JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 5 NO. 2 (NOV 2022)
Available online at www.jfma.math.fsm.undip.ac.id

SOME NEW REMARKS ON POWER SUMS

Leomarich F. Casinillo

Department of Mathematics, Visayas State University, Baybay City, Leyte, Philippines
Email : leomarichcasinillo02011990@gmail.com

Abstract. Power sum is one of the interesting topics in number theory where its application in other sciences is known to be wide. This paper intends to stipulate new remarks on an explicit polynomial solution to power sums. Additionally, it investigates the general solution under odd and even numbers of terms and discusses some examples.
Keywords: power sums; explicit polynomial solution; odd and even terms

I. INTRODUCTION

In number theory, a lot of mathematicians are intrigued by finding an explicit solution to power sums [1] [2] [3] [4] [5] [6]. Power sum is a sum of n consecutive natural numbers that is raised to the power of $p \in \mathbb{Z}^+$ [6]. In particular, this can be written as

$$1^p + 2^p + \ldots + n^p = \sum_{i=1}^{n} i^p \tag{1}$$

Now, if the first term of the sequence is arbitrarily chosen to have a power sum of t terms, then it can be written as follows

$$\lambda^p + (\lambda+1)^p + (\lambda+2)^p + \ldots + (\lambda+t-1)^p = \sum_{i=\lambda}^{\lambda+t-1} i^p \tag{2}$$

where λ and p are natural numbers. Evidently, there are many mathematical results have been published in the literature that deals with the solution of power sums via different methods [7] [8] [9]. Inspired by the existing and fascinating results, this paper intends to construct a new polynomial solution that solves a power sum explicitly.

In fact, this paper is motivated by the paper of Casinillo [10], whose work is focusing on alternating power sums. Hence, an identical method in [10] was used to obtain some new remarks in finding the solution of power sums. Firstly, we consider the different mathematical forms of power sums adapted from [10]. Here it is. Let λ and p be natural numbers. If $t = 2x - 1$ $(x \in \mathbb{Z}^+)$, then

$$S_t^o(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-2} j^p = \lambda^p + (\lambda+1)^p + \cdots + (\lambda+2x-2)^p \tag{3}$$

and for $t = 2x$ $(x \in \mathbb{Z}^+)$, we have

$$S_t^e(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-1} j^p = \lambda^p + (\lambda+1)^p + \cdots + (\lambda+2x-1)^p \tag{4}$$

In this study, the superscripts o and e in the power sum notation above represent an odd and even number of terms of the series, respectively. Next, we need the following notations. First, we let $P_n(x) \in \mathbb{Z}[x]$ be a polynomial in x of degree $n \in \mathbb{Z}^+$. Secondly, we let $f_j(\lambda) \in \mathbb{Z}[\lambda]$ be

JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 5 NO. 2 (NOV 2022)
Available online at www.jfma.math.fsm.undip.ac.id

a polynomial in λ of degree $j \in \mathbb{Z}^+$. In that case, the study focus on solving for a polynomial of the form $P_p(x) = \sum_{j=0}^{p} f_j(\lambda)x^j = f_p(\lambda)x^p + f_{p-1}(\lambda)x^{p-1} + \ldots + f_1(\lambda)x + f_0(\lambda)$ where $f_j(\lambda) \in \mathbb{Z}[\lambda]$ and $j \in \{0, 1, 2, \ldots, p\}$, i.e., $S_t^o(\lambda, x, p) = P_p^o(x)$ and $S_t^e(\lambda, x, p) = P_p^e(x)$. Some illustrations and simulations were also discussed.

II. RESULTS

The following Theorem 2.1 is quick from equation (3) that shows a power sum with odd terms has a polynomial solution as a function of two natural numbers λ and x.

Theorem 2.1. *Let λ, x, and p be natural numbers. If $S_t^o(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-2} j^p$ and $x = \frac{t+1}{2} \leq$ $p + 1$and $t \equiv 1 (mod\ 2)$, then $S_t^o(\lambda, x, p) = \sum_{j=0}^{p} c_j(\lambda)x^j$ where $c_j(\lambda) \in \mathbb{Z}[\lambda]$.*

Proof. To prove Theorem 2.1. we need to simulate equation (3) with $x = 1, 2, \ldots, p, p + 1$. In that case, we obtain the following:

$$\begin{cases} S_t^o(\lambda, 1, p) = \sum_{j=0}^{p} c_j(\lambda)1^j = \lambda^p \\ S_t^o(\lambda, 2, p) = \sum_{j=0}^{p} c_j(\lambda)2^j = \sum_{j=0}^{2}(\lambda + j)^p \\ S_t^o(\lambda, 3, p) = \sum_{j=0}^{p} c_j(\lambda)3^j = \sum_{j=0}^{4}(\lambda + j)^p \\ \quad . \qquad\qquad . \qquad\qquad . \\ \quad . \qquad\qquad . \qquad\qquad . \\ \quad . \qquad\qquad . \qquad\qquad . \\ S_t^o(\lambda, p + 1, p) = \sum_{j=0}^{p} c_j(\lambda)(p + 1)^j = \sum_{j=0}^{2p}(\lambda + j)^p \end{cases} \tag{5}$$

It is worth noting that the unknowns $c_0(\lambda)$, $c_1(\lambda)$, $c_2(\lambda)$,…, $c_p(\lambda)$, and the equations are equal in number. Plus, there is no linear dependence between the pairwise equations in (5), hence, we conclude the system of equations in (5) has a unique solution. On the face of it, we obtain the following solutions

$$\begin{cases} c_0(\lambda) = P_0(\lambda) \in \mathbb{Z}[\lambda] \\ c_1(\lambda) = P_1(\lambda) \in \mathbb{Z}[\lambda] \\ c_2(\lambda) = P_2(\lambda) \in \mathbb{Z}[\lambda] \\ \qquad . \\ \qquad . \\ \qquad . \\ c_p(\lambda) = P_p(\lambda) \in \mathbb{Z}[\lambda] \end{cases} \tag{6}$$

where $p, \lambda \in \mathbb{Z}^+$. It is worthy to note that the maximum simulation is $x = p + 1$, hence, the polynomial is only valid for $1 \leq x \leq p + 1$. This completes the proof. □

To make this clear, some examples were presented for the following values $p = 1, 2, 3$.

Illustration 2.1. Considering that $p = 1$ and t is an odd positive integer, then, we have the following equation

$$S_t^o(\lambda, x, 1) = \sum_{j=\lambda}^{\lambda+2x-2} j = (2\lambda+3)x - (\lambda+3) \tag{7}$$

where $1 \leq x \leq 2$.
To solve equation (7), we let $S_t^o(\lambda, x, 1) = c_1(\lambda)x + c_2(\lambda)$, where $c_1(\lambda) \in \mathbb{Z}[\lambda]$ and $c_2(\lambda) \in \mathbb{Z}[\lambda]$. By Theorem 2.2, we simulate x from 1 to 2 and obtain the following system of equation

$$\begin{cases} c_1(\lambda) + c_2(\lambda) = \lambda \\ 2c_1(\lambda) + c_2(\lambda) = \displaystyle\sum_{j=\lambda}^{\lambda+2} j \end{cases}$$

In that case, we get the following

$$\begin{cases} c_1(\lambda) = 2\lambda + 3 \\ c_2(\lambda) = -\lambda - 3 \end{cases}$$

Hence, equation (7) is obtained and valid for $1 \leq x \leq 2$.

Example 1. Solving for $S_3^o(5, 2, 1)$. Applying equation (7), we get

$$S_3^o(5, 2, 1) = \sum_{j=5}^{7} j = 18.$$

Illustration 2.2. Considering that $p = 2$ and t is an odd natural number, then, we have

$$\begin{aligned} S_t^o(\lambda, x, 2) &= \sum_{j=\lambda}^{\lambda+2x-2} j^2 \\ &= (4\lambda+10)x^2 + (2\lambda^2 - 6\lambda - 25)x + (-\lambda^2 + 2\lambda + 15) \end{aligned} \tag{8}$$

where $1 \leq x \leq 3$.

In solving equation (8), we assume $S_t^o(\lambda, x, 2) = c_1(\lambda)x^2 + c_2(\lambda)x + c_3(\lambda)$, where $c_1(\lambda), c_2(\lambda), c_3(\lambda) \in \mathbb{Z}[\lambda]$. In view of Theorem 2.2, we simulate x from 1 to 3 and obtain the following

$$\begin{cases} c_1(\lambda) + c_2(\lambda) + c_3(\lambda) = \lambda^2 \\ 4c_1(\lambda) + 2c_2(\lambda) + c_3(\lambda) = \displaystyle\sum_{j=\lambda}^{\lambda+2} j^2 \\ 9c_1(\lambda) + 3c_2(\lambda) + c_3(\lambda) = \displaystyle\sum_{j=\lambda}^{\lambda+4} j^2 \end{cases}$$

So, we get the following solution

$$\begin{cases} c_1(\lambda) = 4\lambda + 10 \\ c_2(\lambda) = 2\lambda^2 - 6\lambda - 25 \\ c_3(\lambda) = -\lambda^2 + 2\lambda + 15 \end{cases}$$

Hence, equation (8) holds and is valid for $1 \leq x \leq 3$.

Example 2. Solving for $S_5^o(2, 3, 2)$. Applying equation (8), we get

$$S_5^o(2,3,2) = \sum_{j=2}^{6} j^2 = 90.$$

Illustration 2.3. Considering that $p = 3$ and t be an odd natural number, then, we obtain

$$\begin{aligned} S_t^o(\lambda,x,3) &= \sum_{j=\lambda}^{\lambda+2x-2} j^3 \\ &= (8\lambda+28)x^3 + (6\lambda^2 - 18\lambda - 127)x^2 + (2\lambda^3 - 9\lambda^2 + 13\lambda + 194)x \\ &\quad + (-\lambda^3 + 3\lambda^2 - 3\lambda - 95) \end{aligned} \tag{9}$$

where $1 \le x \le 4$. Again, to solve for equation (9), let $S_t^o(\lambda,x,3) = c_1(\lambda)x^3 + c_2(\lambda)x^2 + c_3(\lambda)x + c_4(\lambda)$, where $c_1(\lambda), c_2(\lambda), c_3(\lambda), c_4(\lambda) \in \mathbb{Z}[\lambda]$. So, by Theorem 2.2, simulate for the following values $x = 1,2,3,4$, and we have

$$\begin{cases} c_1(\lambda) + c_2(\lambda) + c_3(\lambda) + c_4(\lambda) = \lambda^3 \\ 8c_1(\lambda) + 4c_2(\lambda) + 2c_3(\lambda) + c_4(\lambda) = \sum_{j=\lambda}^{\lambda+2} j^3 \\ 27c_1(\lambda) + 9c_2(\lambda) + 3c_3(\lambda) + c_4(\lambda) = \sum_{j=\lambda}^{\lambda+4} j^3 \\ 64c_1(\lambda) + 16c_2(\lambda) + 4c_3(\lambda) + c_4(\lambda) = \sum_{j=\lambda}^{\lambda+6} j^3 \end{cases}$$

In that case, we obtain the following solution

$$\begin{cases} c_1(\lambda) = 8\lambda + 28 \\ c_2(\lambda) = 6\lambda^2 - 18\lambda - 127 \\ c_3(\lambda) = 2\lambda^3 - 9\lambda^2 + 13\lambda + 194 \\ c_4(\lambda) = -\lambda^3 + 3\lambda^2 - 3\lambda - 95 \end{cases}$$

Thus, equation (9) is obtained and it is valid for $1 \le x \le 4$.

Example 3. Solving for $S_7^o(5,4,3)$. Applying equation (9), we have

$$\begin{aligned} S_7^o(5,4,3) &= \sum_{j=5}^{11} j^3 = 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 + 11^3 \\ &= [8(5)+28](4)^3 + [6(5)^2 - 18(5) - 127](4)^2 + [2(5)^3 - 9(5)^2 \\ &\quad + 13(5) + 194](4) + [-(5)^3 + 3(5)^2 - 3(5) - 95] \\ &= 4256. \end{aligned}$$

Secondly, the following result is also quick from equation (4).

Theorem 2.2. *Let λ, x, and p be natural numbers. If $S_t^e(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-1} j^p$ and $x = \frac{t}{2} \leq p+1$ and $t \equiv 0 (mod\ 2)$, then $S_t^e(\lambda, x, p) = \sum_{j=0}^{p} d_j(\lambda) x^j$ where $d_j(\lambda) \in \mathbb{Z}[\lambda]$.*

Proof. To prove the above Theorem 2.2, we simulate $S_t^e(\lambda, x, p)$ for the following values of $x = 1, 2, \ldots, p, p+1$. Then, we get

$$\begin{cases} S_t^e(\lambda,1,p) = \sum_{j=0}^{p} d_j(\lambda)1^j = \sum_{j=0}^{1}(\lambda+j)^p \\ S_t^e(\lambda,2,p) = \sum_{j=0}^{p} d_j(\lambda)2^j = \sum_{j=0}^{3}(\lambda+j)^p \\ S_t^e(\lambda,3,p) = \sum_{j=0}^{p} d_j(\lambda)3^j = \sum_{j=0}^{5}(\lambda+j)^p \\ \quad . \qquad\qquad . \qquad\qquad . \\ \quad . \qquad\qquad . \qquad\qquad . \\ \quad . \qquad\qquad . \qquad\qquad . \\ S_t^e(\lambda,p+1,p) = \sum_{j=0}^{p} d_j(\lambda)(p+1)^j = \sum_{j=0}^{2p+1}(\lambda+j)^p \end{cases} \tag{10}$$

The number of unknowns and the number of equations is equal in the system of equation (10) that has no linear dependence subsist. Hence, it has a unique solution. Solving the system, we get

$$\begin{cases} d_0(\lambda) = P_0(\lambda) \in \mathbb{Z}[\lambda] \\ d_1(\lambda) = P_1(\lambda) \in \mathbb{Z}[\lambda] \\ d_2(\lambda) = P_2(\lambda) \in \mathbb{Z}[\lambda] \\ \qquad . \\ \qquad . \\ \qquad . \\ d_p(\lambda) = P_p(\lambda) \in \mathbb{Z}[\lambda] \end{cases} \tag{11}$$

where $p, \lambda \in \mathbb{Z}^+$. Again, we have to note that the simulation is from $x = 1$ to $x = p+1$, thus, the polynomial $S_t^e(\lambda, x, p) = \sum_{j=0}^{p} d_j(\lambda) x^j$ only works for $1 \leq x \leq p+1$. And this completes the proof. □

Some illustrations and examples (for $p = 1, 2, 3$) are provided below to exemplify the above Theorem.

Illustration 2.4. Considering that $p = 1$ and t is an even natural number, then, we have

$$S_t^e(\lambda, x, 1) = \sum_{j=\lambda}^{\lambda+2x-1} j = (2\lambda+5)x - 4 \tag{12}$$

where $1 \leq x \leq 2$.In solving equation (12), we let $S_t^e(\lambda, x, 1) = d_1(\lambda)x + d_2(\lambda)$, where $d_1(\lambda) \in \mathbb{Z}[\lambda]$ and $d_2(\lambda) \in \mathbb{Z}[\lambda]$. In view of Theorem 2.2, simulate x from 1 to 2 and we obtain

$$\begin{cases} d_1(\lambda) + d_2(\lambda) = \sum_{j=\lambda}^{\lambda+1} j \\ 2d_1(\lambda) + d_2(\lambda) = \sum_{j=\lambda}^{\lambda+3} j \end{cases}$$

Clearly, we get

$$\begin{cases} d_1(\lambda) = 2\lambda + 5 \\ d_2(\lambda) = -4 \end{cases}$$

Hence, we obtain equation (12) that is valid for $1 \leq x \leq 2$.

Example 4. Solving for $S_4^e(2, 2, 1)$. Applying equation (12), we get

$$S_4^e(2, 2, 1) = \sum_{j=2}^{5} j = 14.$$

Illustration 2.5. Considering that $p = 2$ and t is an even positive integer, then, we get

$$S_t^e(\lambda, x, 2) = \sum_{j=\lambda}^{\lambda+2x-1} j^2 = (4\lambda + 14)x^2 + (2\lambda^2 - 2\lambda - 29)x + 16 \tag{13}$$

where $1 \leq x \leq 3$. To solve equation (13) above, assume that $S_t^e(\lambda, x, 2) = d_1(\lambda)x^2 + d_2(\lambda)x + d_3(\lambda)$, where $d_1(\lambda), d_2(\lambda), d_3(\lambda) \in \mathbb{Z}[\lambda]$. So, by Theorem 2.2, we simulate $S_t^e(\lambda, x, 2)$ for the following values of variable $x = 1, 2, 3$, and obtain

$$\begin{cases} d_1(\lambda) + d_2(\lambda) + d_3(\lambda) = \sum_{j=\lambda}^{\lambda+1} j^2 \\ 4d_1(\lambda) + 2d_2(\lambda) + d_3(\lambda) = \sum_{j=\lambda}^{\lambda+3} j^2 \\ 9d_1(\lambda) + 3d_2(\lambda) + d_3(\lambda) = \sum_{j=\lambda}^{\lambda+5} j^2 \end{cases}$$

In that case, we get

$$\begin{cases} d_1(\lambda) = 4\lambda + 14 \\ d_2(\lambda) = 2\lambda^2 - 2\lambda - 29 \\ d_3(\lambda) = 16 \end{cases}$$

Hence, it is clear that equation (13) holds and works for $1 \leq x \leq 3$.

Example 5. Solving for $S_4^e(2, 2, 2)$. Applying equation (13), we have

JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 5 NO. 2 (NOV 2022)
Available online at www.jfma.math.fsm.undip.ac.id

$$S_4^e(2,2,2) = \sum_{j=2}^{5} j^2 = 2^2 + 3^2 + 4^2 + 5^2$$
$$= [4(2)+14](2)^2 + [2(2)^2 - 2(2) - 29](2) + 16$$
$$= 54.$$

Illustration 2.6. Consider $p = 3$ and t is an even positive integer, then, we get

$$S_t^e(\lambda, x, 3) = \sum_{j=\lambda}^{\lambda+2x-1} j^3$$
$$= (8\lambda + 36)x^3 + (6\lambda^2 - 6\lambda - 139)x^2 + (2\lambda^3 - 3\lambda^2 + \lambda + 200)x - 96 \tag{14}$$

where $1 \leq x \leq 4$. To solve for equation (14), assume that $S_t^e(\lambda, x, 3) = d_1(\lambda)x^3 + d_2(\lambda)x^2 + d_3(\lambda)x + d_4(\lambda)$, where $d_1(\lambda), d_2(\lambda), d_3(\lambda), d_4(\lambda) \in \mathbb{Z}[\lambda]$. In the position of Theorem 2.2, we simulate $S_t^e(\lambda, x, 3)$ for $x = 1, 2, 3, 4$, and we obtain the following system of equation

$$\begin{cases} d_1(\lambda) + d_2(\lambda) + d_3(\lambda) + d_4(\lambda) = \sum_{j=\lambda}^{\lambda+1} j^3 \\ 8d_1(\lambda) + 4d_2(\lambda) + 2d_3(\lambda) + d_4(\lambda) = \sum_{j=\lambda}^{\lambda+3} j^3 \\ 27d_1(\lambda) + 9d_2(\lambda) + 3d_3(\lambda) + d_4(\lambda) = \sum_{j=\lambda}^{\lambda+5} j^3 \\ 64d_1(\lambda) + 16d_2(\lambda) + 4d_3(\lambda) + d_4(\lambda) = \sum_{j=\lambda}^{\lambda+7} j^3 \end{cases}$$

By elimination, we get the following solution

$$\begin{cases} d_1(\lambda) = 8\lambda + 36 \\ d_2(\lambda) = 6\lambda^2 - 6\lambda - 139 \\ d_3(\lambda) = 2\lambda^3 - 3\lambda^2 + \lambda + 200 \\ d_4(\lambda) = -96 \end{cases}$$

Hence, equation (14) holds and is valid for $1 \leq x \leq 4$.

Example 6. Consider $S_8^e(5, 4, 3)$. Then, applying equation (14), we have

$$S_8^e(5,4,3) = \sum_{j=5}^{12} j^3 = 5984.$$

III. CONCLUSION

In this paper, a new polynomial using the simulation method has been developed for power sums' solution, that is, $S_t^o(\lambda, x, p)$, and $S_t^e(\lambda, x, p)$ where $\lambda, x, p \in \mathbb{Z}^+$ and works for $1 \leq x \leq p+1$. Conclusively, as p increases, then the number of terms also increases, i.e., $p \to \infty$ implies $x, t \to \infty$, and $S_t^o(\lambda, x, p) \to \infty$ and $S_t^e(\lambda, x, p) \to \infty$. Moreover, for all values of $\lambda, x, p \in \mathbb{Z}^+$, we have $S_t^o(\lambda, x, p) < S_t^e(\lambda, x, p)$. As for future research, one may consider evaluating the mathematical characteristics (in view of calculus) of the developed polynomial solution for power sums to assess the efficacy of this current paper.

REFERENCES

[1] Bazsó, A., "Polynomial values of (alternating) power sums," *Acta Mathematica Hungarica*, vol. 146, no. 1, pp. 202-219, 2015.

[2] Casinillo, L. F., & Mamolo, L. A., "Alternative Formula for the Series of Consecutive m-Squares under Alternating Signs," *InPrime: Indonesian Journal of Pure and Applied Mathematics*, vol. 2, no. 2, pp. 91-96, 2020.

[3] Casinillo, L. F., "Quadratic polynomial of power sums and alternating power sums," *Journal of Fundamental Mathematics and Applications (JFMA)*, vol. 4, no. 2, pp. 147-153, 2021.

[4] Casinillo, L. F., & Abas, C. L., "Cubic Polynomial for the Series of Consecutive Cubes under Alternating Signs," *InPrime: Indonesian Journal of Pure and Applied Mathematics*, vol. 3, no. 2, pp. 86-91, 2021.

[5] Merca, M., "An alternative to Faulhaber's formula," *The American Mathematical Monthly*, vol. 122, no. 6, pp. 599-601, 2015.

[6] de Bruyn, G. F. C. and de Villiers, J. M., "Formulas for 1^p+2^p+... +n^p.," *The Fibonacci Quarterly,* vol. 32, no. 3, pp. 271-276, 1994.

[7] Howard, F. T., "Sums of powers of integers via generating functions," *Fibonacci Quarterly*, vol. 34, pp. 244-256, 1996.

[8] Kalman, D., "Sums of Powers by Matrix Methods," *The Fibonacci Quarterly,* vol. 28, no. 1, pp. 60- 71, 1990.

[9] Turner, B., "Sums of Powers of Integers via the Binomial Theorem," *Mathematics Magazine, vol.* 53, pp. 92-96, 1980.

[10] Casinillo, L. F., "Another look at a polynomial solution for alternating power sums," *Journal of Fundamental Mathematics and Applications (JFMA)*, vol. 5, no. 1, pp. 16-22, 2021.