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GENERALIZED NON-BRAID GRAPHS OF RINGS

Era Setya Cahyati[1*], Rambu Maya Imung Maharani[2], Sri Nurhayati[3], Yeni Susanti[4]

[1,2,3,4] *Department of Mathematics, Gadjah Mada University, Yogyakarta, Indonesia*
Email: [1]era.setya.cahyati@mail.ugm.ac.id, [2]rambu.imung95@mail.ugm.ac.id,
[3]srinurhayati.sn77@mail.ugm.ac.id, [4]yeni_math@ugm.ac.id
*Corresponding author

Abstract. In this paper, we introduce the definition of generalized non-braid graph of a given ring. Let R be a ring and let k be a natural number. By generalized braider of R we mean the set $B^k(R) := \{x \in R \mid \forall y \in R,\ (xyx)^k = (yxy)^k\}$. The generalized non-braid graph of R, denoted by $G_k(\Upsilon_R)$, is a simple undirected graph with vertex set $R\backslash B^k(R)$ and two distinct vertices x and y are adjacent if and only if $(xyx)^k \neq (yxy)^k$. In particular, we investigate some properties of generalized non-braid graph $G_k(\Upsilon_{\mathbb{Z}_n})$ of the ring $\mathbb{Z}_n$.
Keywords: Graph, Ring, Non-Braid.

I. INTRODUCTION

Study involving algebraic structure and graph theory introduced by Cayley [1] has led to many fascinating results and questions. There are many research papers on assigning a graph to a ring or a group and investigation of algebraic properties of the associated graph. For example, Abdollahi, et. al. [2] introduced the definition of non-commuting graph of a group, see also [3, 4, 5, 6, 7]. As generalization of non-commuting graph of a group, Erfanian, et.al [8] introduced the definition of generalization of the non-commuting graph of a group via a normal subgroup. Erfanian, et. al [9] introduced the definition of non-commuting graph of a ring, see also [10, 11].

Motivated by the concept of non-commuting graph of a ring, Cahyati, et.al [12] defined non-braid graph of ring and explored some properties on completeness and connectedness of non-braid graph of $\mathbb{Z}_n$. In [12] it is also introduced a braider of ring R, denoted by $B(R)$, as the set of all $x \in R$ where $xyx = yxy$ for all $y \in R$. The non-braid graph of R, denoted by Υ_R, is defined as a simple graph with a vertex set $R\backslash B(R)$ and two distinct vertices x and y are adjacent if and only if $xyx \neq yxy$. In this paper we generalize $B(R)$ into $B^k(R)$, that is the set of all $x \in R$ where $(xyx)^k = (yxy)^k$ for all $y \in R$ and call $B^k(R)$ as the generalized braider of ring R. Then the generalized non-braid graphs of R, denoted by $G_k(\Upsilon_R)$, is defined as a simple undirected graph with a vertex set $R\backslash B^k(R)$ and two distinct vertices x and y are adjacent if and only if $(xyx)^k \neq (yxy)^k$. In this paper we present some basic properties of generalized non-braid graph of any ring. Particularly, we give some properties of generalized non-braid graph of ring $\mathbb{Z}_n$ including some sufficient conditions for the graph to be multipartite graph.

II. RESULTS

For this section, we give the following definition.

Definition 1 *Let R be a finite ring and let k be a natural number. Let*

$$B^k(R) = \{x \in R \mid \forall y \in R,\ (xyx)^k = (yxy)^k\}.$$

We call $B^k(R)$ as the generalized braider of R. The generalized non-braid graphs of R, denoted by $G_k(\Upsilon_R)$, is a simple undirected graph with a vertex set $R \backslash B^k(R)$ and two distinct vertices x and y are adjacent, denoted by $x \sim y$, if and only if $(xyx)^k \neq (yxy)^k$.

Example 1 Let $\mathbb{Z}_7$ be a ring. For $k = 4$, the generalized braider of $\mathbb{Z}_7$ is $B^4(\mathbb{Z}_7) = \{\overline{0}\}$ and we have,

$\overline{x}$	$\overline{y}$	$(\overline{xyx})^4$	$(\overline{yxy})^4$	Adjacency
$\overline{1}$	$\overline{2}$	$\overline{2}$	$\overline{4}$	$\overline{1} \sim \overline{2}$
$\overline{1}$	$\overline{3}$	$\overline{4}$	$\overline{2}$	$\overline{1} \sim \overline{3}$
$\overline{1}$	$\overline{4}$	$\overline{4}$	$\overline{2}$	$\overline{1} \sim \overline{4}$
$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{4}$	$\overline{1} \sim \overline{5}$
$\overline{1}$	$\overline{6}$	$\overline{1}$	$\overline{1}$	$\overline{1} \not\sim \overline{6}$
$\overline{2}$	$\overline{3}$	$\overline{2}$	$\overline{4}$	$\overline{2} \sim \overline{3}$
$\overline{2}$	$\overline{4}$	$\overline{2}$	$\overline{4}$	$\overline{2} \sim \overline{4}$
$\overline{2}$	$\overline{5}$	$\overline{1}$	$\overline{1}$	$\overline{2} \not\sim \overline{5}$
$\overline{2}$	$\overline{6}$	$\overline{4}$	$\overline{2}$	$\overline{2} \sim \overline{6}$
$\overline{3}$	$\overline{4}$	$\overline{1}$	$\overline{1}$	$\overline{3} \not\sim \overline{4}$
$\overline{3}$	$\overline{5}$	$\overline{4}$	$\overline{2}$	$\overline{3} \sim \overline{5}$
$\overline{3}$	$\overline{6}$	$\overline{2}$	$\overline{4}$	$\overline{3} \sim \overline{6}$
$\overline{4}$	$\overline{5}$	$\overline{4}$	$\overline{2}$	$\overline{4} \sim \overline{5}$
$\overline{4}$	$\overline{6}$	$\overline{2}$	$\overline{4}$	$\overline{4} \sim \overline{6}$
$\overline{5}$	$\overline{6}$	$\overline{4}$	$\overline{2}$	$\overline{5} \sim \overline{6}$

Table 1. Adjacency of elements in $\mathbb{Z}_7 \backslash B^4(\mathbb{Z}_7)$.

From Table 1. we have $V(G_4(\Upsilon_{\mathbb{Z}_7})) = \mathbb{Z}_7 \backslash B^4(\mathbb{Z}_7) = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$ and set of all edges in graph $G_4(\Upsilon_{\mathbb{Z}_7})$ is

$$\{(\overline{1},\overline{2}),(\overline{1},\overline{3}),(\overline{1},\overline{4}),(\overline{1},\overline{5}),(\overline{2},\overline{3}),(\overline{2},\overline{4}),(\overline{2},\overline{6}),(\overline{3},\overline{5}),(\overline{3},\overline{6}),(\overline{4},\overline{5}),(\overline{4},\overline{6}),(\overline{5},\overline{6})\}.$$

Figure 1. is graph $G_4(\Upsilon_{\mathbb{Z}_7})$.

Let $k \geq 2$ be a natural number and let f_k be a function from $R \setminus B^k(R)$ to R defined by $f_k(x) = x^k$, for all $x \in R$. Let $f_k^{-1}(y)$ for arbitrary y be the set $\{x \in R | f(x) = y\}$, i.e $f_k^{-1}(y)$ is the preimage set of y respect to f_k. Let $I(R)$ be the set of all idempotent elements of R, i.e. $I = \{x \in R | r^2 = r\}$. Let also $U(R)$ be the set of all unit elements of R, and $R^k = \{x^k | x \in R\}$. By definition of f_k, it follows that $f_k(R \setminus B^k(R)) \subseteq R^k$.

Remark 1 The only idempotent elements of any integral domain R are 0 and 1.

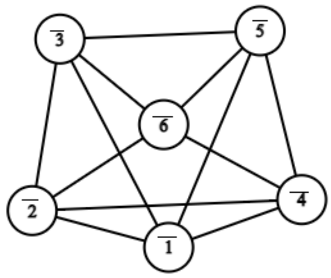

Figure 1. Graph $G_4(\Upsilon_{\mathbb{Z}_7})$

Lemma 1 *Let R be a commutative ring with identity element 1. If $1 \in B^k(R)$, then $R^k \subseteq I(R)$.*

Proof. Let $x^k \in R^k$ be arbitrary. Since $1 \in B^k(R)$, then

$$x^k = (1x1)^k = (x1x)^k = (x^2)^k = (x^k)^2.$$

So, $x^k \in I(R)$. Hence, $R^k \subseteq I(R)$. □

Example 2 Consider ring $\mathbb{Z}_6$. Clearly, $B^2(\mathbb{Z}_6) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$. Moreover, $\overline{1} \in B^2(\mathbb{Z}_6)$ and $\mathbb{Z}_6^2 = \{\overline{1}, \overline{3}, \overline{4}\} \subseteq I(\mathbb{Z}_6) = \{\overline{1}, \overline{3}, \overline{4}\}$.

Theorem 1 *For arbitrary commutative ring R with identity element it follows that the graph $G_k(\Upsilon_R)$ is a null graph if and only if $R^k \subseteq I(R)$.*

Proof. ($\Rightarrow$) Since $G_k(\Upsilon_R)$ is null graph, then for any $x \in R \backslash B^k(R)$, $(1x1)^k = (x1x)^k$. Therefore $1 \in B^k(R)$. By Lemma 1, $R^k \subseteq I(R)$.

($\Leftarrow$) Let $x, y \in V(G_k(\Upsilon_R))$ be arbitrary. We have $x^k, y^k \in R^k \subseteq I(R)$. Moreover, $(x^k)^2 = x^k$ and $(y^k)^2 = y^k$. Note that

$$(xyx)^k = x^k y^k x^k = (x^k)^2 y^k = x^k y^k = x^k (y^k)^2 = y^k x^k y^k = (yxy)^k.$$

Hence, $x \nsim y$ and therefore $G_k(\Upsilon_R)$ is a null graph. □

Example 3 Let $\mathbb{Z}_7$ be a ring. We get $\mathbb{Z}_7^4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{4}\}$ and $I_7 = \{\overline{0}, \overline{1}\}$. Since $\mathbb{Z}_7^4 \nsubseteq I_7$, then $G_4(\Upsilon_{\mathbb{Z}_7})$ is not null graph. For all $\overline{x} \in \mathbb{Z}_7$, $(\overline{0}\overline{x}\overline{0})^4 = (\overline{x}\overline{0}\overline{x})^4$. Hence $\overline{0} \in B^4(\mathbb{Z}_7)$. The generalized non-braid graph of $\mathbb{Z}_7$ is illustrated by Figure 1.

Lemma 2 *Let R be a commutative ring with identity element 1. Arbitrary vertex $x \in R \backslash B^k(R)$ is adjacent to 1 if and only if x^k is not an idempotent element.*

Proof. It is clear that x is adjacent to 1 if only if

$$(x1x)^k \neq (1x1)^k \iff (x^2)^k \neq x^k \iff (x^k)^2 \neq x^k.$$

□

The following result is a direct consequence of Lemma 2.

Corollary 1 *Let R be any commutative ring with identity element 1. Let S be the set*

$$S = \{x \in R \setminus B^k(R) | x^k \notin I(R)\} \cup \{1\}.$$

Then the subgraph of $G_k(\Upsilon_R)$ induced by S has diameter that is at most 2 and moreover contains a star as its subgraph.

Example 4 Consider commutative ring $\mathbb{Z}_6$. We have $V(\mathbb{Z}_6 \setminus B^3(\mathbb{Z}_6)) = \{\overline{1}, \overline{2}, \overline{4}, \overline{5}\}$ and the idempotent elements of $\mathbb{Z}_6$ are $\overline{1}, \overline{3}, \overline{4}$ and $\overline{2}^3 = \overline{2}, \overline{5}^3 = \overline{5}$. Since $\overline{2}^3$ and $\overline{5}^3$ both are not idempotent elements, by Lemma 2 we have $\overline{2} \sim \overline{1}$ and $\overline{5} \sim \overline{1}$ as we can see at Figure 2.

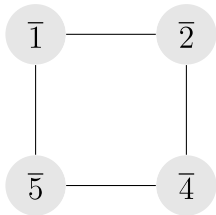

Figure 2. Graph $G_3(\Upsilon_{\mathbb{Z}_6})$.

Lemma 3 *Let R be a commutative ring with identity element and $k \geq 2$ be a natural number. Let $x \in B^k(R)$. If $a \in f_k^{-1}(x)$, then $a \in B^k(R)$.*

Proof. Let $a \in f_k^{-1}(x)$, i.e. $f_k(a) = a^k = x$. Since $x \in B^k(R)$, then each $y \in R$ satisfies $xyx = yxy$. Hence for any $y \in f_k(R \setminus B^k(R))$ we obtain $xyx = yxy$. It follows that for all $b \in f_k^{-1}(y)$,

$$\begin{aligned} xyx &= yxy \\ a^k b^k a^k &= b^k a^k b^k \\ (aba)^k &= (bab)^k \end{aligned}$$

implying $a \in B^k(R)$. □

Theorem 2 *Let R be a commutative ring and $a \in f_k(R \setminus B^k(R))$. If $x, y \in f_k^{-1}(a)$, then $x \nsim y$.*

Proof. Let $x, y \in f_k^{-1}(a)$. Then $f(x) = x^k = a$ and $f(y) = y^k = a$. Note that

$$\begin{aligned} a^3 &= a^3 \\ x^k y^k x^k &= y^k x^k y^k \\ (xyx)^k &= (yxy)^k. \end{aligned}$$

It means, $x \nsim y$. □

From [12] we know that every two unit elements are adjacent in the non-braid graph of a ring. For generalized non-braid graph we have the following.

Theorem 3 *Let R be a commutative ring with identity element. If $a, b \in f_k(R \setminus B^k(R))$ are two distinct unit elements of R, then for any $x \in f_k^{-1}(a)$ and $y \in f_k^{-1}(b)$ where $x, y \notin B^k(R)$ it follows that x is adjacent to y.*

Proof. Let $x \in f_k^{-1}(a)$ and $y \in f_k^{-1}(b)$ be arbitrary element where $x, y \notin B^k(R)$. Then we have $f_k(x) = x^k = a$ and $f_k(y) = y^k = b$. Since a and b are distinct unit elements, then

$$\begin{aligned} aba &\neq bab \\ x^k y^k x^k &\neq y^k x^k y^k \\ (xyx)^k &\neq (yxy)^k. \end{aligned}$$

Thus $x \sim y$. □

Theorem 4 *Let R be a finite commutative ring with identity element. If*

$$a_1, \ldots, a_m \in f_k(R \backslash B^k(R))$$

are distinct unit element of R and $|f_k^{-1}(a_i) \setminus B^k(R)| = r_i$ for $i \in \{1, 2, \ldots, m\}$, then the induced subgraph of $G_k(\Upsilon_R)$ by $\cup f_k^{-1}(x_i) \setminus B^k(R)$, $i \in \{1, 2, \ldots, m\}$ is a complete m-partite graph $K_{r_1,\ldots,r_m}$.

Proof. It is clear that for $i \neq j$, $i, j \in \{1, 2, \ldots, m\}$, we get $f_k^{-1}(x_i) \cap f_k^{-1}(x_j) = \emptyset$. By Theorem 2, all elements in $f_k^{-1}(a_i)$ are not adjacent. By Theorem II., for all $i \neq j$, for all $x \in f_k^{-1}(x_i)$, and for all $b \in f_k^{-1}(x_j)$, it follows that $a \sim b$. Hence, the induced subgraph of $G_k(\Upsilon_R)$ by $\cup f_k^{-1}(x_i) \setminus B^k(R)$, $i \in \{1, 2, \ldots, m\}$ is a complete m-partite graph $K_{r_1,\ldots,r_m}$. □

2.1. Generalized non-braid graph of ring $\mathbb{Z}_n$

In this section, we discuss the generalized non-braid graphs of ring $\mathbb{Z}_n$. Let I_n be the set of all idempotent elements of $\mathbb{Z}_n$ and U_n be the set of all unit elements of $\mathbb{Z}_n$.

Lemma 4 *If $k = l(n-1)$ for some $l \in \mathbb{N}$ and n is a prime number, then $\mathbb{Z}_n^k = \{\overline{0}, \overline{1}\}$. In particular, if $\overline{x} \neq \overline{0}$, then $\overline{x}^k = \overline{1}$.*

Proof. It is obvious that $\{\overline{0}, \overline{1}\} \subseteq \mathbb{Z}_n^k$. Let $\overline{x}^k \in \mathbb{Z}_n^k$. If $\overline{x} = \overline{0}$, then $\overline{x}^k = \overline{0} \in \{\overline{0}, \overline{1}\}$. If $x \neq \overline{0}$, by Fermat Little Theorem, it follows that $x^{n-1} = 1 \mod n$, i.e. $\overline{x}^{n-1} = \overline{1}$. Hence, $\overline{x}^k = (\overline{x}^{n-1})^l = \overline{1}$. Hence, $\mathbb{Z}_n^k \subseteq \{\overline{0}, \overline{1}\}$. □

Example 5 Let $n = 5$ and $l = 1$, then we have $k = l(n-1) = 1(5-1) = 4$. Note that for nonzero element in $\mathbb{Z}_5$ we have $\overline{1}^4 = \overline{2}^4 = \overline{3}^4 = \overline{4}^4 = \overline{1}$. It means $\mathbb{Z}_5^4 = \{\overline{0}, \overline{1}\}$.

The three following Propositions show some properties on generalized braider of ring $\mathbb{Z}_n$ whenever n is a prime number.

Proposition 1 *If n is a prime number, then*

$$B^k(\mathbb{Z}_n) = \begin{cases} \mathbb{Z}_n, & k = l(n-1) \\ \{\overline{0}\}, & \textit{otherwise} \end{cases}$$

for some $l \in \mathbb{N}$.

Proof. The assertion is true for $n = 2$. For $n \geq 3$ we will see two cases:

1. Case $k = l(n-1)$, for some $l \in \mathbb{N}$.
   It is obvious that $B^k(\mathbb{Z}_n) \subseteq \mathbb{Z}_n$. Let $\overline{x} \in \mathbb{Z}_n$. If $\overline{x} = \overline{0}$, then it is obvious that $\overline{0} \in B^k(\mathbb{Z}_n)$. If $\overline{x} \neq \overline{0}$, then by Lemma 4 it follows that $\overline{x}^k = \overline{1}$. For any $\overline{y} \in \mathbb{Z}_n$, by Lemma 4, $\overline{y}^k = \overline{1}$ if $\overline{y} \neq \overline{0}$. If $\overline{y} = \overline{0}$, then $(\overline{xyx})^k = \overline{0} = (\overline{yxy})^k$. If $\overline{y} \neq \overline{0}$, then

$$(\overline{xyx})^k = \overline{x}^k \overline{y}^k \overline{x}^k = \overline{1} = \overline{y}^k \overline{x}^k \overline{y}^k = (\overline{yxy})^k.$$

   So, $\overline{x} \in B^k(\mathbb{Z}_n)$. Hence, $B^k(\mathbb{Z}_n) = \mathbb{Z}_n$.

2. Case $k \neq l(n-1)$ for all $l \in \mathbb{N}$.
   Let $\overline{x} \in B^k(\mathbb{Z}_n)$. Assume that $\overline{x} \neq \overline{0}$. If $\overline{x}^k \neq \overline{1}$, then clearly $\overline{x}^k$ is not an idempotent element. Otherwise, we have a contradiction. Note that for $\overline{1} \in \mathbb{Z}_n$ we have

$$(\overline{x}\overline{1}\overline{x})^k \neq (\overline{1}\overline{x}\overline{1})^k$$

   meaning $\overline{x} \notin B^k(\mathbb{Z}_n)$, a contradiction. If $\overline{x}^k = 1$ then $\overline{x}^k$ is idempotent element, and therefore $\overline{x} = \overline{1}$. Let $\overline{y}^k$ be any element in $\mathbb{Z}_n^k$ that is not idempotent. It follows that

$$\overline{y}^k \neq \overline{y}^{2k} \iff (\overline{1y1})^k \neq (\overline{y1y})^k.$$

   Hence, $\overline{x} \notin B^k(\mathbb{Z}_n)$ and again we have a contradiction. Therefore we conclude that $B^k(\mathbb{Z}_n) = \{\overline{0}\}$.

□

Example 6 Let $n = 7$ and $l = 3$, then we have $k = l(n-1) = 3(7-1) = 18$. For any $\overline{x}, \overline{y} \in \mathbb{Z}_7$ where $\overline{x} \neq 0$ and $\overline{y} \neq 0$ we have $\overline{x}^{18} = \overline{y}^{18} = \overline{1}$. It follows that $(\overline{xyx})^{18} = (\overline{yxy})^{18}$. Hence $B^{18}(\mathbb{Z}_7) = \mathbb{Z}_7$.

As a corollary, we have

Corollary 2 *For any prime number* n *and for any natural number* k*, if* $k = l(n-1)$ *for some natural number* l *then the graph* $G_k(\Upsilon_R)$ *is an empty graph.*

Example 7 Note that $B^{18}(\mathbb{Z}_7) = \mathbb{Z}_7$. It means $G_k(\Upsilon_R) = \emptyset$.

Lemma 5 *If* $n = 2p$ *for a prime number* $p \geq 3$*, then* $\mathbb{Z}_n^k \subseteq I_n$ *for all* k *in the form* $k = l(p-1)$ *for some* $l \in \mathbb{N}$.

Proof. Let $\overline{x}^k \in \mathbb{Z}_n^k$ be arbitrary. Since p is prime, then by Fermat Little Theorem, $x^{p-1} = 1 \mod p$. Hence, there exist $s \in \mathbb{Z}$ such that $x^{p-1} = ps + 1$ and we have $2x^{p-1} = 2ps + 2$ implying $\overline{2x^{p-1}} = \overline{2}$ if and only if $\overline{2}(\overline{x}^{p-1} - \overline{1}) = \overline{0}$. Therefore $\overline{x}^{p-1} - \overline{1} = \overline{0}$ or $\overline{x}^{p-1} - \overline{1} = \overline{p}$.

If $\overline{x}^{p-1} - \overline{1} = \overline{0}$, then

$$\begin{aligned}\overline{x}^{p-1} &= \overline{1}\\ \overline{x}^{p} &= \overline{x}\\ \overline{x}^{2p-p} &= \overline{x}\\ \overline{x}^{2p-p}\overline{x}^{p-2} &= \overline{x}\;\overline{x}^{p-2}\\ \overline{x}^{2p-2} &= \overline{x}^{p-1}\\ (\overline{x}^{p-1})^2 &= \overline{x}^{p-1}.\end{aligned}$$

If $\overline{x}^{p-1} - \overline{1} = \overline{p}$, then

$$\begin{aligned}\overline{x}^{p-1} &= \overline{p} + \overline{1}\\ (\overline{x}^{p-1})^2 &= (\overline{p} + \overline{1})^2\\ &= \overline{p}^2 + 2\overline{p} + \overline{1}\\ &= \overline{p}^2 + \overline{1}.\end{aligned}$$

If $p \geq 3$ is prime number, then $p - 1$ is even number. It means $\frac{p-1}{2} \in \mathbb{Z}$. Note that

$$p^2 = \frac{p-1}{2}\,2p + p.$$

Therefore, $\overline{p}^2 = \overline{p}$. Hence $(\overline{x}^{p-1})^2 = \overline{p} + \overline{1} = \overline{x}^{p-1}$. and thus $(x^{l(p-1)})^2 = x^{l(p-1)}$. Therefore, $x^k \in I_n$. □

Proposition 2 *If $n = 2p$, for a prime number $p \geq 3$, then $B^k(\mathbb{Z}_n) = \mathbb{Z}_n$ for any $k = l(p-1)$ for some $l \in \mathbb{N}$.*

Proof. It is obvious that $B^k(\mathbb{Z}_n) \subseteq \mathbb{Z}_n$. Let $\overline{x} \in \mathbb{Z}_n$. By Lemma 5, $\overline{x}^k$ is idempotent element. Let $\overline{y} \in \mathbb{Z}_n$ be arbitrary. By Lemma 5, $\overline{y}^k$ is also idempotent element. Since every two idempotent elements are not adjacent, then $\overline{x}$ is not adjacent to $\overline{y}$. Hence, $\overline{x} \in B^k(\mathbb{Z}_n)$. □

Example 8 Let $p = 5$ and $l = 2$, then $k = l(p-1) = 2(5-1) = 8$ and $n = 2p = 10$. Note that for any $x, y \in \mathbb{Z}_{10}$, x^k and y^k are idempotent elements. Therefore, we have $B^8(\mathbb{Z}_{10}) = \mathbb{Z}_{10}$.

In the following lemmas, we give some properties on adjacency particularly for ring $\mathbb{Z}_n$

Lemma 6 *Let $\overline{x}, \overline{y} \in V(G_k(\Upsilon_{\mathbb{Z}_n}))$. If $n|xy$, then the vertices $\overline{x}$ and $\overline{y}$ are not adjacent.*

Proof. Since $n|xy$, there is $a \in \mathbb{Z}_n$ such that $xy = na$. Note that $\overline{xy} = \overline{yx} = \overline{0}$, so

$$(\overline{xyx})^k = (\overline{0x})^k = \overline{0} = (\overline{0})^k = (\overline{0y})^k = (\overline{xyy})^k = (\overline{yxy})^k.$$

So, the vertices $\overline{x}$ and $\overline{y}$ are not adjacent. □

Lemma 7 *Let $\overline{x}, \overline{y}, \overline{z} \in V(G_k(\Upsilon_{\mathbb{Z}_n}))$. If $\overline{xy} = \overline{0}$ and $\overline{z} = \overline{x} + \overline{y}$, then vertices x and y are not adjacent to vertex $\overline{z}$.*

Proof. Let $\overline{x}, \overline{y}, \overline{z} \in V(G_k(\Upsilon_{\mathbb{Z}_n}))$, where $\overline{xy} = \overline{0}$ and $\overline{z} = \overline{x} + \overline{y}$. Note that

$$(\overline{xzx})^k = (\overline{x}(\overline{x} + \overline{y})\overline{x})^k = ((\overline{x}^2 + \overline{xy})\overline{x})^k = ((\overline{x}^2 + \overline{0})\overline{x})^k = (\overline{x}^2\overline{x})^k = (\overline{x}^3)^k$$

and

$$\begin{aligned}(\overline{zxz})^k &= ((\overline{x} + \overline{y})\overline{x}(\overline{x} + \overline{y}))^k = ((\overline{x} + \overline{y})(\overline{x}^2 + \overline{xy}))^k \\ &= ((\overline{x} + \overline{y})(\overline{x}^2 + \overline{0}))^k \\ &= ((\overline{x} + \overline{y})(\overline{x}^2))^k \\ &= ((\overline{x}^3 + \overline{x}^2\overline{y})^k \\ &= (\overline{x}^3 + \overline{x}\overline{xy})^k \\ &= (\overline{x}^3 + \overline{x}\overline{0})^k \\ &= (\overline{x}^3)^k.\end{aligned}$$

Thus, $(\overline{xzx})^k = (\overline{zxz})^k$. In similar way, it can be proved that $(\overline{yzy})^k = (\overline{zyz})^k$ Hence, vertex $\overline{y}$ is not adjacent to vertex $\overline{z}$. Therefore $\overline{x}$ and $\overline{y}$ are not adjacent to vertex $\overline{z}$. □

Proposition 3 *Let $\overline{x}, \overline{z} \in V(G_k(\Upsilon_{\mathbb{Z}_{2m}}))$ where m is an odd number. If $\overline{z} = \overline{x} + \overline{m}$, then vertex $\overline{z}$ is not adjacent to vertex $\overline{x}$.*

Proof. Let $\overline{x}, \overline{z} \in V(G_k(\Upsilon_{\mathbb{Z}_{2m}}))$ where $\overline{z} = \overline{x} + \overline{m}$. To show that vertex $\overline{z}$ is not adjacent to vertex $\overline{x}$, we consider the following two cases.
Case 1. For $\overline{x} = \overline{2a}$ where $a \in \mathbb{Z}$, we have $(\overline{xm})^k = \overline{0}$, and by Lemma 7, $\overline{z}$ is not adjacent to vertex $\overline{x}$
Case 2. For $\overline{x} = \overline{2a+1}$ where $a \in \mathbb{Z}$. Note that

$$\begin{aligned}(\overline{zxz})^k &= ((\overline{x} + \overline{m})\overline{x}(\overline{x} + \overline{m}))^k \\ &= ((\overline{x} + \overline{m})(\overline{x}^2 + \overline{xm}))^k \\ &= ((\overline{x} + \overline{m})(\overline{x}^2 + \overline{2a+1}\overline{m}))^k \\ &= ((\overline{x} + \overline{m})(\overline{x}^2 + \overline{m}))^k \\ &= (\overline{x}^3 + \overline{xm} + \overline{x}^2\overline{m} + \overline{m}^2)^k \\ &= (\overline{x}^3 + \overline{2a+1}\overline{m} + \overline{x}^2\overline{m} + \overline{m}^2)^k \\ &= (\overline{x}^3 + \overline{m} + \overline{x}^2\overline{m} + \overline{m}^2)^k \\ &= (\overline{x}^3 + \overline{x}^2\overline{m})^k \\ &= ((\overline{x}^2 + \overline{xm})\overline{x})^k \\ &= (\overline{x}(\overline{x} + \overline{m})\overline{x})^k \\ &= (\overline{xzx})^k.\end{aligned}$$

From Case 1 and Case 2, we conclude that vertex $\overline{z}$ is not adjacent to vertex $\overline{x}$ □

III. CONCLUSIONS AND FUTURE RESEARCH DIRECTION

We have obtained some results on the generalized non-braid graph, such as some conditions for vertices to be adjacent, and necessary and sufficient condition for the graph to be a null graph. But however the structure of the graph in general is not yet obtained. This will be an interesting object for further research in the future.

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