JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 5 NO. 1 (JUN 2022)
Available online at www.jfma.math.fsm.undip.ac.id

IDEMPOTENT MATRIX OVER SKEW GENERALIZED POWER SERIES RINGS

Ahmad Faisol[1*], Fitriani[2]

[1,2]*Department of Mathematics, Universitas Lampung*
Email: [1]ahmadfaisol@fmipa.unila.ac.id, [2]fitriani.1984@fmipa.unila.ac.id

*Corresponding author

Abstract. Given a ring R, a monoid with a strictly ordered $(S, \leq)$, a homomorphism of monoid $\omega : S \rightarrow End(R)$, and a skew generalized power series ring $R[[S, \leq, \omega]]$. We collect all matrices over $R[[S, \leq, \omega]]$, i.e. $M_n(R[[S, \leq, \omega]])$. This set become a ring. In this research, we determine the sufficient conditions for R, $(S, \leq)$ and ω, so the element of $M_n(R[[S, \leq, \omega]])$ is an idempotent matrix.

Keywords: idempotent matrix, strictly ordered monoid, monoid homomorphism, skew generalized power series ring.

I. INTRODUCTION

A matrix is a rectangular array of numbers [1]. When the matrices entries are elements of a ring, the matrices are called matrices over a ring [2]. The ring is defined as a non-empty set that is completed with two binary operations and fulfills certain axioms [3]. One example of a ring is the skew generalized power series rings (SGPSR) $R[[S, \leq, \omega]]$ [4]. If we collect all functions f such that support of f is Artinian and narrow, then the set from a strictly ordered monoid $(S, \leq)$. Research related to the properties of SGPSR $R[[S, \leq, \omega]]$, among others, can be found in Mazurek et al. [5],[6],[7] and Faisol et al. [8],[9],[10],[11],[12],[13].

SGPSR $R[[S, \leq, \omega]]$ is a generalization of The Generalized Power Seris Rings (GPSR) $R[[S]]$ introduced by Ribenboim [14]. He used a strictly ordered monoid structure and applied Artinian and narrow concepts of set theory to construct this ring. This ring is a generalization of the ring semigroup $R[S]$[15]. On the other hand, we can express GPSR as the general form of the ring polynomial $R[X]$ and the ring power series $R[[X]]$[16]. Besides Ribenboim [17],[18],[19],[20],[21], a study related to the structure of GPSR $R[[S]]$ can also be seen in the results of Faisol et al. [22],[23],[24],[25],[26].

A set of $n \times n$ matrices with entries in a ring R that form a ring under matrix addition and matrix multiplication is called a matrix ring [27], denoted by $M_n(R)$. In 2021, Rugayah et al. [28] have constructed the set of all matrices over SGPSR, denoted by $M_n(R[[S, \leq, \omega]])$. Furthermore, they introduce the definition and the properties of the ideal matrix ring over SGPSR.

If R is a ring with an identity element, then an element e of R is idempotent if $e^2 = e$ [29]. A square matrix C is said to be idempotent in matrices theory when it has the property that $C^2 = C$ [30]. This concept gives us some motivation to investigate the sufficient conditions for the matrices over SGPSR to be an idempotent matrix. So, in this research, we will determine the sufficient conditions for an element of $M_n(R[[S, \leq, \omega]])$ to be an idempotent matrix over SGPSR $R[[S, \leq, \omega]]$.

II. MAIN RESULT

First, we review the construction of SGPSR as follows from Mazurek, and Ziembowski [4].

In this article, we express R as a ring with an identity element, $(S, \leq)$ as a strictly ordered monoid, and a homomorphism of monoid ω, where $\omega : S \to End(R)$, and ω_s as a homomorphic image of $\omega(s)$, for every $s \in S$. Furthermore, $\omega_{st} = \omega(st) = \omega(s)\omega(t) = \omega_s\omega_t$, for every $s, t \in S$. If $1 \in S$ is an identity element of S, then $\omega_1 = id_R$ is an identity element of $End(R)$.

Let $R^S = \{h | h : S \to R\}$ and

$$R[[S, \leq, \omega]] = \{h \in R^S | supp(h) \text{ is narrow and Artinian}\}$$

where $supp(h)$ is the set of all $s \in S$, such that $h(s) \neq 0$. For every $h, k \in R[[S, \leq, \omega]]$, $supp(h + k)$ is a subset of $supp(h) \cup supp(k)$, $supp(-h) = supp(h)$, and $supp(hk)$ is a subset of $supp(h) + supp(k)$. Now, we define:

$$(hk)(s) = \sum_{(x,y) \in \chi_s(h,k)} h(x)\omega_x(k(y)), \tag{1}$$

for all $h, k \in R[[S, \leq, \omega]]$, where the set

$$\chi_s(h, k) = \{(x, y) \in supp(h) \times supp(k) | xy = s\}$$

is finite. With this operation, $R[[S, \leq, \omega]]$ is a ring that we call Skew Generalized Power Series Ring (SGPSR).

For any $t \in R$ and $s \in S$, we define the maps $c_t, f_s \in R[[S, \leq, \omega]]$ as follows:

$$c_t(u) = \begin{cases} t & \text{if } u = 1 \\ 0 & \text{if } u \neq 1, \end{cases} \tag{2}$$

and

$$f_s(u) = \begin{cases} 1 & \text{if } u = s \\ 0 & \text{if } u \neq s, \end{cases} \tag{3}$$

for every $u \in S$.

Based on (2) and (3), $t \mapsto c_t$ is a ring monomorphism from R to $R[[S, \leq, \omega]]$, and $s \mapsto f_s$ is a monoid monomorphism from S to the multiplicative monoid (with composition operation) of SGPSR $R[[S, \leq, \omega]]$. Moreover, $f_s c_t = c_{\omega_s(t)} f_s$.

Example 1 Now, we give some examples of SGPSR $R[[S, \leq, \omega]]$.

1. We take $S = \mathbb{N} \cup \{0\}$ as a monoid with pointwise addition, a trivial ordered $\leq$ on S, and $\omega : S \to End(R)$ as a homomorfism of monoid with $\omega_s = id_R$, for all $s \in S$, then SGPSR $R[[S, \leq, \omega]]$ is the polynomial ring $R[X]$.

2. If we choose $S = \mathbb{N} \cup \{0\}$ as a monoid with pointwise addition, an usual ordered $\leq$ on S, $\omega : S \to End(R)$ as a homomorfism of monoid with $\omega_s = id_R$ for every $s \in S$, then SGPSR $R[[S, \leq, \omega]]$ become power series ring $R[[X]]$.

3. If we only choose $\omega : S \to End(R)$ as a monoid homomorfism with $\omega_s = id_R$ for all $s \in S$, then SGPSR $R[[S, \leq, \omega]]$ become the GPSR $R[[S]]$.

Now, we collect all idempotents of R, we donote by $E(R) = \{e \in R | e^2 = e\}$. If $\sigma(e) = e$ for any $e \in E(R)$, then σ is idempotent-stabilizing, where σ is an endomorphism of a ring R. Next, we provide the conditions for an element of SGPSR $R[[S, \leq, \omega]]$ to be an idempotent in Proposition 1.

Proposition 1 *Let $(S, \leq)$ be a strictly ordered monoid, and $\omega : S \to End(R)$ is a homomorphism of monoid, whereω_s is idempotent-stabilizing for all $s \in S$. If $f \in R[[S, \leq, \omega]]$, then f is an idempotent element of $R[[S, \leq, \omega]]$ if and only if there is an idempotent element e in R such that $f = c_{\omega_s(e)}$ for every $s \in S$.*

Proof.

1. We will show that there exist an idempotent element e in R such that $f = c_{\omega_s(e)}$.

   (*case 1: f is equal to* 0) It is clear that $f = 0$ is an idempotent of $R[[S, \leq, \omega]]$, because

   $$(ff)(s) = (00)(s) = \sum_{xy=s} 0(x)\omega_x(0(y)) = \sum_{xy=s} 0\omega_x(0) = 0 = 0(s) = f(s),$$

   for every $s \in S$.

   Since ω_s is idempotent-stabilizing for every $s \in S$, for an idempotent $e = 0 \in R$, we have $\omega_s(0) = \omega_s(e) = e = 0$ and

   $$c_{\omega_s(e)}(u) = c_0(u) = \begin{cases} 0 & \text{if } u = 1 \\ 0 & \text{if } u \neq 1, \end{cases}$$

   for every $u \in S$. Therefore, $c_{\omega_s(e)} = 0 = f$.

   (*case 2: $f \neq 0$*) Since f is an idempotent of $R[[S, \leq, \omega]]$,

   $$(ff)(u) = \sum_{xy=u} f(x)\omega_x(f(y)) = f(u), \tag{4}$$

   for every $u \in S$.

   If $u = 1 \in S$, then

   $$(ff)(1) = \sum_{xy=1} f(x)\omega_x(f(y)) = f(1)\omega_1(f(1)) = f(1)id_R(f(1)) = f(1)f(1).$$

   Therefore, based on (4), we get $f(1)f(1) = (ff)(1) = f(1)$. In the other word, there is an idempoten $e = f(1) \in R$.

If $u \neq 1 \in S$, then

$$(ff)(u) = \sum_{xy=u} f(x)\omega_x(f(y))$$

and

$$f(u) = \sum_{xy=u} f(x)\omega_x(f(y))$$

This conditions, only holds for $f(u) = 0$.

Since ω_s is idempotent-stabilizing for every $s \in S$, we obtain

$$f(u) = \begin{cases} e & \text{if } u = 1 \\ 0 & \text{if } u \neq 1 \end{cases} = \begin{cases} \omega_s(e) & \text{if } u = 1 \\ 0 & \text{if } u \neq 1, \end{cases} = c_{\omega_s(e)}(u)$$

for every $u, s \in S$. Therefore, $f = c_{\omega_s(e)}$.

In the other side, we will show that if e is an idempotent of a ring R dan $c_{\omega_s(e)} = f$ for all $s \in S$, then f is an idempotent element of $R[[S, \leq, \omega]]$.

For every $u, s \in S$, we get:

$$\begin{aligned} (ff)(u) &= (c_{\omega_s(e)}c_{\omega_s(e)})(u) \\ &= \sum_{xy=u} c_{\omega_s(e)}(x)\omega_x(c_{\omega_s(e)}(y)) \\ &= c_{\omega_s(e)}(1)\omega_1(c_{\omega_s(e)}(u)) + 0 \\ &= \omega_s(e) id_R(c_{\omega_s(e)}(u)) \\ &= \omega_s(e)c_{\omega_s(e)}(u) \end{aligned}$$

Since ω_s is idempotent-stabilizing, if $u = 1$, then

$$(ff)(1) = \omega_s(e)c_{\omega_s(e)}(1) = e\omega_s(e) = ee = e = \omega_s(e).$$

If $u \neq 1$, then

$$(ff)(u) = \omega_s(e)c_{\omega_s(e)}(u) = e0 = 0.$$

Therefore, we obtain $(ff)(u) = c_{\omega_s(e)}(u) = f(u)$, which is prove that f is an idempotent element of $R[[S, \leq, \omega]]$.

□

We can see the definition of set of all matrices over SGPSR $R[[S,\leq,\omega]]$ as follows from Rugayah et al. [28].

Proposition 2 *Let R be a ring with $1_R \in R$, e an idempotent element of R, $(S,\leq)$ a strictly ordered monoid, and ω is a homomorphism of monoid from S to $End(R)$, where ω_s is idempotent-stabilizing for every $s \in S$. If $E_n(R[[S,\leq,\omega]]) = [f_{ij}] \in M_n(R[[S,\leq,\omega]])$ such that*

$$f_{ij} = \begin{cases} c_{\omega_s(e)} & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

for all $s \in S$, then $E_n(R[[S,\leq,\omega]])$ is an idempotent matrix over $R[[S,\leq,\omega]]$.

Proof. For $1 \leq i \leq n$ and $1 \leq j \leq n$, we get $E_n^2(R[[S,\leq,\omega]]) = [f_{ij}][f_{ij}] = [\alpha_{ij}]$ where

$$\alpha_{ij} = \begin{cases} c^2_{\omega_s(e)} & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

Based on Proposition 1, $f = c_{\omega_s(e)}$ is an idempotent element of $R[[S,\leq,\omega]]$. Therefore,

$$\alpha_{ij} = \begin{cases} c_{\omega_s(e)} & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases} = f_{ij}.$$

In the other word, $E_n^2(R[[S,\leq,\omega]]) = [f_{ij}][f_{ij}] = [\alpha_{ij}] = E_n(R[[S,\leq,\omega]])$. Hence, $E_n(R[[S,\leq,\omega]])$ is an idempotent matrix over $R[[S,\leq,\omega]]$. □

III. CONCLUSIONS AND FUTURE RESEARCH DIRECTION

According to the main results, if the homomorphic image of a monoid homomorphism ω is idempotent-stabilizing, then an element f of SGPSR $R[[S,\leq,\omega]]$ is idempotent element if and only if ring R has an idempotent element e such that $f = c_{\omega_s(e)}$, for every $s \in S$. Furthermore, a matrix over SGPSR $R[[S,\leq,\omega]]$ is an idempotent matrix if it is a diagonal matrix with entries on the main diagonal is equal to $c_{\omega_s(e)}$.

For further research, there is an opportunity to study the necessary and sufficient conditions for a matrix over SGPSR $R[[S,\leq,\omega]]$ to be a unit element and a nilpotent element of $M_n(R[[S,\leq,\omega]])$.

ACKNOWLEDGEMENT

The authors would like to thank the LPPM of Universitas Lampung for the support and funding for this study under research contract number: 661/UN 26.21/PN/2022.

REFERENCES

[1] H. Anton and C. Rorres, *Elementary Linear Algebra: Applications Version, 9th Edition*. New Jersey, 2005.

[2] W. C. Brown, *Matrices Over Commutative Rings*. New York: : Marcel Dekker Inc., 1993.

[3] D. S. Dummit and R. M. Foote, *Abstract Algebra, Third Edit*. John Wiley and Sons, Inc., 2004.

[4] R. Mazurek and M. Ziembowski, Uniserial Rings of Skew Generalized Power Series, *J. Algebra*, vol. 318, no. 2, pp. 737-764, 2007.

[5] R. Mazurek and M. Ziembowski, On Von Neumann Regular Rings of Skew Generalized Power Series, *Communications in Algebra*, vol. 36, no. 5, pp. 1855-1868, 2008.

[6] R. Mazurek and M. Ziembowski, The ascending chain condition for principal left or right ideals of skew generalized power series rings, *J. Algebr.*, vol. 322, no. 4, pp. 983-994, 2009.

[7] R. Mazurek and M. Ziembowski, Weak dimension and right distributivity of skew generalized power series rings, *J. Math. Soc. Japan*, vol. 62, no. 4, pp. 1093-1112, 2010.

[8] A. Faisol, Homomorfisam Ring Deret Pangkat Teritlak Miring, *J. Sains MIPA*, vol. 15, no. 2, pp. 119-124, 2009.

[9] A. Faisol, Pembentukan Ring Faktor Pada Ring Deret Pangkat Teritlak Miring, in Prosiding Semirata FMIPA Univerisitas Lampung, 2013, pp. 1-5.

[10] A. Faisol, Endomorfisma Rigid dan Compatible pada Ring Deret Pangkat Tergeneralisasi Miring, *J. Mat.*, vol. 17, no. 2, pp. 45-49, 2014.

[11] A. Faisol, B. Surodjo, and S. Wahyuni, Modul Deret Pangkat Tergeneralisasi Skew T-Noether, in Prosiding Seminar Nasional Aljabar, Penerapan dan Pembelajarannya, 2016, pp. 95-100.

[12] A. Faisol, B. Surodjo, and S. Wahyuni, The Impact of the Monoid Homomorphism on The Structure of Skew Generalized Power Series Rings, *Far East J. Math. Sci.*, vol. 103, no. 7, pp. 1215-1227, 2018.

[13] A. Faisol and Fitriani, The Sufficient Conditions for Skew Generalized Power Series Module $M[[S,w]]$ to be $T[[S,w]]$-Noetherian $R[[S,w]]$-module, *Al-Jabar J. Pendidik. Mat.*, vol. 10, no. 2, pp. 285-292, 2019.

[14] P. Ribenboim, Generalized Power Series Rings, in Lattices, Semigroups, and Universal Algebra, 1990.

[15] R. Gilmer, *Commutative Semigroup Rings*. Chicago and London: The University of Chicago, 1984.

[16] T. W. Hungerford, *Algebra*. New York: Springer-Verlag New York, Inc., 1974.

[17] P. Ribenboim, Rings of Generalized Power Series: Nilpotent Elements, *Abh. Math. Sem. Univ. Hambg.*, vol. 61, pp. 15-33, 1991.

[18] P. Ribenboim, Noetherian Rings of Generalized Power Series, *J. Pure Appl. Algebr.*, vol. 79, no. 3, pp. 293-312, 1992.

[19] P. Ribenboim, Rings of Generalized Power Series II: Units and Zero-Divisors, *J. Algebr.*, vol. 168, pp. 71-89, 1994.

[20] P. Ribenboim, Special Properties of Generalized Power Series, *J. Algebr.*, vol. 173, pp. 566-586, 1995.

[21] P. Ribenboim, Semisimple Rings and Von Neumann Regular Rings of Generalized Power Series, *J. Algebr.*, vol. 198, no.2, pp. 327-338, 1997.

[22] A. Faisol, B. Surodjo, and S. Wahyuni, $T[[S]]$-Noetherian Property on Generalized Power Series Modules, *JP J. Algebr. Number Theory Appl.*, vol. 43, no. 1, pp. 1-12, 2019.

[23] A. Faisol, B. Surodjo, and S. Wahyuni, The Sufficient Conditions for $R[X]$-module $M[X]$ to be $S[X]$-Noetherian, *European J. Math. Sci.*, vol. 5, no. 1, pp. 1-13, 2019.

[24] A. Faisol, B. Surodjo, and S. Wahyuni, The Relation between Almost Noetherian Module, Almost Finitely Generated Module and T-Noetherian Module, in Journal of Physics: Conference Series, Sep. 2019, vol. 1306, no. 1.

[25] W.A. Pardede, A. Faisol, and Fitriani, The $X[[S]]$-Sub-Exact Sequence of Generalized Power Series Rings, *Al-Jabar: Jurnal Pendidikan Matematika*, vol. 11, no. 2, pp. 299-306, 2020.

[26] A. Faisol, Fitriani, and Sifriyani, Determining the Noetherian Property of Generalized Power Series Modules by Using X-Sub-Exact Sequence, in Journal of Physics: Conference Series, Jan. 2021, vol. 1751, no. 1.

[27] T. Y. Lam, *A First Course in Noncommutative Rings*. Springer-Verlag, New York, Inc., 1991.

[28] S. Rugayah, A. Faisol, and Fitriani, Matriks atas Ring Deret Pangkat Tergeneralisasi Miring, *Barekeng: Jurnal Ilmu Matematika dan Terapan*, vol. 15, no. 1, pp. 157-166, 2021.

[29] W. A. Adkins and S. H. Weintraub, *Algebra: An Approach via Module Theory*. New York: Springer-Verlag, New York, Inc., 1992.

[30] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*.SIAM, 2000.