JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 6 NO. 1 (JUN 2023)
Available online at www.jfma.math.fsm.undip.ac.id

A CLOSER LOOK AT A PATH DOMINATION NUMBER IN GRID GRAPHS

Leomarich F. Casinillo*

[1]Department of Mathematics, Visayas State University, Baybay City, Leyte, Philippines
Email : [1]leomarich.casinillo@vsu.edu.ph
*Corresponding author

Abstract. This article exposes the combinatorial formula that determines the path domination number in a grid graph and discusses some of its properties. Seven properties are derived regarding the path domination number of grid graphs. Furthermore, some additional properties as direct consequences of the derived main properties are also discussed.
Keywords: Grid graph, path domination number, combinatorial formula

I. INTRODUCTION

Domination in graphs is a wide and interesting topic in graph theory [1], [2]. In fact, a lot of discrete mathematicians have devoted themselves to discovering new results in the domination number in graphs [3], [4], [5], [6], [7], [8]. Over the years, various parameters of domination numbers in graphs have been published in the literature. One of these parameters is called path domination of undirected graphs [9], [10]. Here, this article aims to investigate the path domination number of a particular graph known as a two-dimensional grid. This research is inspired by the study by Casinillo [11] which deals with combinatorial properties of domination numbers in triangular grid graphs. Hence, the main focus of this study is to give a closer look at the combinatorial properties of path domination in lattice or grid graphs. To understand clearly the concepts of path domination, we need some definitions of terms concerning the properties of graphs as presented below [9], [10], [12].

Here, we let $G = (V(G), E(G))$ be an undirected graph as a function of vertex set $V(G)$ and edge set $E(G)$. The order and size of graph G are the cardinality $|V(G)|$ of $V(G)$ and cardinality $|E(G)|$ of $E(G)$, respectively. An *open neighborhood* of a vertex $u \in G$ is the set $N_G(u) = N(u) = \{v \in V(G): uv \in E(G)\}$. On the other hand, the *closed neighborhood* of a vertex $u \in G$ is the set $N_G[u] = N[u] = \{u\} \cup N(u)$. Let $S \subseteq V(G)$. Then, the *open neighborhood* of set S is $N_G(S) = N(S) = \bigcup_{u \in S} N_G(u)$. Additionally, the *closed neighborhood* of set S is the set $N_G[S] = N[S] = S \cup N(S)$. A path of order $n \geq 2$ is a graph of a finite sequence of edges that joins a sequence of vertices, that is, if a path has order $n \geq 3$, then two vertices are of degree 1, and the other $n-2$ vertices of degree 2. A path graph is denoted by P_n with order n and size $n-1$. Hence, if $n = 1$, then graph P_1 is a trivial path. A rectangular grid graph is an $m \times n$ lattice graph that is a Cartesian product of two paths P_m and P_n denoted by $P_m \square P_n$ [13], [14]. In particular, Figure 1 below shows the graph of $P_4 \square P_7$.

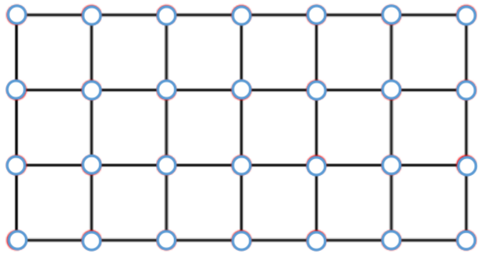

Figure 1. Graph $P_4 \square P_7$.

Let $G = P_m \square P_n$ be a grid graph where $m > n \geq 4$. Let $D \subseteq V(G)$. Then D is a *dominating set* of graph G if for all $u \in V(G) \backslash D, \exists v \in D$ such that $uv \in E(G)$, that is, $N[D] = V(G)$. A domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of dominating set D. A dominating set $D_{kp} \subseteq V(G)$ is called a k-path dominating set of a graph G if the subgraph $\langle D_{kp} \rangle$ induced by D_{kp} is a set of k paths. A k-path domination number of G, denoted by $\gamma_{k-path}(G)$, is the minimum cardinality of k paths dominating set D_{kp}. Figure 2 shows an example of k-path dominating set for graph $G = P_7 \square P_4$ where the k-path domination number $(k = 3)$ is $\gamma_{3-path}(G) = 12$.

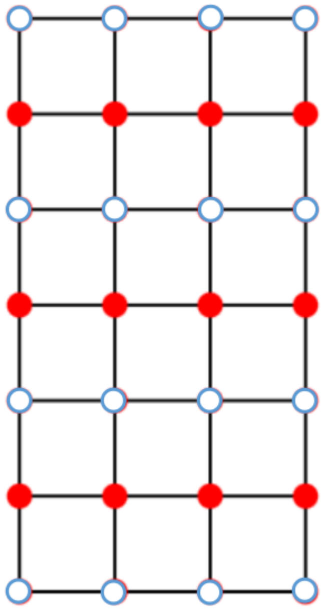

Figure 2. The 3-path domination in graph $P_7 \square P_4$.

In general, the k-path dominating set configuration is horizontal if $m > n$ and vertical if $m < n$ and not placed at the first and last paths of graph G (see Figure 2). Moreover, A dominating set $D_p \subseteq V(G)$ is called a path dominating set of a graph G if the subgraph $\langle D_p \rangle$ induced by D_p is a set of the path. A path domination number of G, denoted by $\gamma_{path}(G)$, is the minimum cardinality of path dominating set D_p. Below is an example of a path dominating set for graph $G = P_7 \square P_4$ where the path domination number is $\gamma_{path}(G) = 14$.

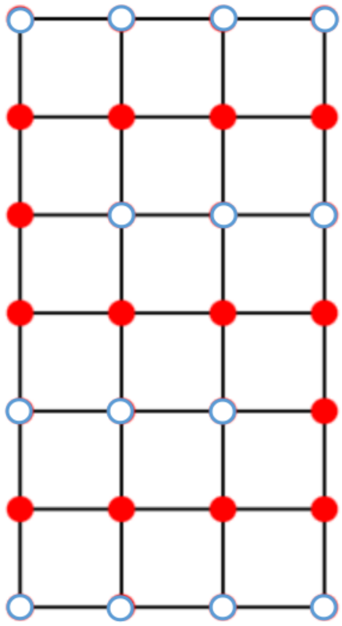

Figure 3. The path domination in graph $P_7 \square P_4$.

The distance $d_G(u, v)$ between two the vertices u and v in G is defined as the length of the shortest path between u and v. The degree of a vertex $v \in V(G)$ is the number of incident edges and it is denoted by $deg_G(v)$. For more details on definitions in graph theory, the readers may refer to [15]. Hence, in this article, we expose new formula that determines the path domination number in graph $G = P_m \square P_n$ and we present some of its combinatorial properties.

II. RESULTS AND DISCUSSION

First, we present the following Remark as very useful in constructing our results. The remark is well-known that determines the domination number of a path graph for any order $n \in \mathbb{N}$.

Remark 2.1. [17], [18] *Let $H = P_m$ be a path of order $m \in \mathbb{N}$. Then,*

$$\gamma(H) = \begin{cases} \frac{m}{3} & if\ m \equiv 0 (mod\ 3) \\ \frac{m+2}{3} & if\ m \equiv 1 (mod\ 3) \\ \frac{m+1}{3} & if\ m \equiv 2 (mod\ 3) \end{cases}$$

The Lemma below presents the combinatorial formula for finding the k-path domination number of the grid graph with the dimension $m > n \geq 4$.

Lemma 2.2. *Let $G = P_m \square P_n$ be a grid graph where $m > n \geq 4$. Then,*

$$\gamma_{k-path}(G) = \begin{cases} \frac{mn}{3} & if\ m \equiv 0(mod\ 3) \\ \frac{mn+2n}{3} & if\ m \equiv 1(mod\ 3) \\ \frac{mn+n}{3} & if\ m \equiv 2(mod\ 3) \end{cases}$$

Proof: Suppose that $G = P_m \square P_n$ and $m > n \geq 4$. Let $H = P_m$. Then, it clearly follows that $\gamma_{k-path}(G) = nk$, where $k = \gamma(H)$ by Remark 1. Hence, the hypothesis follows and completes the proof. □

It is worth noting that if the restriction is $n > m \geq 4$, then we can just simply change m to n in Lemma 2.2. Our next result shows the formula for determining the path domination number of the grid graph.

Theorem 2.3. *Let $G = P_m \square P_n$ be a grid graph where $m > n \geq 4$. Then,*

$$\gamma_{path}(G) = \begin{cases} \frac{mn-2m+6}{3} & if\ m \equiv 0(mod\ 3) \\ \frac{mn+2n+2m-8}{3} & if\ m \equiv 1(mod\ 3) \\ \frac{mn+n+2m-7}{3} & if\ m \equiv 2(mod\ 3) \end{cases}$$

Proof: Let $G = P_m \square P_n$ where $m > n \geq 4$ and let $H = P_m$. Then, consider the following cases:
Case 1. Suppose that $m \equiv 0(mod\ 3)$. Hence, by Lemma 2.2, we get $\gamma_{k-path}(G) = nk$, where $k = \gamma(H)$. Note that in graph H, every dominating set $u \in D, N(u) = \{x, y\}$ where $x, y \in H \backslash D$. Hence, there is only one configuration for dominating set in G. Now, the distance of each $k = \gamma(H)$ path in G is 2, then it follows that $\gamma_{path}(G) = nk + 2(k-1)$. Hence, by Remark 1, case 1 holds.
Case 2. Secondly, we suppose that $m \equiv 1(mod\ 3)$. Then, by Lemma 2.2, we also get $\gamma_{k-path}(G) = nk$, where $k = \gamma(H)$. It follows that there are $k-2$ dominating paths where their distance is equal to 2 and 3 of the dominating paths have a distance of 1. Note that this configuration is unique to have a minimum path dominating set in graph G. Thus, it is clear that $\gamma_{path}(G) = nk + 2(k-3) + 2$. Hence, by Remark 1, case 2 also holds.
Case 3. Lastly, we suppose that $m \equiv 0(mod\ 3)$. By Lemma 2.2, we obtain $\gamma_{k-path}(G) = nk$, where $k = \gamma(H)$. Then, there are $k-1$ dominating paths with a distance equal to 2, and 2 of the dominating paths have a distance of 1. Again, this configuration is unique to have a minimum path dominating set in G. So, it follows that $\gamma_{path}(G) = nk + 2(k-2) + 1$. By Remark 1, case 3 holds.
Combining the 3 cases completes the proof. □

Suppose we let $G = P_m \square P_n$ be a grid graph and $m > n \geq 4$. Then, the following results below are immediate from Theorem 2.3.

Corollary 2.4. *The difference $\gamma_{path}(G) - \gamma_{k-path}(G)$ can be made arbitrarily large.*

Proof: Let $k \in \mathbb{N}$. Suppose that $\gamma_{path}(G) - \gamma_{k-path}(G) = k$ where $k = f(m,n)$ for all $m > n \geq 4$. Then, increasing the values of m and n, the hypothesis follows. ☐

Corollary 2.5. Let $m \geq 2$ *and* $n = 2$ *or* 3. *Then,* $\gamma_{k-path}(G) = \gamma_{path}(G) = m$ *where* $k = 1$.

Proof: Suppose that $G = P_m \square P_n$ where $m \geq 2$ *and* $n = 2$ *or* 3. Then, it is clear that for $n = 2$, it is either the first or second path with order m is the dominating vertex set of graph G. Now, if $n = 3$, then it follows that the middle path of order m is the dominating vertex set. Hence, the hypothesis follows. ☐

Remark 2.6. *If* $k \geq 1$, *then* $\gamma_{k-path}(G) \leq \gamma_{path}(G)$.

Proof: Quick from Theorem 2.3. ☐

Remark 2.7. *If* $m = n \geq 4$, *then*

$$\gamma_{path}(G) = \begin{cases} \dfrac{m^2 - 2m + 6}{3} & if\ m \equiv 0 (mod\ 3) \\ \dfrac{m^2 + 4m - 8}{3} & if\ m \equiv 1 (mod\ 3) \\ \dfrac{m^2 + 3m - 7}{3} & if\ m \equiv 2 (mod\ 3) \end{cases}$$

Proof: Quick from Theorem 2.3. ☐

It is worth noting that m can be replaced by n in Remark 2.7. Let $N_{\gamma_{k-path}}(G)$ and $N_{\gamma_{path}}(G)$ be the number of ways of putting a k-path dominating set and path dominating set in graph G, respectively. For the readers, the number of ways of putting the dominating set in the path and cycle graph can be read in [18]. Then, the following result is quick from Remark 2.1 and Theorem 2.3.

Lemma 2.8. *Let* $G = P_m \square P_n$ *be a grid graph where* $m > n \geq 4$. *Then,*

$$N_{\gamma_{path}}(G) \leq N_{\gamma_{k-path}}(G) = N_{\gamma}(P_m)$$

Proof: Suppose that $G = P_m \square P_n$ be a grid graph where $m > n \geq 4$. Then, by Remark 2.1, it is immediately that $N_{\gamma_{k-path}}(G) = N_{\gamma}(P_m)$. It is worth noting that dominating set must be minimum, i.e., optimum. Thus, k dominating paths should not be placed at the endpoints in graph G to maintain the minimal distance. Hence, in view of Theorem 2.3, it suffices to show that $N_{\gamma_{path}}(G) \leq N_{\gamma_{k-path}}(G)$. This completes the proof. ☐

The next result is a direct consequence of Lemma 2.2 above. This shows the formula for determining how many configurations can be made to form a path dominating set in graph $G = P_m \square P_n$.

Theorem 2.9. *Let $G = P_m \square P_n$. If $m > n \geq 4$, then*

$$N_{\gamma_{path}}(G) = \begin{cases} 2 & if\ m \equiv r (mod\ 3) \\ 4 & if\ m \equiv 2(mod\ 3) \end{cases}$$

where $r \in \{0, 1\}$.

Proof: Let $G = P_m \square P_n$. Suppose that $m > n \geq 4$. Then, we consider the following cases:
Case 1. It is worth noting that if $m \equiv 0(mod\ 3)$, then by Lemma 2.8, it follows that $N_{\gamma_{k-path}}(G) = 1$. Since there are two possible ways to connect the k-path in G, then $N_{\gamma_{path}}(G) = 2N_{\gamma_{k-path}}(G) = 2$.
Case 2. For $m \equiv 1(mod\ 3)$, there is also 1 way to put k-path dominating set without dominating the vertex in the first and last path position in graph G. Hence, it follows the same argument from Case 1.
Case 3. Let $m \equiv 2(mod\ 3)$. Then, there are 2 configurations of k-path dominating set in graph G without dominating vertex in the first and last path position, that is, $N_{\gamma_{k-path}}(G) = 2$. Since there are also two ways to connect the k-path dominating set, thus, $N_{\gamma_{path}}(G) = 2N_{\gamma_{k-path}}(G) = 4$.
Combining the 3 cases completes the proof. □

The following remarks are a direct consequence of Theorem 2.9 above.

Remark 2.10. *If $m \geq 2$ and $n = 3$, the $N_{\gamma_{path}}(G) = 1$.*

Remark 2.11. *If $m \geq 4$ and $n = 2$, the $N_{\gamma_{path}}(G) = 2$.*

Let $\bar{n}$ be the number of ways (shortest routes) in getting from v_1 to v_n, where v_1, $v_n \in D_p$, and $\deg(v_1) = deg(v_n) = 1$. Hence, the following results below are immediate.

Theorem 2.12. *Let $G = P_m \square P_n$ where $m > n \geq 4$. If $\gamma(P_m) \equiv 0(mod\ 2) \geq 3$, then*

$$\bar{n} = \binom{m+n-4}{m-3} = \binom{m+n-4}{n-1} \ and\ d_G(v_1, v_n) = m + n - 4$$

where v_1, $v_2 \in D_p$ and $deg(v_1) = deg(v_n) = 1$.

Proof: Let $G = P_m \square P_n$ where $m > n \geq 4$. Suppose that $\gamma(P_m) \equiv 0(mod\ 2) \geq 3$, then we let A be the set of all shortest routes from v_1 to v_n where v_1, $v_2 \in D_p$, and $\deg(v_1) = deg(v_n) = 1$. Then, it follows that $d_G(v_1, v_n) = (m-3) + (n-1) = m + n - 4$. Next, we construct a binary sequence to represent the shortest route. So, we let "0" denote a vertical segment and "1" denote a vertical segment in graph G. In that case, every route from v_1 to v_n can be represented by a binary sequence of length $m + n - 4$. Thus, we can establish a mapping φ from A to set B of all binary sequences of length $m + n - 4$, that is, $\varphi: A \to B$. On the face of it, it is clear that φ is one-to-one and onto, hence φ is bijective. By the Bijection Principle (BP), it follows that $\bar{n} = |A| = |B| = \binom{m+n-4}{m-3} = \binom{m+n-4}{n-1}$. This completes the proof. ☐

Let $D_p^3 \subset D_p$ be a subset of path dominating sets with degree 3 and let $D_p^4 \subset D_p$ be a subset of path dominating set with degree 4, that is, $D_p^3 \cup D_p^4 = D_p$ and $D_p^3 \cap D_p^4 = \emptyset$. Hence, the following results are quick from Theorem 2.3.

Theorem 2.13. *Let* $G = P_m \Box P_n$. *If* $m > n \geq 4$, *then*

$$|D_p^3| = \begin{cases} \frac{4m-3}{3} & if\ m \equiv 0(mod\ 3) \\ \frac{4m-4}{3} & if\ m \equiv 1(mod\ 3) \\ \frac{4m-5}{3} & if\ m \equiv 2(mod\ 3) \end{cases}$$

and

$$|D_p^4| = \begin{cases} \frac{mn-2m}{3} & if\ m \equiv 0(mod\ 3) \\ \frac{mn+2n-2m-4}{3} & if\ m \equiv 1(mod\ 3) \\ \frac{mn+n-2m-2}{3} & if\ m \equiv 2(mod\ 3). \end{cases}$$

Proof: By Theorem 2.3, it follows directly that $D_p = D_p^3 \cup D_p^4$. Hence, $|D_p^3| = |D_p \backslash D_p^4|$ and $|D_p^4| = |D_p \backslash D_p^3|$. Thus, the hypothesis follows. ☐

Corollary 2.14. *Let* $G = P_m \Box P_n$. *If* $m > n \geq 4$, *then*

$$[N(D_p^3) \cap N(D_p^4)] \backslash D_p \subset V(G) \backslash D$$

Proof: Suppose that $G = P_m \Box P_n$ where $m > n \geq 4$. Then, it follows that $[N(D_p^3) \cap N(D_p^4)] \backslash D_p \neq \emptyset$. Since $[N(D_p^3) \cap N(D_p^4)] \subset V(G)$, then the hypothesis follows. ☐

Remark 2.15. *Let* $G = P_m \Box P_n$. *If* $m > n \geq 4$, *then*

$$|N(D_p^3) \cap N(D_p^4)| = \begin{cases} \frac{2m-6}{3} & if\ m \equiv 0(mod\ 3) \\ \frac{2m-8}{3} & if\ m \equiv 1(mod\ 3) \\ \frac{2m-7}{3} & if\ m \equiv 2(mod\ 3) \end{cases}$$

III. CONCLUSION

This article introduced new results involving a combinatorial formula that counts the k-path and path dominating set of grid graph $G = P_m \square P_n$ where $m > n \geq 4$ denoted by $\gamma_{k-path}(G)$ and $\gamma_{path}(G) = |D_p|$, respectively. The difference between $\gamma_{k-path}(G)$ and $\gamma_{path}(G) = |D_p|$ can be made arbitrarily large in relation to the dimension of grid graph G. Furthermore, the number of the configuration of path dominating set in G is 2 when $m \equiv r(mod\ 3)$ where $r \in \{0, 1\}$ or 4 when $m \equiv 2(mod\ 3)$. For future research, it is interesting to investigate the derangement of path dominating sets in grid graphs.

REFERENCES

[1] Cockayne, E. J., & Hedetniemi, S. T. (1977). Towards a theory of domination in graph. *Networks Advanced Topics*, 7, 247–261.

[2] Casinillo, L. F., Lagumbay, E. T. & Abad, H. R. F. (2017). A note on connected interior domination in join and corona of two graphs. *IOSR Journal of Mathematics*, 13(2), 66–69.

[3] Castillano, E. C., Ugbinada, R. L., & Canoy Jr, S. R. (2014). Secure domination in the join of graphs. *Applied Mathematical Sciences*, *8*(105), 5203-5211.

[4] Quadras, J., Mahizl, A. S. M., Rajasingh, I., & Rajan, R. S. (2015). Domination in certain chemical graphs. *Journal of Mathematical Chemistry*, *53*(1), 207-219.

[5] Gnanasoundari, E. A., Raji, M., & Jayalalitha, G. (2020). Equitable Domination in Chemical Structural Graph. *European Journal of Molecular & Clinical Medicine*, *7*(2), 5246-5249.

[6] Borzooei, R. A., & Rashmanlou, H. (2015). Domination in vague graphs and its applications. *Journal of Intelligent & Fuzzy Systems*, *29*(5), 1933-1940.

[7] Rashmanlou, H., Lakdashti, A., & Sabet, S. S. (2018). A Study on Domination Sets in Vague Graphs with Novel Applications. *Annals of Pure and Applied Mathematics*, *16*(1), 223-234.

[8] Casinillo, E. L., Casinillo, L. F., Valenzona, J. S., & Valenzona, D. L. (2020). On Triangular Secure Domination Number. *InPrime: Indonesian Journal of Pure and Applied Mathematics*, *2*(2), 105-110.

[9] Alcón, L. G. (2016). A note on path domination. *Discussiones Mathematicae Graph Theory*, *36*, 1021–1034.

[10] Mekala, A., Kumar, U. V. C., & Murali, R. (2020). Domination and s-path domination in some brick product graphs. *Malaya Journal of Matematik (MJM)*, *8*(1, 2020), 254-257.

[11] Casinillo, L. F. (2020). Odd and even repetition sequences of independent domination number. *Notes on Number Theory and Discrete Mathematics*, *26*(1), 8-20.

[12] Gross, J. L., & Yellen, J. (2003). *Handbook of graph theory*. CRC press.

[13] Hadlock, F. O. (1977). A shortest path algorithm for grid graphs. *Networks*, *7*(4), 323-334.

[14] Itai, A., Papadimitriou, C. H., & Szwarcfiter, J. L. (1982). Hamilton paths in grid graphs. *SIAM Journal on Computing*, *11*(4), 676-686.

[15] Chartrand, G., & Zhang, P. (2012). *A First Course in Graph Theory*. Dover Publication Inc., New York.

[16] Casinillo, E. L., & Casinillo, L. F. (2020). A Note on Full and Complete Binary Planar Graphs. *Indonesian Journal of Mathematics Education*, *3*(2), 70-75.

[17] Casinillo, L. F. (2018). A note on Fibonacci and Lucas number of domination in path. *Electronic Journal of Graph Theory and Applications (EJGTA)*, *6*(2), 317-325.

[18] Casinillo, L. F. (2020). New counting formula for dominating sets in path and cycle graphs. *Journal of Fundamental Mathematics and Applications (JFMA)*, *3*(2), 170-177.