

Theorem 2.2. Let λ , x , and p be natural numbers. If $S_t^e(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-1} j^p$ and $x = \frac{t}{2} \leq p + 1$ and $t \equiv 0 \pmod{2}$, then $S_t^e(\lambda, x, p) = \sum_{j=0}^p d_j(\lambda)x^j$ where $d_j(\lambda) \in \mathbb{Z}[\lambda]$.

Proof. To prove the above Theorem 2.2, we simulate $S_t^e(\lambda, x, p)$ for the following values of $x = 1, 2, \dots, p, p + 1$. Then, we get

$$\left\{ \begin{array}{l} S_t^e(\lambda, 1, p) = \sum_{j=0}^p d_j(\lambda)1^j = \sum_{j=0}^1 (\lambda + j)^p \\ S_t^e(\lambda, 2, p) = \sum_{j=0}^p d_j(\lambda)2^j = \sum_{j=0}^3 (\lambda + j)^p \\ S_t^e(\lambda, 3, p) = \sum_{j=0}^p d_j(\lambda)3^j = \sum_{j=0}^5 (\lambda + j)^p \\ \vdots \\ S_t^e(\lambda, p + 1, p) = \sum_{j=0}^p d_j(\lambda)(p + 1)^j = \sum_{j=0}^{2p+1} (\lambda + j)^p \end{array} \right. \quad (10)$$

The number of unknowns and the number of equations is equal in the system of equation (10) that has no linear dependence subsist. Hence, it has a unique solution. Solving the system, we get

$$\left\{ \begin{array}{l} d_0(\lambda) = P_0(\lambda) \in \mathbb{Z}[\lambda] \\ d_1(\lambda) = P_1(\lambda) \in \mathbb{Z}[\lambda] \\ d_2(\lambda) = P_2(\lambda) \in \mathbb{Z}[\lambda] \\ \vdots \\ d_p(\lambda) = P_p(\lambda) \in \mathbb{Z}[\lambda] \end{array} \right. \quad (11)$$

where $p, \lambda \in \mathbb{Z}^+$. Again, we have to note that the simulation is from $x = 1$ to $x = p + 1$, thus, the polynomial $S_t^e(\lambda, x, p) = \sum_{j=0}^p d_j(\lambda)x^j$ only works for $1 \leq x \leq p + 1$. And this completes the proof. \square

Some illustrations and examples (for $p = 1, 2, 3$) are provided below to exemplify the above Theorem.

Illustration 2.4. Considering that $p = 1$ and t is an even natural number, then, we have

$$S_t^e(\lambda, x, 1) = \sum_{j=\lambda}^{\lambda+2x-1} j = (2\lambda + 5)x - 4 \quad (12)$$

where $1 \leq x \leq 2$. In solving equation (12), we let $S_t^e(\lambda, x, 1) = d_1(\lambda)x + d_2(\lambda)$, where $d_1(\lambda) \in \mathbb{Z}[\lambda]$ and $d_2(\lambda) \in \mathbb{Z}[\lambda]$. In view of Theorem 2.2, simulate x from 1 to 2 and we obtain

$$\begin{cases} d_1(\lambda) + d_2(\lambda) = \sum_{j=\lambda}^{\lambda+1} j \\ 2d_1(\lambda) + d_2(\lambda) = \sum_{j=\lambda}^{\lambda+3} j \end{cases}$$

Clearly, we get

$$\begin{cases} d_1(\lambda) = 2\lambda + 5 \\ d_2(\lambda) = -4 \end{cases}$$

Hence, we obtain equation (12) that is valid for $1 \leq x \leq 2$.

Example 4. Solving for $S_4^e(2, 2, 1)$. Applying equation (12), we get

$$S_4^e(2, 2, 1) = \sum_{j=2}^5 j = 14.$$

Illustration 2.5. Considering that $p = 2$ and t is an even positive integer, then, we get

$$S_t^e(\lambda, x, 2) = \sum_{j=\lambda}^{\lambda+2x-1} j^2 = (4\lambda + 14)x^2 + (2\lambda^2 - 2\lambda - 29)x + 16 \quad (13)$$

where $1 \leq x \leq 3$. To solve equation (13) above, assume that $S_t^e(\lambda, x, 2) = d_1(\lambda)x^2 + d_2(\lambda)x + d_3(\lambda)$, where $d_1(\lambda), d_2(\lambda), d_3(\lambda) \in \mathbb{Z}[\lambda]$. So, by Theorem 2.2, we simulate $S_t^e(\lambda, x, 2)$ for the following values of variable $x = 1, 2, 3$, and obtain

$$\begin{cases} d_1(\lambda) + d_2(\lambda) + d_3(\lambda) = \sum_{j=\lambda}^{\lambda+1} j^2 \\ 4d_1(\lambda) + 2d_2(\lambda) + d_3(\lambda) = \sum_{j=\lambda}^{\lambda+3} j^2 \\ 9d_1(\lambda) + 3d_2(\lambda) + d_3(\lambda) = \sum_{j=\lambda}^{\lambda+5} j^2 \end{cases}$$

In that case, we get

$$\begin{cases} d_1(\lambda) = 4\lambda + 14 \\ d_2(\lambda) = 2\lambda^2 - 2\lambda - 29 \\ d_3(\lambda) = 16 \end{cases}$$

Hence, it is clear that equation (13) holds and works for $1 \leq x \leq 3$.

Example 5. Solving for $S_4^e(2, 2, 2)$. Applying equation (13), we have

$$\begin{aligned}
 S_4^e(2, 2, 2) &= \sum_{j=2}^5 j^2 = 2^2 + 3^2 + 4^2 + 5^2 \\
 &= [4(2) + 14](2)^2 + [2(2)^2 - 2(2) - 29](2) + 16 \\
 &= 54.
 \end{aligned}$$

Illustration 2.6. Consider $p = 3$ and t is an even positive integer, then, we get

$$\begin{aligned}
 S_t^e(\lambda, x, 3) &= \sum_{j=\lambda}^{\lambda+2x-1} j^3 \\
 &= (8\lambda + 36)x^3 + (6\lambda^2 - 6\lambda - 139)x^2 + (2\lambda^3 - 3\lambda^2 + \lambda + 200)x \\
 &\quad - 96
 \end{aligned} \tag{14}$$

where $1 \leq x \leq 4$. To solve for equation (14), assume that $S_t^e(\lambda, x, 3) = d_1(\lambda)x^3 + d_2(\lambda)x^2 + d_3(\lambda)x + d_4(\lambda)$, where $d_1(\lambda), d_2(\lambda), d_3(\lambda), d_4(\lambda) \in \mathbb{Z}[\lambda]$. In the position of Theorem 2.2, we simulate $S_t^e(\lambda, x, 3)$ for $x = 1, 2, 3, 4$, and we obtain the following system of equation

$$\left\{ \begin{aligned}
 d_1(\lambda) + d_2(\lambda) + d_3(\lambda) + d_4(\lambda) &= \sum_{j=\lambda}^{\lambda+1} j^3 \\
 8d_1(\lambda) + 4d_2(\lambda) + 2d_3(\lambda) + d_4(\lambda) &= \sum_{j=\lambda}^{\lambda+3} j^3 \\
 27d_1(\lambda) + 9d_2(\lambda) + 3d_3(\lambda) + d_4(\lambda) &= \sum_{j=\lambda}^{\lambda+5} j^3 \\
 64d_1(\lambda) + 16d_2(\lambda) + 4d_3(\lambda) + d_4(\lambda) &= \sum_{j=\lambda}^{\lambda+7} j^3
 \end{aligned} \right.$$

By elimination, we get the following solution

$$\left\{ \begin{aligned}
 d_1(\lambda) &= 8\lambda + 36 \\
 d_2(\lambda) &= 6\lambda^2 - 6\lambda - 139 \\
 d_3(\lambda) &= 2\lambda^3 - 3\lambda^2 + \lambda + 200 \\
 d_4(\lambda) &= -96
 \end{aligned} \right.$$

Hence, equation (14) holds and is valid for $1 \leq x \leq 4$.

Example 6. Consider $S_8^e(5, 4, 3)$. Then, applying equation (14), we have

$$S_8^e(5, 4, 3) = \sum_{j=5}^{12} j^3 = 5984.$$

III. CONCLUSION

In this paper, a new polynomial using the simulation method has been developed for power sums' solution, that is, $S_t^o(\lambda, x, p)$, and $S_t^e(\lambda, x, p)$ where $\lambda, x, p \in \mathbb{Z}^+$ and works for $1 \leq x \leq p + 1$. Conclusively, as p increases, then the number of terms also increases, i.e., $p \rightarrow \infty$ implies $x, t \rightarrow \infty$, and $S_t^o(\lambda, x, p) \rightarrow \infty$ and $S_t^e(\lambda, x, p) \rightarrow \infty$. Moreover, for all values of $\lambda, x, p \in \mathbb{Z}^+$, we have $S_t^o(\lambda, x, p) < S_t^e(\lambda, x, p)$. As for future research, one may consider evaluating the mathematical characteristics (in view of calculus) of the developed polynomial solution for power sums to assess the efficacy of this current paper.

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