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ROUGH RINGS, ROUGH SUBRINGS, AND ROUGH IDEALS

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Abstract. The basic concept in algebra namely set theory can be expanded into rough set theory. Basic operations on the set such as intersections, unions, differences, and complements still apply to rough sets. In addition, one of the applications on rough sets is the use of rough matrices in decision-making processes. Furthermore, mathematical or informatic researchers who work on rough sets connect the concept of rough sets with algebraic structures (e.g. groups, rings, and modules) so that a concept called rough algebraic structures is obtained. Since the research related to rough sets is mostly carried out at the same time, different concepts have emerged related to rough sets and rough algebraic structures. In this paper, new definitions of the rough ring and rough subring will be given along with related examples and theorems. Furthermore, it will also be defined the left ideal and the right ideal of the rough ring along with examples. Finally, we will discuss the theorem regarding rough ideals.
Keywords: Rough Sets, Rough Rings, Rough Subrings, Rough Ideals

I. INTRODUCTION AND BASIC THEORY

In 1982, Pawlak introduced the notion of rough set [1]. Before we consider the definition of a rough set, we need to consider the definition of approximation space, equivalence relation, lower and upper approximation, and boundary.

Definition 1.1 [1] *Suppose U is a universe and non-empty set and β is an equivalence relation on U. The set (U, β) is said to be an approximation space.*

Definition 1.2 [1] *Suppose U is a universe and β be an equivalence relation on U. We denote the equivalence class of object x in R by $[x]_\beta$.*

Definition 1.3 [1] *Suppose (U, β) is an approximation space and $\mathcal{X}$ is a subset of U. The sets $\overline{\mathcal{X}} = \{x \big| [x]_\beta \cap \mathcal{X} \neq \emptyset\}$, $\underline{\mathcal{X}} = \{x | [x]_\beta \subseteq \mathcal{X}\}$, and $Bnd(\mathcal{X}) = \overline{\mathcal{X}} - \underline{\mathcal{X}}$ are called upper approximation, lower approximation, and boundary of $\mathcal{X}$. In other reference like [2] and [3], the notation of lower and upper approximation are $\overline{Apr}(\mathcal{X})$ and $\underline{Apr}(\mathcal{X})$ respectively.*

Definition 1.4 [1] *Let U is an approximation space and $\mathcal{X}$ is a subset of U. $\mathcal{X}$ is called a rough set in (U, β) if and only if $Bnd(\mathcal{X}) \neq \emptyset$.*

To clarify Definition 1.4, an example will be formed as an explanation of Definition 1.4.

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Example 1.5 Let $U = \{m_1, m_2, m_3, m_4, m_5, m_6\}$ and the equivalence class of U are

$$\begin{cases} \xi_1 = \{m_1, m_3, m_5\} \\ \xi_2 = \{m_2, m_4\} \\ \xi_3 = \{m_6\} \end{cases}$$

Furthermore, define $\mathcal{X} = \{m_1, m_2\}$. Then, we get $\overline{\mathcal{X}} = \{m_1, m_2, m_3, m_4, m_5\}$ and $\underline{\mathcal{X}} = \emptyset$. Because $Bnd(\mathcal{X}) \neq \emptyset$, then we can deduce that $\mathcal{X}$ is a rough set.

Furthermore, we have another definition of a rough set.

Definition 1.6 [2] *Let U is an approximation space. A set $(A, B) \in P(U) \times P(U)$ is said to be a rough set in (U, β) if and only if $(A, B) = Apr(\mathcal{X})$ for some $Y \in P(U)$.*

Note that, the difference is in Definition 1.4, a rough set is $\mathcal{X}$ and in Definition 1.6, a rough set is $Apr(\mathcal{X})$.

Furthermore, there is an application of a rough set. One of the application is in decision-making using rough matrix [4],[5]. Furthermore, in the algebraic structure, we are familiar with the concept of groups, rings, and modules [6],[7]. By linking a concept of a rough set and algebraic structure, then we get a concept of algebraic structure with a rough set called rough algebraic structure. Furthermore, we will remembrance the definition of a rough group.

Definition 1.7 [8],[9] *Assume $K = (U, \beta)$ be an approximation space and $*$ be a binary operation defined on U. Then, $\mathcal{G} \subset U$ is called a rough group if the following properties are satisfied:*
1. *For every $p, q \in \mathcal{G}$, then $p * q \in \overline{\mathcal{G}}$;*
2. *For every $p, q, r \in \mathcal{G}$, then $p * (q * r) = (p * q) * r$ or associative property holds in $\overline{\mathcal{G}}$;*
3. *There exists $e \in \overline{\mathcal{G}}$ such that for every $p \in \mathcal{G}$, then $p * e = e * p = p$;*
4. *For every $p \in \mathcal{G}$, there exists $q \in \mathcal{G}$ such that $p * q = q * p = e$.*

Furthermore, we have the theorem of subset J is said to be a rough subgroup of $\mathcal{G}$.

Theorem 1.8 [8] *Suppose $\mathcal{J} \subseteq \mathcal{G}$ where $\mathcal{G}$ is a rough group. Then, $\mathcal{J}$ is said to be a rough subgroup if this condition is satisfied:*
1. *For every $p, q \in \mathcal{J}$, then $p * q \in \overline{\mathcal{J}}$;*
2. *For every $p \in \mathcal{J}$, then $p^{-1} \in \overline{\mathcal{J}}$.*

Next, we have the definition of rough semigroup

Definition 1.9 [10] *Let $K = (U, \beta)$ be an approximation space and $(*)$ be a binary operation defined on U. A set $T \subseteq U$ is said to be a rough semigroup on approximation space if these conditions is satisfied:*
1. *For every $p, q \in T$, $x * y \in \overline{T}$;*
2. *For every $p, q, r \in T$, $(p * q) * r = p * (q * r)$.*

Furthermore, we have the definition of a rough ring.

Definition 1.10 [3],[11] *Let $" + "$ and $" * "$ are operation in $Apr(\mathcal{R})$. Then, $(Apr(\mathcal{R}), +, *)$ is called a rough ring if these conditions is satisfied:*
1. *$(Apr(\mathcal{R}), +)$ is a rough commutative addition group;*

2. *$(Apr(\mathcal{R}),*)$ is a rough semigroup;*
3. *For every $p, q, r \in Apr(\mathcal{R})$, then $(p+q)*r = p*q + q*r$ and $p*(q+r) = p*q + p*r$.*

Next, in rough algebraic structure, we also have a definition of rough modules [12] and rough G-modules [13]. Furthermore, we can also associate the concept of vector spaces with rough sets. Thus, we have a concept namely rough vector spaces [14].

In this paper, we will modify Definition 1.10 by replacing rough rings $Apr(\mathcal{R})$ with rough rings R and improve its distributive properties. Furthermore, based on a new definition of rough rings R, we will define rough subrings and give some examples of rough rings and rough subrings. Furthermore, we will also attest to the theorem related to subrings. Furthermore, based on a new definition of rough rings and rough subrings, we will define rough ideals and give some examples of rough ideals. Furthermore, we will attest that $\mathfrak{I}_1 \cap \mathfrak{I}_2$ and $\mathfrak{I}_1 + \mathfrak{I}_2$ are rough ideals in $\mathcal{R}$ and $(\mathfrak{I}_1 \cup \mathfrak{I}_2) \subseteq \mathfrak{I}_1 + \mathfrak{I}_2$ but the opposite is not necessarily the case. For more details, the research flow is shown in the following research-resistant diagram.

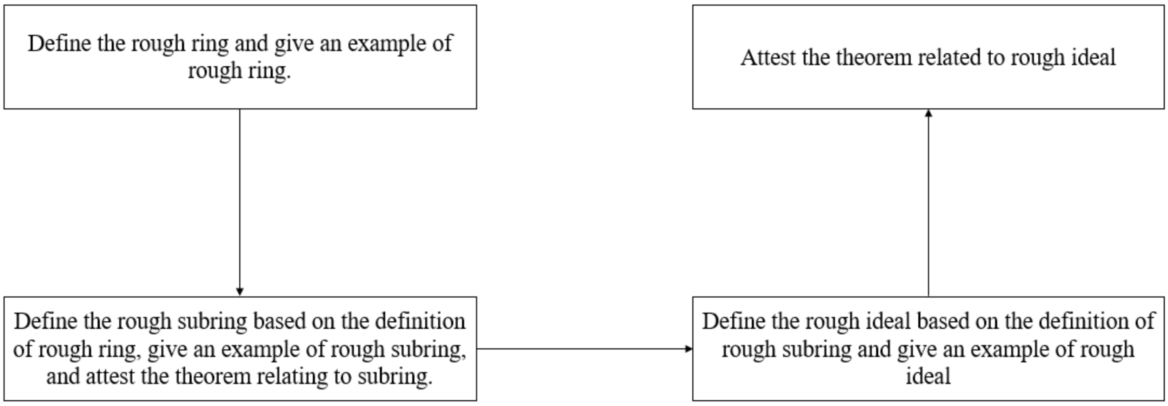

Figure 1. Research-resistant diagram

II. RESULT AND DISCUSSION

In this part, we will discuss another definition and notion of rough rings, rough subrings, and rough ideals.

Definition 2.1 *Suppose $\mathcal{R}$ is a rough set. Define the operation in $\mathcal{R}$ as " $+$ " and " $*$ " with $+$ and $*$ are addition and multiplication in $\mathcal{R}$ respectively. Then, the algebraic system $(\mathcal{R}, +, *)$ is said to be a rough ring if all condition below are satisfied:*
1. *$(\mathcal{R}, +)$ is a rough commutative group;*
2. *$(\mathcal{R}, *)$ is a rough semigroup or $\mathcal{R}$ satisfied associative property;*
3. *For every $p, q, r \in \mathcal{R}$, then $(p+q)*r = p*r + q*r$ and $p*(q+r) = p*q + p*r$ holds in $\overline{\mathcal{R}}$.*

If we compared it to Definition 1.10, there is a slight difference from Definition 2.1. The difference is that, Definition 1.10 uses $Apr(\mathcal{R})$ as a set. Meanwhile, Definition 2.1 uses $\mathcal{R}$ as a set. Furthermore, the right and left distributive properties in Definition 1.10 simply apply without any further explanation. Whereas, in Definition 2.1 the right and left distributive properties should hold in $\overline{\mathcal{R}}$.

Example 2.2 Let $U = \mathbb{Z}_{20}$. For every $a_1, a_2 \in U$, Define an equivalence relation $a_1 - a_2 = 3k$ for some $k \in \mathbb{R}$. Then, the equivalence class of U are

$$\xi_1 = \{\overline{1}, \overline{4}, \overline{7}, \overline{10}, \overline{13}, \overline{16}, \overline{19}\}$$
$$\xi_2 = \{\overline{2}, \overline{5}, \overline{8}, \overline{11}, \overline{14}, \overline{17}\}$$
$$\xi_3 = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}\}$$

Furthermore, let $\mathcal{R} = \{\overline{0}, \overline{1}, \overline{3}, \overline{5}, \overline{7}, \overline{9}, \overline{11}\}$. Then we obtain lower approximation and upper approximation of Y are $\underline{\mathcal{R}} = \emptyset$ and $\overline{\mathcal{R}} = \xi_1 \cup \xi_2 \cup \xi_3$ respectively. Furthermore, because $Bnd(\mathcal{R}) \neq \emptyset$, we can deduce that $\mathcal{R}$ is a rough set. Furthermore, define an operation $+_{20}$ (sum of integer modulo 20) in $\mathcal{R}$ Here is a table of set $\mathcal{R}$ with operation $+_{20}$.

Table 1. operation on element $\mathcal{R}$ with operation $+_{20}$

$+_{20}$	$\overline{0}$	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{7}$	$\overline{9}$	$\overline{11}$	$\overline{13}$	$\overline{15}$	$\overline{17}$	$\overline{19}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{7}$	$\overline{9}$	$\overline{11}$	$\overline{13}$	$\overline{15}$	$\overline{17}$	$\overline{19}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{14}$	$\overline{16}$	$\overline{18}$	$\overline{0}$
$\overline{3}$	$\overline{3}$	$\overline{4}$	$\overline{6}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{14}$	$\overline{16}$	$\overline{18}$	$\overline{0}$	$\overline{2}$
$\overline{5}$	$\overline{5}$	$\overline{6}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{14}$	$\overline{16}$	$\overline{18}$	$\overline{0}$	$\overline{2}$	$\overline{4}$
$\overline{7}$	$\overline{7}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{14}$	$\overline{16}$	$\overline{18}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{6}$
$\overline{9}$	$\overline{9}$	$\overline{10}$	$\overline{12}$	$\overline{14}$	$\overline{16}$	$\overline{18}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{8}$
$\overline{11}$	$\overline{11}$	$\overline{12}$	$\overline{14}$	$\overline{16}$	$\overline{18}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{8}$	$\overline{10}$

We will indicate that $(\mathcal{R}, +_{20})$ is a rough commutative group.

1. For every $p, q \in \mathcal{R}$, then $p +_{20} q \in \overline{\mathcal{R}}$;
2. For every $p, q, r \in \mathcal{R}$, then $(p +_{20} q) +_{20} r = p +_{20} (q +_{20} r)$ or $\mathcal{R}$ is satisfied associative property;
3. There is rough identity element $e \in \overline{\mathcal{R}}$ that is $e = \overline{0}$ such that for each $x \in \mathcal{R}$, $e +_{20} p = p +_{20} e = p$;

Table 2. Inverse element of $\mathcal{R}$

$\overline{0}$	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{7}$	$\overline{9}$	$\overline{11}$	$\overline{13}$	$\overline{15}$	$\overline{17}$	$\overline{19}$
$\overline{0}$	$\overline{19}$	$\overline{17}$	$\overline{15}$	$\overline{13}$	$\overline{11}$	$\overline{9}$	$\overline{7}$	$\overline{5}$	$\overline{3}$	$\overline{1}$

4. Based on Table 2, For each $p \in \mathcal{R}$, there is $q = p^{-1} \in \mathcal{R}$ such that $p +_{20} q = e$;

5. For every $p, q \in \mathcal{R}$, $p +_{20} q = q +_{20} p$.

Thus, $(\mathcal{R}, +_{20})$ is a rough commutative group.

Next, we define the multiplication in $\mathcal{R}$ as $*_{20}$. We will attest that $\mathcal{R}$ is a rough ring.

1. For every $p, q, r \in \mathcal{R}$, then $p *_{20} (q *_{20} r) = (p *_{20} q) *_{20} r$ or $\mathcal{R}$ is a semigroup;

2. For every $p, q, r \in \mathcal{R}$, then $p *_{20} (q +_{20} r) = p *_{20} q + p *_{20} r$ and $(p +_{20} q) *_{20} r = p *_{20} r + q *_{20} r$ holds in $\bar{\mathcal{R}}$.

Thus, we can deduce that $(\mathcal{R}, +_{20}, *_{20})$ is a rough ring.

Furthermore, we will define a notion of rough subrings.

Definition 2.3 *Suppose $\mathcal{X}$ is a rough ring with $\mathcal{Z} \subseteq \mathcal{X}$. $\mathcal{Z}$ is said to be a rough subring of $\mathcal{X}$ if $\mathcal{Z}$ is a rough ring with the same operation as $\mathcal{X}$.*

On the other hand, we have a modified theorem to indicate that $\mathcal{Z}$ is a subring of $\mathcal{X}$. For the subring-related theorem, see [15].

Theorem 2.4 *Suppose $\mathcal{Z}$ is a nonempty rough subset of a rough ring $(\mathcal{R}, +, *)$. $\mathcal{Z}$ is called a rough subring of $\mathcal{R}$ if and only if for every $z_1, z_2 \in \mathcal{Z}$ the following condition is satisfied:*

1. $z_1 - z_2 \in \overline{\mathcal{Z}}$;

2. $z_1 * z_2 \in \overline{\mathcal{Z}}$

Proof: ($\Rightarrow$) We know that $\mathcal{Z}$ is a rough subring of $(\mathcal{R}, +, *)$. Based on the definition of rough subrings, $\mathcal{Z}$ is also a rough ring. Thus, we obtain that for every $z_1, z_2 \in \mathcal{Z}$, then $z_1 - z_2 \in \overline{\mathcal{Z}}$ and $z_1 * z_2 \in \overline{\mathcal{Z}}$.

($\Leftarrow$) For every $z_1, z_2 \in \mathcal{Z}$, $z_1 - z_2 \in \overline{\mathcal{Z}}$ and $z_1 * z_2 \in \overline{\mathcal{Z}}$. Because for all $z_1, z_2 \in \mathcal{Z}$, the first condition shows that $(\mathcal{Z}, +)$ is a rough subgroup of $(\mathcal{R}, +)$. Because $(Z, +)$ is a rough subgroup of $(\mathcal{R}, +)$, then $(\mathcal{Z}, +)$ is a group. Furthermore, $(Y, +)$ is a commutative group. Then, $(\mathcal{Z}, +)$ is a group. Next, the second condition shows that the operation $*$ is closed. Because $\mathcal{Z} \subseteq \mathcal{R}$, it is clear that the rough associative and distribution (right and left) are automatically fulfilled. Thus, we can deduce that $\mathcal{Z}$ is a rough subring in $\mathcal{R}$. ■

To clarify the meaning of Theorem 3.4, an example will be given as follows.

Example 2.5 Based on Example 3.2, a set $\mathcal{Z} = \{\overline{1}, \overline{3}, \overline{5}, \overline{15}, \overline{17}, \overline{19}\}$ is a rough set with $\mathcal{Z} \subseteq \mathcal{R}$. Thus, by definition, it is obvious that $\mathcal{Z}$ is a subring of Y. For every $z_1, z_2 \in \mathcal{Z}$, it is obvious that $z_1 - z_2 \in \overline{\mathcal{Z}}$. Furthermore, here is a multiplication table of $\mathcal{Z}$.

Table 3. Multiplication table of set Y with operation $*_{20}$

$*_{20}$	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{15}$	$\overline{17}$	$\overline{19}$
$\overline{1}$	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{16}$	$\overline{18}$	$\overline{0}$
$\overline{3}$	$\overline{3}$	$\overline{9}$	$\overline{15}$	$\overline{18}$	$\overline{0}$	$\overline{2}$
$\overline{5}$	$\overline{5}$	$\overline{15}$	$\overline{5}$	$\overline{0}$	$\overline{2}$	$\overline{4}$

Based on Table 3, we can deduce that $z_1 * z_2 \in \overline{\mathcal{Z}}$. Thus, $\mathcal{Z}$ is a subring of $\mathcal{R}$.

Furthermore, define rough ideals based on the definition of rough rings and rough subrings. Before we define rough ideals, we will remembrance the definition of ideal in the ring.

Definition 2.6 [15],[16] S*uppose R is a nonzero ring and I is a nonempty subset of R. We say that I is a rough ideal of R if:*

1. For every $j, k \in I$, $j - k \in I$;

2. *For every $j \in I$ and $r \in R$, $jr, rj \in I$.*

Motivated by the definition of ideal in the ring, then the rough ideals on the rough rings are defined as follows.

Definition 2.7 *Suppose that R is a nonzero rough ring and $\mathfrak{I}$ is a rough nonempty subset of $\mathcal{R}$. $\mathfrak{I}$ is said to be the rough ideal of $\mathcal{R}$ if:*
1. *For every $j, k \in \mathfrak{I}$, $j - k \in \overline{\mathfrak{I}}$;*
2. *For every $j \in \mathfrak{I}$ and $r \in R$, $jr, rj \in \overline{\mathfrak{I}}$.*

In general, rings are not required to be commutative. Based on this case, we can define left-rough ideals and right-rough ideals.

Definition 2.8 *Suppose that $\mathcal{R}$ is a rough ring and $\mathfrak{I} \subseteq \mathcal{R}$. A subset $\mathfrak{I}$ is said to be left-rough ideal in $\mathcal{R}$ if*
1. *For every $j, k \in \mathfrak{I}$, $j - k \in \overline{\mathfrak{I}}$;*
2. *For every $r \in \mathcal{R}$ and $j \in \mathfrak{I}$, $rj \in \overline{\mathfrak{I}}$.*

Furthermore, a subset $\mathfrak{I}$ is said to be right-rough ideal in $\mathcal{R}$ if
1. *For every $j, k \in \mathfrak{I}$, $j - k \in \overline{\mathfrak{I}}$;*
2. *For every $r \in \mathcal{R}$ and $j \in \mathfrak{I}$, $i_2 j \in \overline{\mathfrak{I}}$.*

To clarify the definition of rough ideals, an example will be given as follows.

Example 2.9 Let $U = \mathbb{Z}_{10}$. Given an equivalence relation $a - b = 2k$ for some $a, b \in U$ and $k \in \mathbb{R}$. Then, the equivalence class of U are

$$\xi_1 = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}, \overline{9}\}$$
$$\xi_2 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\}$$

Furthermore, let $\mathcal{R} = \mathbb{Z}_5 = \{\overline{0}, \overline{1}, \overline{2}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}\}$. Hence, $\overline{\mathcal{R}} = \xi_1 \cup \xi_2$ and $\underline{\mathcal{R}} = \emptyset$. Because $\overline{\mathcal{R}} - \underline{\mathcal{R}} \neq \emptyset$, we can say that $\mathcal{R}$ is a rough set. Furthermore, it is obvious that $\mathcal{R}$ is a rough ring. Furthermore, define $\mathfrak{I} = \{\overline{0}, \overline{5}\} \subseteq \mathcal{R}$. Hence, $\overline{\mathfrak{I}} = \xi_1 \cup \xi_2$ and $\underline{\mathfrak{I}} = \emptyset$. Since $\overline{\mathfrak{I}} - \underline{\mathfrak{I}} \neq \emptyset$, we can say that $\mathfrak{I}$ is a rough set. Furthermore, we want to indicate that $\mathfrak{I}$ is a rough ideal. For every $j, k \in \mathfrak{I}$, then $j - k \in \overline{\mathfrak{I}}$. Furthermore, for every $j \in \overline{\mathfrak{I}}$ and $y \in \mathcal{R}$, we can attest that $jy, yj \in \overline{\mathfrak{I}}$. Thus, we can deduce that $\mathfrak{I}$ is a rough ideal of $\mathcal{R}$.

Furthermore, we have a theorem for rough ideals.

Theorem 2.10 *Suppose $\mathcal{R}$ is a rough ring. If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are rough ideals in R, then:*
1. *$\mathfrak{I}_1 \cap \mathfrak{I}_2$ is a rough ideal in $\mathcal{R}$ with sufficient condition $\overline{\mathfrak{I}}_1 \cap \overline{\mathfrak{I}}_2 = \overline{\mathfrak{I}_1 \cap \mathfrak{I}_2}$;*
2. *$\mathfrak{I}_1 + \mathfrak{I}_2 = \{j + k | j \in \mathfrak{I}_1$ and $k \in \mathfrak{I}_2\}$ is a rough ideal in $\mathcal{R}$;*
3. *$(\mathfrak{I}_1 \cup \mathfrak{I}_2) \subseteq \mathfrak{I}_1 + \mathfrak{I}_2$.*

Proof:

1. We will attest that $\mathfrak{I}_1 \cap \mathfrak{I}_2$ is a rough ideal in $\mathcal{R}$. Take $r \in \mathcal{R}$ and $j, k \in \mathfrak{I}_1 \cap \mathfrak{I}_2$ arbitrary. Because $j, k \in \mathfrak{I}_1 \cap \mathfrak{I}_2$, then $j, k \in \mathfrak{I}_1$ and $j, k \in \mathfrak{I}_2$. Furthermore, because $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are rough ideal in $\mathcal{R}$, then

$$\begin{cases} j - k \in \overline{\mathfrak{I}}_1, j - k \in \overline{\mathfrak{I}}_2 \\ rj \in \overline{\mathfrak{I}}_1, rj \in \overline{\mathfrak{I}}_2 \\ jr \in \overline{\mathfrak{I}}_1, jr \in \overline{\mathfrak{I}}_2 \end{cases}$$

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so, we get that $j - k \in \overline{\mathfrak{I}}_1 \cap \overline{\mathfrak{I}}_2$ and $rj, jr \in \overline{\mathfrak{I}}_1 \cap \overline{\mathfrak{I}}_2$. Furthermore, based on the statement, we know that $\overline{\mathfrak{I}}_1 \cap \overline{\mathfrak{I}}_2 = \overline{\mathfrak{I}_1 \cap \mathfrak{I}_2}$. So, we can deduce that $j - k \in \overline{\mathfrak{I}_1 \cap \mathfrak{I}_2}$ and $rj, jr \in \overline{\mathfrak{I}_1 \cap \mathfrak{I}_2}$. Thus, $\mathfrak{I}_1 \cap \mathfrak{I}_2$ is a rough ideal in $\mathcal{R}$. ■

2. We will attest that $\mathfrak{I}_1 + \mathfrak{I}_2$ is a rough ideal in $\mathcal{R}$. Take $r \in \mathcal{R}$ and $j, k \in \mathfrak{I}_1 + \mathfrak{I}_2$ arbitrarily. This is mean that $j = j_1 + k_1$ and $k = j_2 + k_2$ for each $j_1, k_1 \in \mathfrak{I}_1$ and $i_2, j_2 \in \mathfrak{I}_2$. Because $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are rough ideals in $\mathcal{R}$, then we have $j_1 - k_1 \in \overline{\mathfrak{I}}_1$ and $j_2 - k_2 \in \overline{\mathfrak{I}}_2$. Based on this, we have

$$p - q = (j_1 + k_1) - (j_2 + k_2) = (j_1 - j_2) + (k_1 - k_2) \in \overline{\mathfrak{I}}_1 + \overline{\mathfrak{I}}_2$$

Furthermore, because $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are rough ideals in $\mathcal{R}$, then we have $rj, jr \in \overline{\mathfrak{I}}_1$ and $rj, jr \in \overline{\mathfrak{I}}_2$. Based on this, we have

$$\begin{cases} rj = r(j_1 + j_2) = rj_1 + rj_2 \in \overline{\mathfrak{I}}_1 + \overline{\mathfrak{I}}_2 \\ jr = (j_1 + j_2)r = j_1 r + j_2 r \in \overline{\mathfrak{I}}_1 + \overline{\mathfrak{I}}_2 \end{cases}$$

Thus, $I_1 + I_2$ is an ideal in R. ■

3. We will attest that $(\mathfrak{I}_1 \cup \mathfrak{I}_2) \subseteq \mathfrak{I}_1 + \mathfrak{I}_2$. Take $p \in \mathfrak{I}_1 \cup \mathfrak{I}_2$ arbitrary. This is mean that $j \in \mathfrak{I}_1$ or $j \in \mathfrak{I}_2$. If $j \in \mathfrak{I}_1$, then because $0_{\mathcal{R}} \in \mathfrak{I}_2$ we have

$$j = j + 0 \in \mathfrak{I}_1 + \mathfrak{I}_2$$

Furthermore, if $a \in I_2$, then because $0_{\mathcal{R}} \in I_1$ we have

$$j = 0 + j \in \mathfrak{I}_1 + \mathfrak{I}_2$$

Thus, we can deduce that $(\mathfrak{I}_1 \cup \mathfrak{I}_2) \subseteq \mathfrak{I}_1 + \mathfrak{I}_2$. ■

Next, if $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are rough ideals in rough ring R, then we will indicate that $\mathfrak{I}_1 + \mathfrak{I}_2 \nsubseteq (\mathfrak{I}_1 \cup \mathfrak{I}_2)$

We will attest with a counterexample. Based on Example 3.10, let $\mathfrak{I}_1 = \{\overline{0}, \overline{5}\}$ and $\mathfrak{I}_2 = \{\overline{0}, \overline{1}, \overline{5}, \overline{9}\}$. We can attest that $\mathfrak{I}_2$ is a rough ideal in $\mathcal{R}$. Furthermore, we see that

$$\begin{cases} \mathfrak{I}_1 + \mathfrak{I}_2 = \{\overline{0}, \overline{1}, \overline{4}, \overline{5}, \overline{6}, \overline{9}\} \\ \quad \mathfrak{I}_1 \cup \mathfrak{I}_2 = \{\overline{0}, \overline{1}, \overline{5}, \overline{9}\} \end{cases}$$

From here, we can deduce that $\mathfrak{I}_1 + \mathfrak{I}_2 \supseteq \mathfrak{I}_1 \cup \mathfrak{I}_2$ and $\mathfrak{I}_1 + \mathfrak{I}_2 \nsubseteq (\mathfrak{I}_1 \cup \mathfrak{I}_2)$.

Here is an example for Theorem 2.10 numbers 1 and 2.

Example 2.11 Based on the example in Theorem 2.10, we have $\mathfrak{I}_1 \cap \mathfrak{I}_2 = \{\overline{0}, \overline{5}\}$ and it is obvious that $\mathfrak{I}_1 \cap \mathfrak{I}_2$ is a rough ideal of $\mathcal{R}$. Furthermore, Let $\mathfrak{I}_1 = \mathfrak{I}_2 = \{\overline{0}, \overline{5}\}$. Then, we have $\mathfrak{I}_1 + \mathfrak{I}_2 = \{\overline{0}, \overline{5}\}$ and this is an ideal of $\mathcal{R}$.

III. CONCLUSION

Based on the explanation above, we can modify the definition of a rough ring and provide examples related to the new definition of rough rings. Furthermore, based on the definition of rough rings, we define rough subrings, attest the theorem to attest that the set is subrings, and give an example related to rough rings. Next, based on the predefined definitions of rough rings and rough subrings, we can define the rough ideals, the left rough ideals, and the right rough ideals. And then, we can also give an example related to rough ideals. Finally, we can attest that $\mathfrak{I}_1 \cap \mathfrak{I}_2$ and $\mathfrak{I}_1 + \mathfrak{I}_2$ are rough ideals in $\mathcal{R}$. Besides that, we can attest too that $(\mathfrak{I}_1 \cup \mathfrak{I}_2) \subseteq \mathfrak{I}_1 + \mathfrak{I}_2$ but the opposite is not necessarily the case.

For further research, by using the concept of rough rings and rough ideals, concepts related to rough quotient rings can be obtained. Furthermore, by using the concept of rough rings, we can construct a notion of rough rings homomorphism and rough rings isomorphism.

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