invertible exclusive-submatrices if A is invertible. We will now prove this in the following result.

Theorem 3. *Let A be an invertible matrix of order n. Then A contains $2^{n+1} - 4$ relatively-invertible exclusive-submatrices.*

PROOF: Suppose that A is a square matrix of order n. To prove this theorem, we must count the number of exclusive-submatrices in A. If A is of order 1, then it has $2^2 - 4 = 0$ is true because a matrix of order 1 contains 0 exclusive-submatrix (see Illustration 2.8).Let X and Y be the sets containing the rows and columns of matrix A as an element, respectively. Let r_i and c_j denote the ith row and jth column of A with $1 \le i,\ j \le n$, respectively. Then

$$X = \{r_1, r_2, r_3, \ldots, r_n\} \text{ and } Y = \{c_1, c_2, c_3, \ldots, c_n\}.$$

By taking the power set of X and Y,

$$P(X) = \left|\{\varnothing, \{r_1\}, \{r_2\}, ..., \{r_n\}, \{r_1, r_2\}, \{r_1, r_3\}, \ldots, \{r_1, r_n\}, \ldots, X\}\right| = 2^n$$

and

$$P(Y) = \left|\{\varnothing, \{c_1\}, \{c_2\}, ..., \{c_n\}, \{c_1, c_2\}, \{c_1, c_3\}, \ldots, \{c_1, c_n\}, \ldots, Y\}\right| = 2^n.$$

We can represent the elements in $P(X)$ and $P(Y)$ as the chosen deletion. For instance, the set $\{r_1\}$ containing the element r_1 represents the deleted first row in matrix A and the set $\{c_3, c_4, \ldots, c_n\}$ containing the elements $c_3, c_4, \ldots, c_n$, which can be found somewhere in $P(Y)$, represent the deleted columns $3, 4, \ldots, n$. Note that the empty set represents the entire matrix A; while the set X and Y represent the deletion of the entire rows or columns of matrix A, respectively, and is violating the definition of the exclusive-submatrix. Hence, we must remove these elements, so we obtain

$$P(X) = \left|\{\{r_1\}, \{r_2\}, ...\{r_n\}, \{r_1, r_2\}, \{r_1, r_3\}, \ldots, \{r_1, \mathrm{r}_n\}, \ldots, \{r_2, \mathrm{r}_3, \ldots, r_n\}\}\right| = 2^n - 2$$

and

$$P(Y) = \left|\{\{c_1\}, \{c_2\}, ...\{c_n\}, \{c_1, c_2\}, \{c_1, c_3\}, \ldots, \{c_1, c_n\}, \ldots, \{c_2, c_3, \ldots, c_n\}\}\right| = 2^n - 2.$$

Thus, we have

$$P(X) + P(Y) = 2^n - 2 + 2^n - 2 = 2 \cdot 2^n - 4 = 2^{n+1} - 4.$$

Therefore, a square matrix A of order n contains $2^{n+1} - 4$ exclusive-submatrices.
Moreover, since A is invertible, it follows from Theorem 3.1 that there are $2^{n+1} - 4$ relatively-invertible exclusive-submatrices.

Remark 2. *Supported with the above proof, it follows that an* $m \times n$ *matrix* A *contains* $2^m + 2^n - 4$ *exclusive-submatrices.*

Proposition 1. *If* B *is an exclusive-submatrix of an invertible matrix* A, *then* $B^{\sim}$ *is unique with respect to the given matrix* A.

PROOF: Suppose that B is an exclusive-submatrix of an invertible matrix A. Then B is relatively-invertible, say right relatively-invertible. Suppose, further, that B has relative-inverses $B'^{\sim}$ and $B''^{\sim}$ with respect to A such $BB'^{\sim}=I$ and $BB''^{\sim}=I$. Then

$$BB'^{\sim}-BB''^{\sim}=0$$

and, by the Distributive law of matrices,

$$B\left(B'^{\sim}-B''^{\sim}\right)=0,$$

$B\neq 0$ since it is relatively-invertible, it must be $B'^{\sim}-B''^{\sim}=0$ or

$$B'^{\sim}=B''^{\sim},$$

so, the inverse of B with respect to A is unique.

Proposition 2. *If A is an invertible matrix of order m and B is an $m\times n$ relatively-invertible matrix such that $m>n$,then AB is relatively-invertible and $\left(AB\right)^{\sim}=B^{\sim}A^{-1}$.*

PROOF: Suppose that A is an invertible matrix of order m and B is an $m\times n$ relatively-invertible matrix. Since $m>n$, B is left relatively-invertible by Corollary 2.1 and that $B^{\sim}$ is of order $n\times m$. Then, we have

$$\left(B^{\sim}A^{-1}\right)\left(AB\right)=B^{\sim}\left(A^{-1}A\right)B=B^{\sim}\left(I_mB\right)=B^{\sim}B=I_n,$$

so AB is relatively-invertible and $\left(AB\right)^{\sim}=B^{\sim}A^{-1}$ is its relative-inverse.

Remark 3. *The case for $m<n$ also holds with the product BA.*

Illustration 7. Let $A=\begin{bmatrix}1&0&1\\-1&4&2\\1&3&1\end{bmatrix}$ and let $B=\begin{bmatrix}1&4\\7&2\\0&5\end{bmatrix}$.Observe that $\det(A)=-9$ and matrix B is relatively-invertible by Illustration 3.1. Then, we have

$$A^{-1}=\begin{bmatrix}\frac{2}{9}&\frac{-1}{3}&\frac{4}{9}\\\frac{-1}{3}&0&\frac{1}{3}\\\frac{7}{9}&\frac{1}{3}&\frac{-4}{9}\end{bmatrix},$$

and we take $B^{\sim}=\begin{bmatrix}\frac{-1}{13}&\frac{2}{13}&0\\\frac{7}{26}&\frac{-1}{26}&0\end{bmatrix}$. So, by Proposition 2.2, we have

$$
\begin{aligned}
\left(B^{\sim}A^{-1}\right)\left(AB\right) &= \left(\begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & 0 \\ \frac{7}{26} & \frac{-1}{26} & 0 \end{bmatrix}\begin{bmatrix} \frac{2}{9} & \frac{-1}{3} & \frac{4}{9} \\ \frac{-1}{3} & 0 & \frac{1}{3} \\ \frac{7}{9} & \frac{1}{3} & \frac{-4}{9} \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 0 & 1 \\ -1 & 4 & 2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 7 & 2 \\ 0 & 5 \end{bmatrix}\right) \\
&= \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & 0 \\ \frac{7}{26} & \frac{-1}{26} & 0 \end{bmatrix}\left(\begin{bmatrix} \frac{2}{9} & \frac{-1}{3} & \frac{4}{9} \\ \frac{-1}{3} & 0 & \frac{1}{3} \\ \frac{7}{9} & \frac{1}{3} & \frac{-4}{9} \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 4 & 2 \\ 1 & 3 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & 4 \\ 7 & 2 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & 0 \\ \frac{7}{26} & \frac{-1}{26} & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 7 & 2 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & 0 \\ \frac{7}{26} & \frac{-1}{26} & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 7 & 2 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$

System of Linear Equations

Theorem 5. *Let $AX = B$ be the system of linear equations with non-square coefficient matrix A of order $m \times n$; let $m > n$; let C be a matrix of order m such that A is an exclusive-submatrix of C. If rank $A = n$ and C is invertible, then the system has a unique solution of the form $X = A^{\sim}B$, where $A^{\sim}$ is the relative-inverse of A.*

Proof: Let $AX = B$ be the system of linear equations with non-square coefficient matrix A of order $m \times n$; let $m > n$; let C be matrix of order m such that A is an exclusive-submatrix of C. Suppose that rank $A = n$. Then the system has a unique solution. Suppose, further, that C is invertible. Then Theorem 2.1 implies that the relative-inverse $A^{\sim}$ of A exists. Also, Corollary 2.1 tells us that $A^{\sim}$ is the left relative-inverse of because $m > n$. We claim that $X = A^{\sim}B$ is a solution to the system. To see this, we have

$$AX = B$$
$$A^{\sim}AX = A^{\sim}B$$
$$IX = A^{\sim}B$$
$$X = A^{\sim}B$$

and this shows that our claim is true which establishes the theorem.

Illustration 8. To illustration Theorem 2.5, consider the system of linear equations below.

$$x + y = 1$$
$$2x - y = 5$$
$$3x + 4y = 2$$

Let A be the coefficient matrix of the system, then we have

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

Applying the Gaussian elimination in A to get

$$A' = \begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}.$$

Since A' is row equivalent to A, then rank $A =$ rank $A' = 2$ which is equal to the number of variables. Choose $C = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \\ 3 & 4 & 2 \end{bmatrix}$ with $C^{-1} = \begin{bmatrix} \frac{6}{7} & \frac{2}{7} & \frac{-1}{7} \\ \frac{1}{7} & \frac{-2}{7} & \frac{1}{7} \\ \frac{-11}{7} & \frac{1}{7} & \frac{3}{7} \end{bmatrix}$. So, the first three sufficient conditions of Theorem 2.1 are satisfied. Furthermore, by the condition (iv) of Theorem 2.1, we get

$$A^{\sim} = \begin{bmatrix} \frac{6}{7} & \frac{2}{7} & \frac{-1}{7} \\ \frac{1}{7} & \frac{-2}{7} & \frac{1}{7} \end{bmatrix}.$$

Finally, by Theorem 2.5, the unique solution of the system is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{6}{7} & \frac{2}{7} & \frac{-1}{7} \\ \frac{1}{7} & \frac{-2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

To verify the solution, we have:

$$\begin{aligned} x + y = 1 &\Rightarrow (2) + (-1) = 1 \\ 2x - y = 5 &\Rightarrow 2(2) - (-1) = 5 \\ 3x + 4y = 2 &\Rightarrow 3(2) + 4(-1) = 2 \end{aligned}$$

Theorem 6. *Let $AX = B$ be the system of linear equations with non-square coefficient matrix A of order $m \times n$; let $m < n$; let C be a matrix of order n such that A is an exclusive-submatrix of C. If rank $A < n$ and C is invertible, then the system has infinitely many solution of the form $X = A^{\sim}B$, where $A^{\sim}$ is the relative-inverse of A and $X = A^{\sim}B$ is unique with respect to the given matrix C.*

PROOF: Let $AX = B$ be the system of linear equations with non-square coefficient matrix A of order $m \times n$; let $m < n$; let C be matrix of order n such that A is an exclusive-submatrix of C. Suppose that rank $A < n$, then the system has infinitely many solutions. Suppose, further, that matrix C is invertible. Then Theorem 2.1 tells us that the relative-inverse $A^{\sim}$ of A exists. Corollary 2.1 implies that $A^{\sim}$ is the right relative-inverse of A since $m < n$. To show that the equation $X = A^{\sim}B$ is a solution to the system, we have

$$AX = A\left(A^{\sim}B\right) = \left(AA^{\sim}\right)B = IB = B,$$

which shows that it is indeed a solution. It also follows from Proposition 2.1 that $A^{\sim}$ is unique with respect to the given matrix C and so is $X = A^{\sim}B$. This proves the theorem.

Illustration 9. To illustrate Theorem 2.6, consider the system of linear equations below.

$$\begin{aligned} x + y - 2z &= 5 \\ 2x + 3y + 4z &= 2 \end{aligned}$$

SOLUTION: Let A be the coefficient matrix of the system, then we have

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 4 \end{bmatrix}.$$

Applying the Gaussian elimination in A to get

$$A' = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 8 \end{bmatrix}.$$

Since A' is row equivalent to A, then rank A= rank $A'=2$ which is less than the number of variables. Choose $C=\begin{bmatrix}1&1&-2\\0&1&0\\2&3&4\end{bmatrix}$ with $C^{-1}=\begin{bmatrix}\frac{1}{2}&\frac{-5}{4}&\frac{1}{4}\\0&1&0\\\frac{-1}{4}&\frac{-1}{8}&\frac{1}{8}\end{bmatrix}$. So, the first three sufficient conditions of Theorem 2.1 are satisfied. Moreover, by the condition (iv) of Theorem 2.1, we obtain

$$A^{\sim}=\begin{bmatrix}\frac{1}{2}&\frac{1}{4}\\0&0\\\frac{-1}{4}&\frac{1}{8}\end{bmatrix}.$$

Thus, by Theorem 2.6, one of a solution to the system is given by

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&\frac{1}{4}\\0&0\\\frac{-1}{4}&\frac{1}{8}\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix}=\begin{bmatrix}3\\0\\-1\end{bmatrix}.$$

To verify the solution, we have:

$$\begin{aligned}x+\ y-2z=5 &\ \Rightarrow\ \quad 3+\ 0-2(-1)\ =5\\2x+3y+4z=2 &\ \Rightarrow\ 2(3)+3(0)+4(-1)=2\end{aligned}$$

III. CONCLUSION

In this paper, the following conclusions were drawn. The notion of relative-inverse reduces the equation $A^{-1}A=I_n$ to $B^{\sim}B=I_m$ if $m<n$ and $AA^{-1}=I_m$ to $BB^{\sim}=I_n$ if $m>n$, where matrices B and $B^{\sim}$ are exclusive submatrices of A and A^{-1}, respectively. Some properties of the other existing one-sided inverse hold in the notion relative-inverse. In particular, a notable result which is similar to the existence of infinitely many quasi-inverses holds in the notion of relative-inverse. Consequently, the notion of relative-inverse can provide a solution to a consistent linear system $AX=B$, where A is a non-square matrix.

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