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A PYTHON CODE FOR GENERATING ALL PROPER SUBGROUPS OF DIHEDRAL GROUP

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Abstract. The dihedral group of order $2n$ denoted by D_{2n} is the symmetry group of a regular n-polygon consisting of rotation and reflection elements and the composition of both elements. Like any other group, the dihedral group also has a subgroup whose numbers differ depending on the value of n. This research explores and develop all the forms of proper subgroups of D_{2n} and all of these proper subgroups of D_{2n} are generated and counted with the help of Python program.

Keywords: dihedral group, prime numbers, python, object-oriented programming.

I. INTRODUCTION

At the end of the 17th century Joseph-Louis Lagrange tried to study the method of solving polynomial equations which became the initial foundation of group theory. A few years later in 1830, Evariste Galois used the term group to describe a set of one-on-one functions of a finite set that can be grouped to form a set closed to the composition operation [1]. Furthermore, the modern definition of group that is known today is the result of the accumulation of a long process of several subsequent studies.

In general, a group is a set equipped with a binary operation such that it is closed to this operation, associative, has an identity element, and every element has an inverse. Every group has subgroups, which are subsets of the group that form a group with respect to the same binary operations as the group. A proper subgroup is a subgroup that is distinct from the group itself. Many author previously have consider such variety of proper subgroup as can be seen in [2, 3, 4, 5, 6, 7].

One type of group which is interesting to discuss is the dihedral group denoted by D_{2n}, which is a polygon symmetry group consisting of rotation and reflection elements [8]. The dihedral group is widely utilized in the art world and is closely related to various objects in nature. Dihedral groups are applied by mineralogists and chemists to study molecular structures or crystal structures based on symmetry. Syarifudin and Wardhana have classify many types of subgroups in the dihedral group D_{2n} [9].

In this paper, we develop a code in Python language to help generating and counting all the proper subgroups of dihedral group; the basic theories for dihedral groups are based on studies in [10]. Python is one of the most popular object-oriented high-level programming languages today and is widely used as a tool for mathematical computing and data analysis [11]. Python is one of the easiest programming languages to learn because it has a compact and simple syntax. Previously, Python has been used to generate all subgroups of a particular group, for example Z_n [12].

II. METHODOLOGY

The following is the overview of steps used to finding and generating all proper subgroups of dihedral group:

1. Studying the literature related to the concept of dihedral group.
2. Determine the subgroups of the dihedral group.
3. Create an algorithm to generate these subgroups of the dihedral group.
4. Convert the algorithm into Python programming language.

III. RESULTS AND DISCUSSIONS

First, we provide some theoretical foundation including definitions and theorems used this research. Then, we provide the algorithm to generate all the proper subgroups of dihedral group. This algorithm then is written in Python language, and finally, we provide some examples of the results.

3.1 Theoretical Foundation and Results

Definition 1 (Dihedral Group) [9]. *Group G is called a dihedral group of order $2n, n \geq 3$ if G is generated by two elements $a, b \in G$ with the following properties*

$$G = \langle a, b \mid a^n = e, b^2 = e, bab^{-1} = a^{-1} \rangle$$

The dihedral group of order $2n$ is hereafter symbolized by D_{2n}.

From the definition it is easy to see that $|D_{2n}| = 2n$ and D_{2n} can be written as the set $D_{2n} = \{e, a, a^2, a^3, \dots, a^{n-1}, b, ab, a^2b, a^3b, \dots, a^{n-1}b\}$. Element a is commonly referred to as the rotation element and b is referred to as the reflection element. For example, if we take $n = 4$, we get $D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$.

Definition 2 (Subgroup) [13]. *Suppose G is a group. A subset H of G is a subgroup of G if H is nonempty, closed with respect to the same binary operation as G, has the same identity element as G, and every element has an inverse in H. If H is a subgroup of G, then it is denoted $H \leq G$. Proper subgroups are all H such that $H \leq G$ but $H \neq G$.*

For example, one of the subgroups of D_8 is $H = \{e, a, a^2, a^3\}$. Next, to determine the subgroups of D_{2n}, all the factors of n must first be known. Many factors of n can be determined by utilizing the following theorem.

Theorem 1 (The Fundamental Theorem of Arithmetic) [14]. *Every integer $n \geq 2$ is prime or can be expressed as a product of primes. The factorization into primes is unique except for the order of the factors.*

From the previous theorem, we are be able to infer the number of positive factors of n from its prime factorization as the following proposition shows.

Proposition 1. *Suppose the prime factorization of* n *is* $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ *where* $p_1, \dots, p_k$ *are distinct primes with* $p_1 < \dots < p_k$ *and each* a_i *is a positive integer. The number of positive factors of* n, *denoted by* $l(n)$ *is* $l(n) = (a_1 + 1)(a_2 + 1) \dots (a_k + 1)$.

Proof. Since the prime factorization of n is $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, the positive factors of *n* can be represented as

$$\boldsymbol{d} = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$$

where $b_i \in \{0,1, \dots, a_i\}$. It follows by combinatorial argument that because each b_i can be chosen from a set with order $a_i + 1$ and each b_i is chosen independent from each other, then all possible value of $\boldsymbol{d}$ are as many as $(a_1 + 1)(a_2 + 1) \dots (a_k + 1)$. (QED)

To simplify the determination of the subgroups of the dihedral group, the subgroups will be divided into three categories: reflexive subgroups, rotational subgroups, and mixed subgroups. A reflexive subgroup is a subgroup built by ab elements. Rotational subgroups are subgroups built by rotation elements of order more than equal to two. While the mixed subgroup is a subgroup built by rotation and reflection elements [9].

Suppose a dihedral group $\boldsymbol{D_{2n}} = \{\boldsymbol{e}, \boldsymbol{a}, \dots, \boldsymbol{a^{n-1}}, \boldsymbol{b}, \boldsymbol{ab}, \dots, \boldsymbol{a^{n-1}b}\}$ is given. Then suppose the number n have positive factors denoted by $\boldsymbol{d_n} = \{\boldsymbol{1}, \boldsymbol{k_1}, \dots, \boldsymbol{k_i}, \boldsymbol{n}\}$. The subgroups of the dihedral group are of the following form [15]:

1. $S_1 = \{e, a^k b\}$, with $k = 0, 1, \dots, n-1$
2. $S_2 = \{e, a^k, a^{2k}, \dots, a^{n-k}\}$, with $k = 1, k_1, \dots, k_i$
3. $S_3 = \{e, a^k, a^{2k}, \dots, a^{n-k}, a^{l_k} b, a^{l_k + k} b, a^{l_k + 2k} b, \dots, a^{l_k + n - k} b\}$, with $k = k_1, \dots, k_i$ and $l_k = 0, 1, \dots, k-1$

Subgroup S_1 is referred to as a reflexive subgroup, subgroup S_2 is referred to as a rotational subgroup, while subgroup S_3 is referred to as a mixed subgroup.

Subgroup S_1 comes from the fact that $(a^k b)(a^k b) = (a^k b a^k) b = bb = e$. The proper subgroup S_2 is the group that's generated by the element a^k or can be written as $S_2 = \langle a^k \rangle$. The next subgroup, S_3, is the combined version of S_1 and S_2. Other proper subgroups that are not included in the three categories is just the trivial one i.e., $\{e\}$. From the previous explanation we can compute the total number of proper subgroups of dihedral group, that is

$$|S_1| + |S_2| + |S_3| = n + (i + 1) + (k_1 + \dots + k_i).$$

Example. Let $n = 4$ so that $D_{2 \cdot 4} = D_8 = \{e, a, a^2, a^3, b, ab, a^2 b, a^3 b\}$. There are 4 proper subgroups of the form S_1 as follows $\{e, b\}, \{e, ab\}, \{e, a^2 b\}, \{e, a^3 b\}$. On the other hand, the positive factors of 4 are {1,2,4}. This means there are 2 proper subgroups of the form S_2 as follows $\{e, a, a^2, a^3\}, \{e, a^2\}$. Furthermore, the proper subgroups of the form S_3 are as follows $\{e, a^2, b, a^2 b\}, \{e, a^2, ab, a^3 b\}$. In total there are 8 proper subgroups of D_8.

3.2 Code and Implementation

A *code/program* is a collection of consecutive statements in a computer. The recipe for what the computer is supposed to do in a program is called *algorithm*. We will attempt to set up the algorithm first and then implement it in a program. This is useful when the algorithm is much more compact than the resulting program code [11]. The use of program will help to generate all proper subgroups of dihedral groups when n is large.

The previous section is the basis for the following algorithm. Note that the subgroups of a dihedral group can be easily generated by utilizing the factors of n. In general, the algorithm for generating such subgroups is as follows:

1. Initially given a value of n, form the group D_{2n}.
2. Determine all the positive factors of n.
3. Loop through each factor of n that is not equal to n to first generate a subgroup S_2.
4. For each existing subgroup S_2, use looping again to extend it to create subgroup S_3.
5. Finally, use separate looping for k from 1 to $n-1$ to generate subgroup S_1.

The described algorithm is then applied to the Python programming language. The final Python code is as following.

```
from math import *

# Initialization
print("============================================")
print("Generating all the proper subgroups of D_2n")
print("============================================")
n = int(input("Input n: "))
print("D_%d = " %(2*n), end="")

# Showing the contents of group D_2n
keys = []
for i in range(2*n):
    if i==0:
        keys.append("e")
    elif i<n:
        keys.append("a^" + str(i))
    else: keys.append("a^" + str(i-n) +"b")
print(keys, "\n")

# Creating dictionary to pair each element of group to the number 0 to n-1
dict = {}
for i in range(2*n):
    dict[i] = keys[i]

# Determining the factors of 2n
factor = []
i = 1
while (i * i < n):
    if (n % i == 0):
        factor.append(i)
    i += 1
```

```
for i in range(int(sqrt(n)), 0, -1):
    if (n % i == 0):
        factor.append(n // i)

# Generating all the proper subgroups and print it
print("Proper Subgroups")
print("---------------")
total = 0
for i in factor:
    start = 0
    subgroup = []
    while start < n:
        subgroup.append(dict[start])
        start = start + i
    print(subgroup, "--> orde", len(subgroup))
    total = total + 1

    if i != 1:
        for j in range(i):
            subgroup2 = []
            start2 = n + j
            while start2 < 2*n:
                subgroup2.append(dict[start2])
                start2 = start2 + i
            print(subgroup + subgroup2, "--> orde", len(subgroup+subgroup2))
            total = total + 1
print()
print("In total there are %d proper subgroups" %total)
```

The algorithm can be represented as a flowchart depicted in Fig. 1.

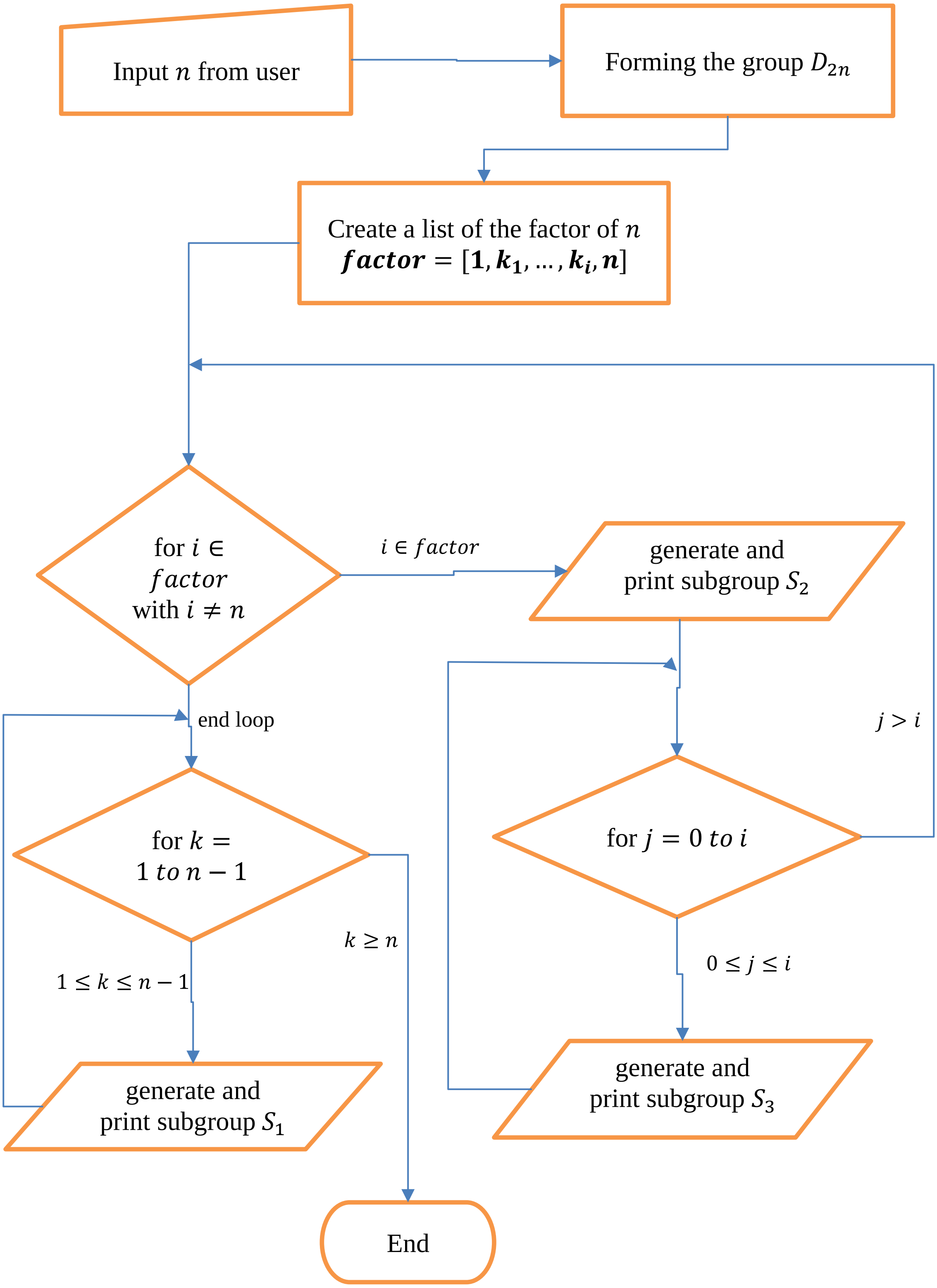

Figure 1. Flowchart of Algorithm

Outputs depicted in Fig. 2 and Fig. 3 are examples for $n = 6$ and $n = 29$.

```
============================================
Generating all the proper subgroups of D_2n
============================================
Input n: 6
D_12 = ['e', 'a^1', 'a^2', 'a^3', 'a^4', 'a^5', 'a^0b', 'a^1b', 'a^2b', 'a^3b', 'a^4b', 'a^5b']

Proper Subgroups
--------------
['e', 'a^1', 'a^2', 'a^3', 'a^4', 'a^5'] --> orde 6
['e', 'a^2', 'a^4'] --> orde 3
['e', 'a^2', 'a^4', 'a^0b', 'a^2b', 'a^4b'] --> orde 6
['e', 'a^2', 'a^4', 'a^1b', 'a^3b', 'a^5b'] --> orde 6
['e', 'a^3'] --> orde 2
['e', 'a^3', 'a^0b', 'a^3b'] --> orde 4
['e', 'a^3', 'a^1b', 'a^4b'] --> orde 4
['e', 'a^3', 'a^2b', 'a^5b'] --> orde 4
['e'] --> orde 1
['e', 'a^0b'] --> orde 2
['e', 'a^1b'] --> orde 2
['e', 'a^2b'] --> orde 2
['e', 'a^3b'] --> orde 2
['e', 'a^4b'] --> orde 2
['e', 'a^5b'] --> orde 2

In total there are 15 proper subgroups
```

Figure 2. All proper Subgroups of D_{12}

```
============================================
Generating all the proper subgroups of D_2n
============================================
Input n: 29
D_58 = ['e', 'a^1', 'a^2', 'a^3', 'a^4', 'a^5', 'a^6', 'a^7', 'a^8', 'a^9', 'a^10', 'a^11', 'a^12', 'a^13', 'a^14', 'a^15', 'a^16', 'a^17', 'a^18', 'a^19', 'a^20', 'a^21', 'a^22', 'a^23', 'a^24', 'a^25
', 'a^26', 'a^27', 'a^28', 'a^0b', 'a^1b', 'a^2b', 'a^3b', 'a^4b', 'a^5b', 'a^6b', 'a^7b', 'a^8b', 'a^9b', 'a^10b', 'a^11b', 'a^12b', 'a^13b', 'a^14b', 'a^15b', 'a^16b', 'a^17b', 'a^18b', 'a^19b', 'a^2
0b', 'a^21b', 'a^22b', 'a^23b', 'a^24b', 'a^25b', 'a^26b', 'a^27b', 'a^28b']

Proper Subgroups
--------------
['e', 'a^1', 'a^2', 'a^3', 'a^4', 'a^5', 'a^6', 'a^7', 'a^8', 'a^9', 'a^10', 'a^11', 'a^12', 'a^13', 'a^14', 'a^15', 'a^16', 'a^17', 'a^18', 'a^19', 'a^20', 'a^21', 'a^22', 'a^23', 'a^24', 'a^25', 'a^2
6', 'a^27', 'a^28'] --> orde 29
['e'] --> orde 1
['e', 'a^0b'] --> orde 2
['e', 'a^1b'] --> orde 2
['e', 'a^2b'] --> orde 2
['e', 'a^3b'] --> orde 2
['e', 'a^4b'] --> orde 2
['e', 'a^5b'] --> orde 2
['e', 'a^6b'] --> orde 2
['e', 'a^7b'] --> orde 2
['e', 'a^8b'] --> orde 2
['e', 'a^9b'] --> orde 2
['e', 'a^10b'] --> orde 2
['e', 'a^11b'] --> orde 2
['e', 'a^12b'] --> orde 2
['e', 'a^13b'] --> orde 2
['e', 'a^14b'] --> orde 2
['e', 'a^15b'] --> orde 2
['e', 'a^16b'] --> orde 2
['e', 'a^17b'] --> orde 2
['e', 'a^18b'] --> orde 2
['e', 'a^19b'] --> orde 2
['e', 'a^20b'] --> orde 2
['e', 'a^21b'] --> orde 2
['e', 'a^22b'] --> orde 2
['e', 'a^23b'] --> orde 2
['e', 'a^24b'] --> orde 2
['e', 'a^25b'] --> orde 2
['e', 'a^26b'] --> orde 2
['e', 'a^27b'] --> orde 2
['e', 'a^28b'] --> orde 2

In total there are 31 proper subgroups
```

Figure 3. All proper Subgroups of D_{58}

IV. CONCLUSIONS

The conclusions that can be drawn from this research are as follows. All subgroups of a dihedral group D_{2n} can be classified as either a rotation, reflection, or mixed subgroup. All subgroups of a dihedral group D_{2n} is able to be generated with Python program. The algorithm to generate all the subgroups of group D_{2n} utilizes the factors of n.

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