

# ANOTHER LOOK AT A POLYNOMIAL SOLUTION FOR ALTERNATING POWER SUMS

Leomarich F. Casinillo

*Department of Mathematics, Visayas State University, Baybay City, Leyte, Philippines*  
 Email : [leomarichcasinillo02011990@gmail.com](mailto:leomarichcasinillo02011990@gmail.com)

**Abstract.** This paper aims to provide a new general explicit polynomial solution on power sums of consecutive positive integers under alternating signs. Moreover, it examines the solution under odd and even number of terms in the series and provides some examples.

**Keywords:** General polynomial, power sums; alternating signs; odd and even terms.

## I. INTRODUCTION

In the paper of Casinillo [1], the alternative general form for alternating series for consecutive positive integer with power  $p \in \mathbb{Z}^+$  are presented and explored. Although the series is well-known and well-research throughout the years, still discrete mathematicians are intrigued to find alternative solutions [2][3][4][5][6]. In most of the previous results, the series is expressed as a polynomial in relation to Bernoulli numbers [2][3], and generating functions [4]. However, this current paper develops a polynomial solution for the series as an extension of the existing papers of Casinillo and Mamolo [5], and Casinillo and Abas [6] where they had solved the series for  $p = 2$  and  $p = 3$ , respectively. From the paper of Casinillo [1], here are the different forms of the series in relation to odd and even terms. Let  $\lambda$  and  $p$  be a positive integer. If  $\lambda$  is even and  $t = 2x - 1$  ( $x \in \mathbb{Z}^+$ ), we have

$$A_t^o(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-2} (-1)^j j^p = \lambda^p - (\lambda + 1)^p + \dots + (\lambda + 2x - 2)^p \quad (1)$$

and for  $t = 2x$  ( $x \in \mathbb{Z}^+$ ), we also have

$$A_t^e(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-1} (-1)^j j^p = \lambda^p - (\lambda + 1)^p + \dots - (\lambda + 2x - 1)^p. \quad (2)$$

Considering if  $\lambda$  is odd, then, we have

$$A_t^o(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-2} (-1)^{j+1} j^p = \lambda^p - (\lambda + 1)^p + \dots + (\lambda + 2x - 2)^p \quad (3)$$

where  $t = 2x - 1$  ( $x \in \mathbb{Z}^+$ ), and lastly

$$A_t^e(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-1} (-1)^{j+1} j^p = \lambda^p - (\lambda + 1)^p + \dots - (\lambda + 2x - 1)^p \quad (4)$$

where  $t = 2x$  ( $x \in \mathbb{Z}^+$ ). The superscripts  $o$  and  $e$  in the alternating series represent odd and even terms, respectively. Let  $P_d(x) \in \mathbb{Z}[x]$  be a polynomial in  $x$  of degree  $d \in \mathbb{Z}^+$ . For instance, we let  $x = 3$ , so we have,  $P_3(x) = 2x^3 + x^2 - 5$ ,  $P_3(x) = 7x^3 + 4x^2 - x$ , and  $P_3(x) = -3x^3$ , among others. In that case, the focus of this paper is to construct a new explicit

polynomial solution in  $x$  for alternating power sum where  $p \geq 1$ . The coefficient of variable  $x$  in the polynomial is a function of  $\lambda$ , that is,  $P_a(x) = f_0(\lambda)x^d + f_1(\lambda)x^{d-1} + \dots + f_{p-1}(\lambda)x + f_p(\lambda)$  where  $f_i(\lambda) \in \mathbb{Z}[\lambda]$  for  $i = 0, 1, 2, \dots, p$ . Moreover, the polynomial solution will be investigated in relation to the parity of a number of terms in the series.

## II. RESULTS

The first result is immediate from equations (1) and (3) above. This represents that alternating series can be transformed into an explicit polynomial under an odd number of terms. The explicit polynomial of this series can be solved by simulation that forms a system of equations.

**Theorem 2.1.** *Let  $\lambda$ ,  $x$ , and  $p$  be a positive integer. If  $A_t^o(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-2} (-1)^j j^p$  and  $x = \frac{t+1}{2}$  and  $t \equiv 1 \pmod{2}$ , then  $A_t^o(\lambda, x, p) = \sum_{j=0}^p a_j(\lambda)x^j > 0$  where  $a_j(\lambda) \in \mathbb{Z}[\lambda]$ .*

*Proof.* Without loss of generality (WLOG), we let  $\lambda$  be an even positive integer. Then, we simulate equation (1) with the following values  $x = 1, 2, \dots, p, p+1$ . So we obtain the following system of equations

$$\left\{ \begin{array}{l} A_t^o(\lambda, 1, p) = \sum_{j=0}^p a_j(\lambda) = \lambda^p \\ A_t^o(\lambda, 2, p) = \sum_{j=0}^p a_j(\lambda)2^j = \sum_{j=0}^2 (-1)^j (\lambda + j)^p \\ A_t^o(\lambda, 3, p) = \sum_{j=0}^p a_j(\lambda)3^j = \sum_{j=0}^4 (-1)^j (\lambda + j)^p \\ \vdots \\ \vdots \\ A_t^o(\lambda, p+1, p) = \sum_{j=0}^p a_j(\lambda)x^j = \sum_{j=0}^{2p} (-1)^j (\lambda + j)^p \end{array} \right. \quad (5)$$

Since the number of unknowns ( $a_0(\lambda), a_1(\lambda), a_2(\lambda), \dots, a_p(\lambda)$ ) and the number of equations are equal, and no linear dependence exists between pairwise equations, then the system of equations has a unique solution. In that case, using Gaussian elimination and backward substitution, we get the following results,

$$\left\{ \begin{array}{l} a_0(\lambda) = P_p(\lambda) \in \mathbb{Z}[\lambda] \\ a_1(\lambda) = P_{p-1}(\lambda) \in \mathbb{Z}[\lambda] \\ a_2(\lambda) = P_{p-2}(\lambda) \in \mathbb{Z}[\lambda] \\ \vdots \\ \vdots \\ a_p(\lambda) = P_0(\lambda) \in \mathbb{Z}[\lambda] \end{array} \right. \quad (6)$$

In (6), it is noted that for  $\lambda \geq 2$ , we have  $a_j(\lambda) > 0$  for all  $j \in \{0, 1, 2, \dots, p\}$  and for  $\lambda = 1$ , we also have  $a_0(\lambda) + a_1(\lambda) + a_2(\lambda) + \dots + a_p(\lambda) > 0$ . Hence, it is concluded that  $A_t^o(\lambda, x, p) >$

0 for all values of  $x \in \mathbb{Z}$ . After that, we verify that the result holds for all positive integer  $x$ . So, we let  $M(x)$  be the equation,

$$A_t^o(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-2} (-1)^j j^p = \sum_{j=0}^p a_j(\lambda) x^j \quad (7)$$

For  $x = 1$ ,  $M(x)$  holds, that is,

$$\lambda^p = \sum_{j=0}^p P_{p-j}(\lambda)(1)^j = \lambda^p \quad (8)$$

We have to note that the induction hypothesis is the statement  $M(x)$  and we want to show that  $M(x + 1)$  also holds. So, it follows that

$$A_{t+2}^o(\lambda, x + 1, p) = \sum_{j=0}^p a_j(\lambda) x^j - (\lambda + 2x - 1)^p + (\lambda + 2x)^p \quad (9)$$

and we have

$$A_{t+2}^o(\lambda, x + 1, p) = \sum_{j=0}^p P_{p-j}(\lambda) x^j - (\lambda + 2x - 1)^p + (\lambda + 2x)^p \quad (10)$$

By expanding and simplifying equation (10), we get

$$A_{t+2}^o(\lambda, x + 1, p) = \sum_{j=0}^p P_{p-j}(\lambda)(x + 1)^j \quad (11)$$

Hence,  $M(x)$  holds for all values of  $x \in \mathbb{Z}$ . This completes the proof.  $\square$

**Illustration 2.1.** Let  $p = 1$  and  $\lambda$  be odd. We have

$$A_t^o(\lambda, x, 1) = \sum_{j=\lambda}^{\lambda+2x-2} (-1)^{j+1} j = x + (\lambda - 1) \quad (12)$$

To obtain equation (12), we assume  $A_t^o(\lambda, x, 1) = ax + b$ , where  $a = f_1(\lambda) \in \mathbb{Z}[\lambda]$  and  $b = f_2(\lambda) \in \mathbb{Z}[\lambda]$ . In that case, we simply simulate  $x$  from 1 to 2. Then, we obtain

$$\begin{cases} a + b = \lambda \\ 2a + b = \lambda + 1 \end{cases}$$

Solving the parameters  $a$  and  $b$ , we get

$$\begin{cases} a = 1 \\ b = \lambda - 1 \end{cases}$$

By substitution, it follows that equation (12) holds. By, mathematical induction, it is clear that equation (12) is valid for all  $x \in \mathbb{Z}$ .

*Example 1.* Consider  $A_5^o(5, 3, 1)$ . Then, applying equation (12), we have

$$A_5^o(5, 3, 1) = \sum_{j=5}^9 (-1)^{j+1} j = 5 - 6 + 7 - 8 + 9 = 3 + (5 - 1) = 7$$

**Illustration 2.2.** Let  $p = 2$  and  $\lambda$  be odd. We have

$$A_t^o(\lambda, x, 2) = \sum_{j=\lambda}^{\lambda+2x-2} (-1)^{j+1} j^2 = 2x^2 + (2\lambda - 3)x + (\lambda - 1)^2 \quad (13)$$

The proof can be found in [5].

*Example 2.* Consider  $A_5^0(5, 3, 2)$ . Then, applying equation (13), we have

$$A_5^0(5, 3, 2) = \sum_{j=5}^9 (-1)^{j+1} j^2 = 5^2 - 6^2 + 7^2 - 8^2 + 9^2 = 2(3)^2 + [2(5) - 3]3 + (5 - 1)^2 = 55$$

**Illustration 2.3.** Let  $p = 3$  and  $\lambda$  be odd. We have

$$A_t^0(\lambda, x, 3) = \sum_{j=\lambda}^{\lambda+2x-2} (-1)^{j+1} j^3 = 4x^3 + (6\lambda - 9)x^2 + (3\lambda^2 - 9\lambda + 6)x + (\lambda - 1)^2 \quad (14)$$

The proof can be found in [6].

*Example 3.* Consider  $A_5^0(5, 3, 3)$ . Then, applying equation (14), we have

$$A_5^0(5, 3, 3) = \sum_{j=5}^9 (-1)^{j+1} j^3 = 5^3 - 6^3 + 7^3 - 8^3 + 9^3 = 4(3)^3 + [6(5) - 9](3)^2 + [3(5)^2 - 9(5) + 6](3) + (5 - 1)^3 = 469$$

The second result is also immediate from equations (2) and (4) above. This also represents that alternating series can be transformed into an explicit polynomial under an even number of terms. Again, the solution for this polynomial can be obtained by simulation and solving a system of equations.

**Theorem 2.2.** Let  $\lambda$ ,  $x$ , and  $p$  be a positive integer. If  $A_t^e(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-1} (-1)^j j^p$  and  $x = \frac{t}{2}$  and  $t \equiv 0 \pmod{2}$ , then  $A_t^e(\lambda, x, p) = \sum_{j=0}^p b_j(\lambda) x^j > 0$  where  $b_j(\lambda) \in \mathbb{Z}[\lambda]$ .

*Proof.* WLOG, we let  $\lambda$  be an even natural number. Next, for values of  $x = 1, 2, \dots, p, p + 1$ , we simulate equation (2). Then, we get the following system of equations below,

$$\left\{ \begin{array}{l} A_t^e(\lambda, 1, p) = \sum_{j=0}^p b_j(\lambda) = \sum_{j=0}^1 (-1)^j (\lambda + j)^p \\ A_t^e(\lambda, 2, p) = \sum_{j=0}^p b_j(\lambda) 2^j = \sum_{j=0}^3 (-1)^j (\lambda + j)^p \\ A_t^e(\lambda, 3, p) = \sum_{j=0}^p b_j(\lambda) 3^j = \sum_{j=0}^5 (-1)^j (\lambda + j)^p \\ \vdots \\ A_t^e(\lambda, p + 1, p) = \sum_{j=0}^p b_j(\lambda) x^j = \sum_{j=0}^{2p+1} (-1)^j (\lambda + j)^p \end{array} \right. \quad (15)$$

Now, it is worthy to note that the number of unknowns ( $b_0(\lambda), b_1(\lambda), b_2(\lambda), \dots, b_p(\lambda)$ ) and the number of equations in (15) is equal, and additionally, no linear dependence exists, then it is clear that the system of equations has a singular solution. Using the Gaussian elimination and applying the method of backward substitution, we obtain

$$\begin{cases} b_0(\lambda) = 0 = P_p(\lambda) \in \mathbb{Z}[\lambda] \\ b_1(\lambda) = P_{p-1}(\lambda) \in \mathbb{Z}[\lambda] \\ b_2(\lambda) = P_{p-2}(\lambda) \in \mathbb{Z}[\lambda] \\ \vdots \\ b_p(\lambda) = P_0(\lambda) \in \mathbb{Z}[\lambda] \end{cases} \quad (16)$$

In the system of equations in (16), we have to note that for  $\lambda \geq 1$ , we have  $b_j(\lambda) \leq 0$  for all  $j \in \{0, 1, 2, \dots, p\}$ . This implies that  $A_t^e(\lambda, x, p) < 0$  for all  $x \in \mathbb{Z}$ . Next, we verify the result that it holds for all positive integer  $x$ . We let  $N(x)$  be,

$$A_t^e(\lambda, x, p) = \sum_{j=\lambda}^{\lambda+2x-1} (-1)^j j^p = \sum_{j=0}^p b_j(\lambda) x^j \quad (17)$$

For  $x = 1$ ,  $N(x)$  holds, that is,

$$\lambda^p - (\lambda + 1)^p = \sum_{j=0}^p P_{p-j}(\lambda) (1)^j = \lambda^p - (\lambda + 1)^p \quad (18)$$

It is worth noting that the induction hypothesis is  $N(x)$ . So, we want to show that  $N(x + 1)$  is also true. We have

$$A_{t+2}^e(\lambda, x + 1, p) = \sum_{j=0}^p b_j(\lambda) x^j + (\lambda + 2x)^p - (\lambda + 2x + 1)^p \quad (19)$$

and we also obtain,

$$A_{t+2}^e(\lambda, x + 1, p) = \sum_{j=0}^p P_{p-j}(\lambda) x^j + (\lambda + 2x)^p - (\lambda + 2x + 1)^p \quad (20)$$

In that case, by expanding and simplifying equation (20), we end up with

$$A_{t+2}^e(\lambda, x + 1, p) = \sum_{j=0}^p P_{p-j}(\lambda) (x + 1)^j \quad (21)$$

Hence, it is concluded that  $N(x)$  holds for all values of  $x \in \mathbb{Z}$ . This completes the proof.  $\square$

**Illustration 2.4.** Let  $p = 1$  and  $\lambda$  be odd. Then, we have

$$A_t^e(\lambda, x, 1) = \sum_{j=\lambda}^{\lambda+2x-1} (-1)^{j+1} j = -x \quad (22)$$

To arrive at equation (22), we need to assume that  $A_t^e(\lambda, x, 1) = cx + d$ , where  $c = f_3(\lambda) \in \mathbb{Z}[\lambda]$  and  $d = f_4(\lambda) \in \mathbb{Z}[\lambda]$ . Next, we simulate  $x = 1, 2$ . So, we have

$$\begin{cases} c + d = \lambda - (\lambda + 1) = -1 \\ 2c + d = \lambda - (\lambda + 1) + (\lambda + 2) - (\lambda + 3) = -2 \end{cases}$$

In that case, we solve for the parameters  $c$  and  $d$ , and we get

$$\begin{cases} c = -1 \\ d = 0 \end{cases}$$

Hence, equation (22) holds, and clearly, it is valid for all  $x \in \mathbb{Z}$  by mathematical induction.

*Example 4.* Consider  $A_6^e(5, 3, 1)$ . Applying equation (22), we have

$$A_6^e(5, 3, 1) = \sum_{j=5}^{10} (-1)^{j+1} j = 5 - 6 + 7 - 8 + 9 - 10 = -3$$

**Illustration 2.5.** Let  $p = 2$  and  $\lambda$  be odd. We have

$$A_t^e(\lambda, x, 2) = \sum_{j=\lambda}^{\lambda+2x-1} (-1)^{j+1} j^2 = -2x^2 - (2\lambda - 1)x \quad (23)$$

The proof can be found in [5].

*Example 5.* Consider  $A_6^e(5, 3, 2)$ . Applying equation (23), we have

$$\begin{aligned} A_6^e(5, 3, 2) &= \sum_{j=5}^{10} (-1)^{j+1} j^2 = 5^2 - 6^2 + 7^2 - 8^2 + 9^2 - 10^2 = -2(3)^2 - [2(5) - 1]3 \\ &= -45 \end{aligned}$$

**Illustration 2.6.** Let  $p = 3$  and  $\lambda$  be odd. We have

$$\begin{aligned} A_t^e(\lambda, x, 3) &= \sum_{j=\lambda}^{\lambda+2x-1} (-1)^{j+1} j^3 \\ &= -4x^3 + (-6\lambda + 3)x^2 + (-3\lambda^2 + 3\lambda)x \end{aligned} \quad (24)$$

The proof can be found in [6].

*Example 6.* Consider  $A_6^e(5, 3, 2)$ . Applying equation (24), we have

$$\begin{aligned} A_6^e(5, 3, 1) &= \sum_{j=5}^{10} (-1)^{j+1} j^3 = 5^3 - 6^3 + 7^3 - 8^3 + 9^3 - 10^3 \\ &= -4(3)^3 + [-6(5) + 3](3)^2 + [-3(5)^2 + 3(5)](3) = -531. \end{aligned}$$

### III. CONCLUSION

This study had developed new polynomial solutions for alternating power sums, that is,  $A_t^o(\lambda, x, p)$  and  $A_t^e(\lambda, x, p)$  where  $\lambda, x, p \in \mathbb{Z}$ . It is concluded that polynomial solutions are different in regards to the parity of a number of terms (odd and even) in the series. Moreover, it is found that  $A_t^o(\lambda, x, p) > 0$  and  $A_t^e(\lambda, x, p) < 0$ . Furthermore, one may consider developing a new polynomial solution for power sums based on the current paper as future research.

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