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CHARACTERIZATION OF WEAKLY PRIME SUBMODULES

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Abstract. Let R is a commutative ring with identity and M is a unital R-module. El-bast and Smith (1988) have researched and introduced the multiplication module. Prime submodule has been studied by Ameri (2002). Then Atani and Farzalipour (2007) have extended the prime submodule to weakly prime submodule. Some properties were proved on each paper. This paper will study about characterization of weakly prime submodules.
Keywords: Multiplication Modules, Weakly Prime Submodule, Prime Submodule.

I. INTRODUCTION

Weakly prime submodules in multiplication module with identity have been introduced and studied by Atani and Farzalipour in [1]. Before we state the result let us introduce some notation and terminology. Throughout this paper all rings will be commutative with identity and M is a unital R-modules. If S is a submodule of M, then we can define an ideal of R, it is $\{r \in R | rM \subseteq S\}$ and denoted by $(S : M)$. We can also denote an annihilator of M, it is $(0 : M)$. Multiplication modules have been defined by [3]. Let M is unital R-module, M is called a multiplication module if for each S submodule of M is the product of K and M, where K is an ideal of R. In such a case, we say that $K = (S : M)$. The concept of prime ideals is extended to a more general concept by several authors; it is module. Ameri (2002) has studied and introduced the prime submodules of M, where M is multiplication R-modules. Ameri (2002) also defined the product of two submodules of multiplication R-modules.

Definition 1 [2] *Let M is unital R-module. We can define that S submodule of M is equal to KM for some ideal K of R. K that fits the property then is called presentation ideal of S.*

Note that not all submodules have presentation ideal. To understand more about presentation ideal, see an example below.

Example 1 $\left\{ \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \middle| z \in \mathbb{Z} \right\}$ as module over itself. Clearly that M is a multiplication R-module because every submodule of M has a presentation ideal.

Definition 2 [2] *Let M is multiplication R-module and S and T are submodules of M. Let $S = KM$ and $T = LM$ for some ideals K and L of R, then the product of S and T is defined by $ST = KLM$.*

Clearly that ST is a submodule of M and contained in $S \cap T$. For $x, y \in M$, by xy, it means that the product of Rx and Ry are equal to KLM for some presentation ideals K and L of x and y, respectively. The concept about prime ideals has been expanded to prime submodules by Ameri (2002).

Definition 3 [2] *Let M is a multiplication R-module and W is proper submodule of M. Then W is called prime submodule when it fits the property as follows*

$$rx \in W \text{ (with } r \in R \text{ and } x \in M) \Rightarrow r \in (W : M) \text{ or } x \in W.$$

Theorem 1 [2] *Let M is multiplication R-module and W is a proper submodule of M. Then W is called prime submodule if and only if*

$$ST \subseteq W \Rightarrow S \subseteq W \text{ or } T \subseteq W$$

for each submodule S and T of M.

For Definition 3, we have Corollary.

Corollary 1 [2] *Let M is a multiplication R-module and W is a proper submodule of M. Then W is called prime submodule if and only if*

$$xy \subseteq W \Rightarrow x \in W \text{ or } y \in W$$

for every $x, y \in M$.

Prime ideal is extended to weakly prime ideal by Anderson and Smith in [4]. It motivates Atani and Farzalipour to extend prime submodule to weakly prime submodule.

Definition 4 [1] *Let M is unital R-module and W is a proper submodule of M. Then W is called weakly prime submodule when it fits the property*

$$0 \neq rx \in W \text{ (with } r \in R \text{ and } x \in M) \Rightarrow r \in (W : M) \text{ or } x \in W.$$

By definition, when we have a module then it is clear that every prime submodule is weakly prime submodule. However, we know that since 0 is always weakly prime, so this proves that a weakly prime submodule is not always prime submodule.

Example 2 Given M is Z_{12} as module over ring Z and $W = \{0, 3, 6, 9\}$ is proper submodule of M with ideal $(W : M) = \{r \in Z | rZ_{12} \subseteq N\} = 3Z$. Therefore, for $rm \in W$ we get $m = 0, m = 3, m = 6$ and $m = 9$ which are elements in W or $r \in 3Z$. So, W is prime submodule of M. It is clear that W is also weakly prime submodule.

Example 3 Given Z_n as module over itself with n is composite numbers. Let $N = \{0\}$ is submodule of M, so N is weakly prime but not prime.

Theorem 2 [1] *Let M is unital R-module and W is a weakly prime submodule of M. Then the following statements apply:*

(i) If W is not prime, then $(W : M)W = 0$.

(ii) If M is a multiplication R-module and W is not prime, then $W^2 = 0$.

II. RELATION BETWEEN WEAKLY PRIME SUBMODULE AND PRIME SUBMODULE

In this theorem, characteristics of weakly prime submodule is given by Atani and Farzalipour (2007).

Theorem 1 [1] *Let M is unital R-module and W is a proper submodule of M. Then the following statements are equivalent:*

(i) *K is an ideal of R and S is submodule of M with $0 \neq KS \subseteq W \Rightarrow KM \subseteq W$ or $S \subseteq W$.*

(ii) *W is a weakly prime submodule of M.*

(iii) *For every $x \in M - W$ then applies $(W : Rx) = (W : M) \cup (0 : Rx)$.*

(iv) *For every $x \in M - W$ then applies $(W : Rx) = (W : M)$ or $(W : Rx) = (0 : Rx)$.*

Next given properties the product of submodule of multiplication R-module. Atani and Farzalipour (2007) have proved that if M is a multiplication R-modules and S and T are submodules of M then (i), (ii), and (iii) apply. Then we prove that the converse is also true.

Lemma 1 *Let M is a multiplication R-module and let S and T are submodules of M. Then the following statements apply:*

i *For every $s \in S$, we have $sT = 0$ if and only if $ST = 0$.*

ii *For every $t \in T$, we have $St = 0$ if and only if $ST = 0$.*

iii *For every $s \in S, t \in T$, we have $st = 0$ if and only if $ST = 0$.*

Proof. (i) $\Rightarrow$ Given in [1]
$\Leftarrow$ Let K and L are the presentation ideals of S and T, respectively. Then $0 = ST = KLM$, so $KL \subseteq (0 : M)$. Assuming that $s \in S$, we have $\Sigma_{s \in S} K_s M = \Sigma_{s \in S} Rs = S = KM$. Then

$$sT = K_s LM \subseteq \Sigma_{s \in S} K_s LM = KLM \subseteq (0 : M)M = 0.$$

(ii) This evidence is similar to that in case (i) and we remove it.

(iii) $\Rightarrow$ Given in [1]
$\Leftarrow$ Similar to that case (i), so $KL \subseteq (0 : M)$. Assuming that $s \in S, t \in T$, then

$$st = RsRt = K_s L_t M \subseteq \Sigma_{s \in S} \Sigma_{t \in T} K_s L_t M = KLM \subseteq (0 : M)M = 0.$$

□

Atani and Farzalipour (2007) have extended the result of Ameri (2002). This proves that, for certain conditions the properties of weakly prime ideal is similar to the weakly prime submodule.

Theorem 2 [1] *Let M is a multiplication R-module and M is finitely generated. Let W is a proper submodule of M. Then the following statements are equivalent.*

(i) W is weakly prime submodule of M.

(ii) Let S and T are submodules of M with $0 \neq ST \subseteq W \Rightarrow S \subseteq W$ or $T \subseteq W$.

We are motivated to continue the research about the weakly prime submodules, also to extend the result of Ameri (2002). The properties of weakly prime ideal are also similar to weakly prime submodule with some conditions.

Theorem 3 *Let M is a multiplication R-module and M is finitely generated. Let W is a proper submodule of M and S and T are submodules of M. Then the following statement applies:*

$$0 \neq ST \subseteq W \text{ satisfy either } S \subseteq W \text{ or } T \subseteq W \Rightarrow \text{for every } x, y \in M \text{ with } 0 \neq x, y \subseteq W \text{ satisfy either } x \in W \text{ or } y \in W.$$

Proof. If W is prime, then follows [2]. So, we can assume that W is not prime. Let $S = Rx$ and $T = Ry$ with $0 \neq ST = xy = RxRy \subseteq W$, so either $Rx \subseteq W$ or $Ry \subseteq W$; hence either $rx \in W$ or $ry \in W$ for every $r \in R$. Since ring R has non-zero identity, so either $x \in W$ or $y \in W$. □

However, the converse of Theorem 4 is not always true, unless for every submodule of multiplication R-modules is cyclic. Now we can prove the characters of weakly prime submodules.

Theorem 4 *Let M is a multiplication R-module and M is finitely generated and every submodule of M is cylic. If W is a proper submodule of M, then the following statements are equivalent.*

(i) W is weakly prime submodule of M.

(ii) Let S and T are submodules of M with $0 \neq ST \subseteq W \Rightarrow S \subseteq W$ or $T \subseteq W$.

(iii) For every $x, y \in M$ with $0 \neq xy \subseteq W \Rightarrow x \in W$ or $y \in W$.

Proof. The equivalence of (i) and (ii) given in [1], and $(ii) \Rightarrow (iii)$ given in Theorem 4. Now we will prove $(iii) \Rightarrow (i)$.
Suppose that $0 \neq KS \subseteq W$, where K is an ideal of R and S is a submodule of M. Because submodules of M are cyclic, we can assume that $S = Ry$ and take $Rx = KM$. Then

$$0 \neq xy = RxRy = I(Ry : M)M = KRy = KS \subseteq W.$$

So either $x \in W$ or $y \in W$ by (iii). Therefore, we have either $KM \subseteq W$ or $Ry \subseteq W$. Thus, W is weakly prime submodule by Theorem 2. □

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III. CONCLUSIONS

By the last theorem, the equivalence of weakly prime submodule is not always true for any modules. It is true when R-module M is multiplication and every submodule of M is cyclic. Consequently, to show the proper submodule of module is weakly prime is similar to show the proper ideal of ring is weakly prime by [4].

REFERENCES

[1] S. E. Atani and F. Farzalipour, ”On weakly prime submodules,” *Tamkang J. of Math.*, Vol. 38, no.3, pp. 247-252, 2007.

[2] R. Ameri, ”On the prime submodules of multiplication modules,” *Inter. J. of Math. and Mathematical Sciences*, Vol.27, pp. 1715-1724, 2002.

[3] Z. A. El-bast and P. F. Smith, ”Multiplication modules,” *Comm. in Algebra*, Vol.16, no.4, pp. 755-779, 1988.

[4] D. D. Anderson and E. Smith, ”Weakly prime ideals,” *Houston J. of Math.* Vol. 29, pp. 831-840, 2003.