JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 4 NO. 2 (NOV 2021)
Available online at www.jfma.math.fsm.undip.ac.id

SOME CARTESIAN PRODUCTS OF A PATH AND PRISM RELATED GRAPHS THAT ARE EDGE ODD GRACEFUL

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Abstract. Let G be a connected undirected simple graph of size q and let k be the maximum number of its order and its size. Let f be a bijective edge labeling which codomain is the set of odd integers from 1 up to $2q-1$. Then f is called an edge odd graceful on G if the weights of all vertices are distinct, where the weight of a vertex v is defined as the sum $mod(2k)$ of all labels of edges incident to v. Any graph that admits an edge odd graceful labeling is called an edge odd graceful graph. In this paper, some new graph classes that are edge odd graceful are presented, namely some cartesian products of path of length two and some circular related graphs.

Keywords: edge odd graceful graphs, edge odd graceful labeling, cycle, prism graphs, antiprism graphs, path, cartesian product.

I. INTRODUCTION

Labeling, according to Walis, [18], is a function from the set of graph elements (vertices or edges) to a set of numbers, that are usually integers. A labeling which domain is the set of edges (vertices or edges and vertices), is respectively called edge (vertex or total) labeling. It is known from the survey presented in [6] that some sorts of labeling have been defined since 1960. Many kind of labelings, including harmonious labeling, graceful labeling, anti magic labeling, magic labeling, geometric labeling, mean labeling and irregular labeling, etc. are mentioned in the survey. And for those mentioned labeling, many classes of graphs have been investigated, for instance in [1], [2], [4], [5], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]. One other labeling called edge odd graceful labeling is also introduced, by Solairaju et.all [16]. Let $G = (V(G), E(G))$ be a simple undirected graph of order n and of size m and let $k = max\{n, m\}$ and let $f : E(G) \longrightarrow \{1, 3, ..., 2m-1\}$ be an edge labeling on G. For any vertex $v \in V(G)$ it is defined the weight $wt_f(v)$ of v respect to f by $wt_f(v) = (\sum_{vw \in E(G)} f(vw))mod(2k)$. Then f is called edge odd graceful labeling on G if for every two vertices v_1 and v_2, the weights $wt_f(v_1)$ and $wt_f(v_2)$ are distinct. Graph G is called edge odd graceful graph if G admits an edge odd graceful labeling In [5], it is shown that some classes of graphs are edge odd graceful, namely, wheel graphs, double wheel graphs, web graphs, helm graphs, gear graphs, fan graphs, double fan graphs and polar grid graphs.

II. Preliminaries

In this section we give some preliminary terminologies and definitions. All graphs in this paper is simple and undirected. Let G_1 and G_2 be two graphs. The cartesian product of G_1 and G_2, denoted by $G_\times G_2$ is a graph with vertex set $V(G_1) \times V(G_2)$ and two arbitrary vertices

(x_1, y_1) and (x_2, x_2) in $V(G_1 \times G_2)$ are connected by an edge if and only if $x_1 = x_2$ and $y_1y_2 \in E(G_2)$ or $y_1 = y_2$ and $x_1x_2 \in E(G_2)$. As an example, see Figure 4.. By path graph P_n of order n, we mean a connected graph of n vertices, such that two of them are of degree 1 and the remaining vertices are of degree 2. (See Figure 1.). By a cycle C_n of order n we mean a connected 2-regular graph as we can see in Figure 2..

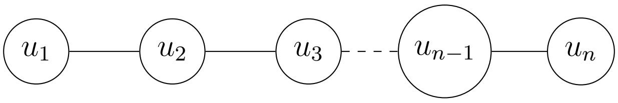

Figure 1. Path P_n

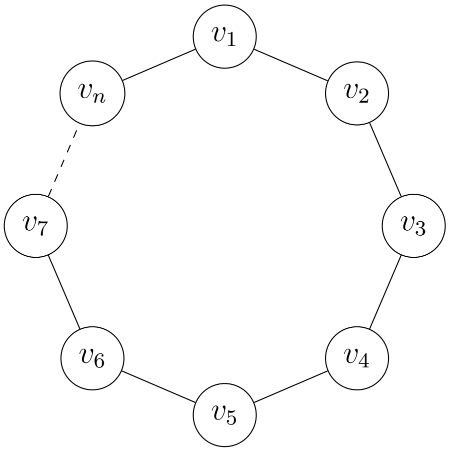

Figure 2. Cycle C_n

Let the vertex sets of P_2 and C_3 be $V(P_2) = \{u_1, u_2\}$ and $V(C_3) = \{v_1, v_2, v_3\}$, respectively. The cartesian product $P_2 \times C_3$ is given in the Figure 3..

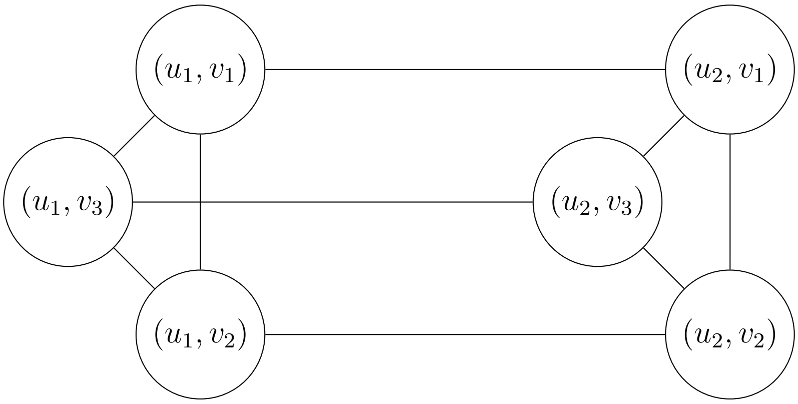

Figure 3. Cartesian Product Graph $P_2 \times C_3$

In this paper, we consider some cartesian products of path of length two and some graphs, namely cycle, prism graphs, antiprism graphs and double sun flower graphs. We prove that the resulted cartesian product graphs are edge odd graceful.

III. RESULTS

For the first observation, we investigate the cartesian product of path of length two with a cycle.

3.1. Cartesian Product of Path P_2 and Cycle C_n

Below we prove that the cartesian product of path P_2 and cycle C_n, $n \geq 3$ is edge odd graceful.

Theorem 1 *Let $n \geq 3$ be an arbitrary positive integer. Then the graph $P_2 \times C_n$ is edge odd graceful.*

Proof. Let the vertex set and the edge set of $P_2 \times C_n$ be

$$V(P_2 \times C_n) = \{a_i, b_i | i = 1, 2, \ldots, n\}$$

and

$$E(P_2 \times C_n) = \{a_i a_{(i+1) mod n}, b_i b_{(i+1) mod n}, a_i, b_i | i = 1, 2, \ldots, n\},$$

respectively. It is clear that $|V(P_2 \times C_n)| = 2n$ and $|E(P_2 \times C_n)| = 3n$ so that we have $k = max|V(P_2 \times C_n)|, |E(P_2 \times C_n)| = 3n$. We define an edge odd labeling

$$f_1 : E(P_2 \times C_n) \to \{1, 3, \ldots, 6n - 1\}$$

in the following manner :

$$\begin{aligned} f(a_i a_{(i+1) mod n}) &= 2i - 1 \\ f(b_i b_{(i+1) mod n}) &= 2n + 2i - 1 \\ f(a_i b_i) &= 6n - 2i + 1 \end{aligned}$$

for all $i = 1, 2, \ldots, n$. It is clear that

$$\begin{aligned} \{f(a_i a_{(i+1) mod n}) | i = 1, 2, \ldots, n\} &= \{1, 3, \ldots, 2n - 1\} \\ \{f(b_i b_{(i+1) mod n}) | i = 1, 2, \ldots, n\} &= \{2n + 1, 2n + 3, \ldots, 4n - 1\} \\ \{f(a_i b_i) | i = 1, 2, \ldots, n\} &= \{4n + 1, 4n + 3, \ldots, 6n - 1\}. \end{aligned}$$

Therefore, f_1 is a bijection. Moreover, we have the following vertex weights $mod\ 6n$:

$$\begin{aligned} wt_{f_1}(a_1) &= 2n - 1 \\ wt_{f_1}(b_1) &= 6n - 1 \\ wt_{f_1}(a_i) &= 2i - 3, && \text{for } i = 2, 3, \ldots, n \\ wt_{f_1}(b_i) &= 4n + 2i - 3, && \text{for } i = 2, 3, \ldots, n, \end{aligned}$$

which are all different. Therefore f_1 is an edge odd graceful labeling. This prove that $P_2 \times C_n$ is an edge odd graceful. □

On Figure 4. we give an example of the labeling for $P_2 \times C_3$. The weights of the vertices are the following:

$$\begin{aligned} wt_{f_1}(a_1) &= 5 \qquad & wt_{f_1}(b_1) &= 11 \\ wt_{f_1}(a_2) &= 1 \qquad & wt_{f_1}(b_2) &= 17 \\ wt_{f_1}(a_3) &= 3 \qquad & wt_{f_1}(b_3) &= 15. \end{aligned}$$

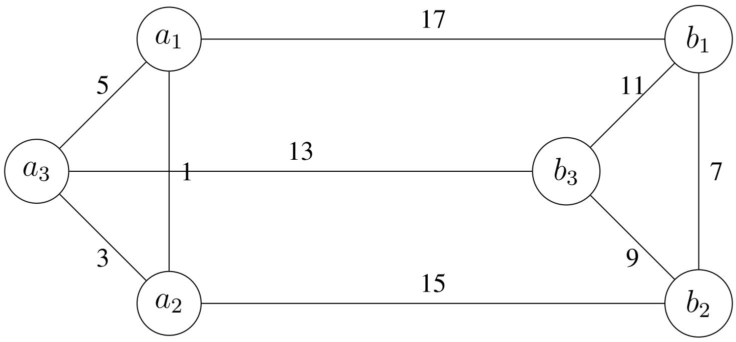

Figure 4. Cartesian Product Graph $P_2 \times C_3$

3.2. Cartesian Product of Path P_2 and Antiprism Graph A_n

By prism graph D_n, we mean a graph of $2n$ vertices, comprising an outer and an inner cycles of size n and some additional n edges connecting vertices of the outer cycle and vertices of the inner cycle (see Figure 5.). By antiprism graph A_n we mean a graph obtained from the prism graph D_n by adding some diagonal edges. As example of an antiprism graph, see Figure 6..

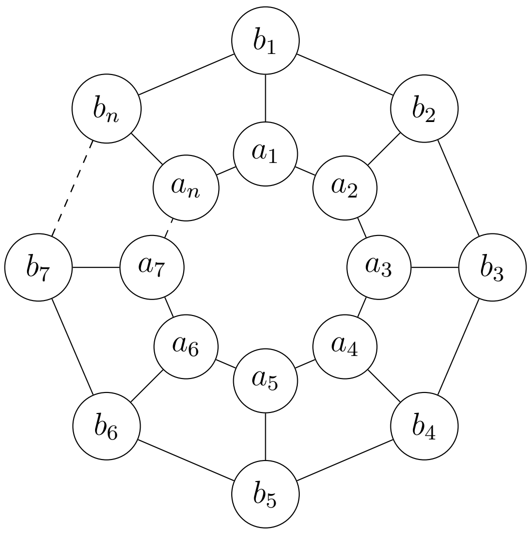

Figure 5. Prism Graph D_n

In the following theorem we prove that for arbitrary positive integer $n \geq 3$, cartesian product of path P_2 and antiprism graph A_n is an edge odd graceful graph.

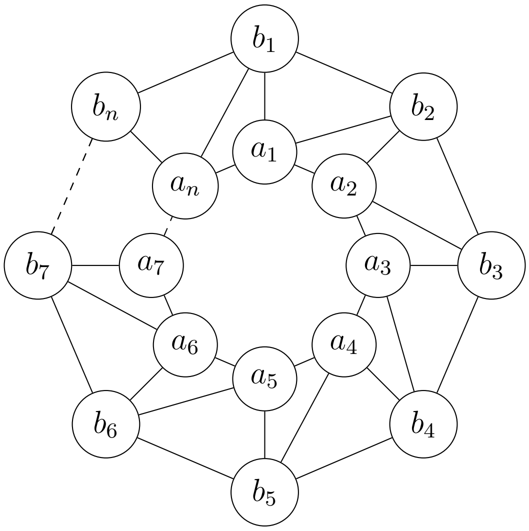

Figure 6. Antiprism Graph A_n

Theorem 2 *For arbitrary positive integer $n \geq 3$, the cartesian product $P_2 \times A_n$ is edge odd graceful.*

Proof. Let the vertices and the edges of $P_2 \times A_n$ be

$$V(P_2 \times A_n) = \{a_i, b_i, A_i, B_i | i = 1, 2, \ldots, n\}$$

and

$$E(P_2 \times A_n) = \{a_i a_{(i+1)modn} | i = 1, 2, \ldots, n\} \cup \{b_i b_{(i+1)modn} | i = 1, 2, \ldots, n\} \cup$$
$$\{a_i b_i, a_i b_{(i+1)modn} | i = 1, 2, \ldots, n\} \cup \{A_i A_{(i+1)modn} | i = 1, 2, \ldots, n\} \cup$$
$$\{B_i B_{(i+1)modn} | i = 1, 2, \ldots, n\} \cup \{A_i B_i, A_i B_{(i+1)modn} | i = 1, 2, \ldots, n\} \cup$$
$$\{a_i A_i, b_i B_i | i = 1, 2, \ldots, n\}.$$

It is clear that $|V(P_2 \times A_n)| = 4n$ and $|E(P_2 \times A_n)| = 10n$ and hence we have $k = max\{|V(P_2 \times A_n)|, |E(P_2 \times A_n)|\} = 10n$. Now we construct an edge odd labeling

$$f_2 : E(P_2 \times A_n) \rightarrow \{1, 2, \ldots, 20n - 1\}$$

by

$$\begin{aligned}
f_2(a_ia_{(i+1)modn}) &= 2n-2i+1\\
f_2(b_ib_{(i+1)modn}) &= 8n-2i+1\\
f_2(a_ib_{(i+1)modn}) &= 14n+2i-1\\
f_2(a_ib_i) &= 8n+2i-1\\
f_2(A_iA_{(i+1)modn}) &= 4n+2i-1\\
f_2(B_iB_{(i+1)modn}) &= 12n-2i+1\\
f_2(A_iB_{(i+1)modn}) &= 18n+2i-1\\
f_2(A_iB_i) &= 14n-2i+1\\
f_2(a_iA_i) &= 4n-2i+1\\
f_2(b_iB_i) &= 16n+2i-1
\end{aligned}$$

for all $i = 1, 2, \ldots, n$. It is clear that

$$\begin{aligned}
\{f_2(a_ia_{(i+1)modn})|i=1,2,\ldots,n\} &= \{1,3,\ldots,2n-1\}\\
\{f_2(b_ib_{(i+1)modn})|i=1,2,\ldots,n\} &= \{6n+1,6n+3,\ldots,8n-1\}\\
\{f_2(a_ib_{(i+1)modn})|i=1,2,\ldots,n\} &= \{14n+1,14n+3,\ldots,16n-1\}\\
\{f_2(a_ib_i)|i=1,2,\ldots,n\} &= \{8n+1,8n+3,\ldots,10n-1\}\\
\{f_2(A_iA_{(i+1)modn})|i=1,2,\ldots,n\} &= \{4n+1,4n+3,\ldots,6n-1\}\\
\{f_2(B_iB_{(i+1)modn})|i=1,2,\ldots,n\} &= \{10n+1,10n+3,\ldots,12n-1\}\\
\{f_2(A_iB_{(i+1)modn})|i=1,2,\ldots,n\} &= \{18n+1,18n+3,\ldots,20n-1\}\\
\{f_2(A_iB_i)|i=1,2,\ldots,n\} &= \{12n+1,12n+3,\ldots,14n-1\}\\
\{f_2(a_iA_i)|i=1,2,\ldots,n\} &= \{2n+1,2n+3,\ldots,4n-1\}\\
\{f_2(b_iB_i)|i=1,2,\ldots,n\} &= \{16n+1,16n+3,\ldots,18n-1\}
\end{aligned}$$

confirming that f_2 is a bijection. Furthermore we then obtain the following vertex weights $mod\ 2k$ for $2k = 20n$ under the labeling f_2:

$$\begin{aligned}
&wt_{f_3}(a_1) = 8n+1 &&\\
&wt_{f_3}(a_i) = 10n-2i+3 && \text{for } i = 2,3,\ldots,n\\
&wt_{f_3}(b_1) = 14n+1 &&\\
&wt_{f_3}(b_i) = 14n+2i-1 && \text{for } i = 2,2,\ldots,n\\
&wt_{f_3}(A_1) = 6n-1 &&\\
&wt_{f_3}(A_i) = 4n+2i-3 && \text{for } i = 2,3,\ldots,n\\
&wt_{f_3}(B_1) = 12n-1 &&\\
&wt_{f_3}(B_i) = 12n-2i+1 && \text{for } i = 2,3,\ldots,n.
\end{aligned}$$

It is easy to check that

$$
\begin{aligned}
\{wt_{f_3}(a_1)\} &= \{8n+1\}\\
\{wt_{f_3}(a_i)|i=2,3,\ldots,n\} &= \{8n+3, 8n+5, \ldots, 10n-1\}\\
\{wt_{f_3}(b_1)\} &= \{14n+1\}\\
\{wt_{f_3}(b_i)|i=2,3,\ldots,n\} &= \{14n+3, 14n+5, \ldots 16n-1\}\\
\{wt_{f_3}(A_1)\} &= 6n-1\\
\{wt_{f_3}(A_i)|i=2,3,\ldots,n\} &= \{4n+1, 4n+3, \ldots, 6n-3\}\\
\{wt_{f_3}(B_1)\} &= 12n-1\\
\{wt_{f_3}(B_i)|i=2,3,\ldots,n\} &= \{10n+1, 10n+3, \ldots, 12n-3\}
\end{aligned}
$$

showing that the weights are all distinct. Hence f_2 is an edge odd graceful on $P_2 \times A_n$ and therefore $P_2 \times A_n$ is an edge oddd graceful. □

In Figure 7., we give an illustration of an edge odd labeling on $P_2 \times A_8$. We separate the graph into several part to make the labeling look clear.

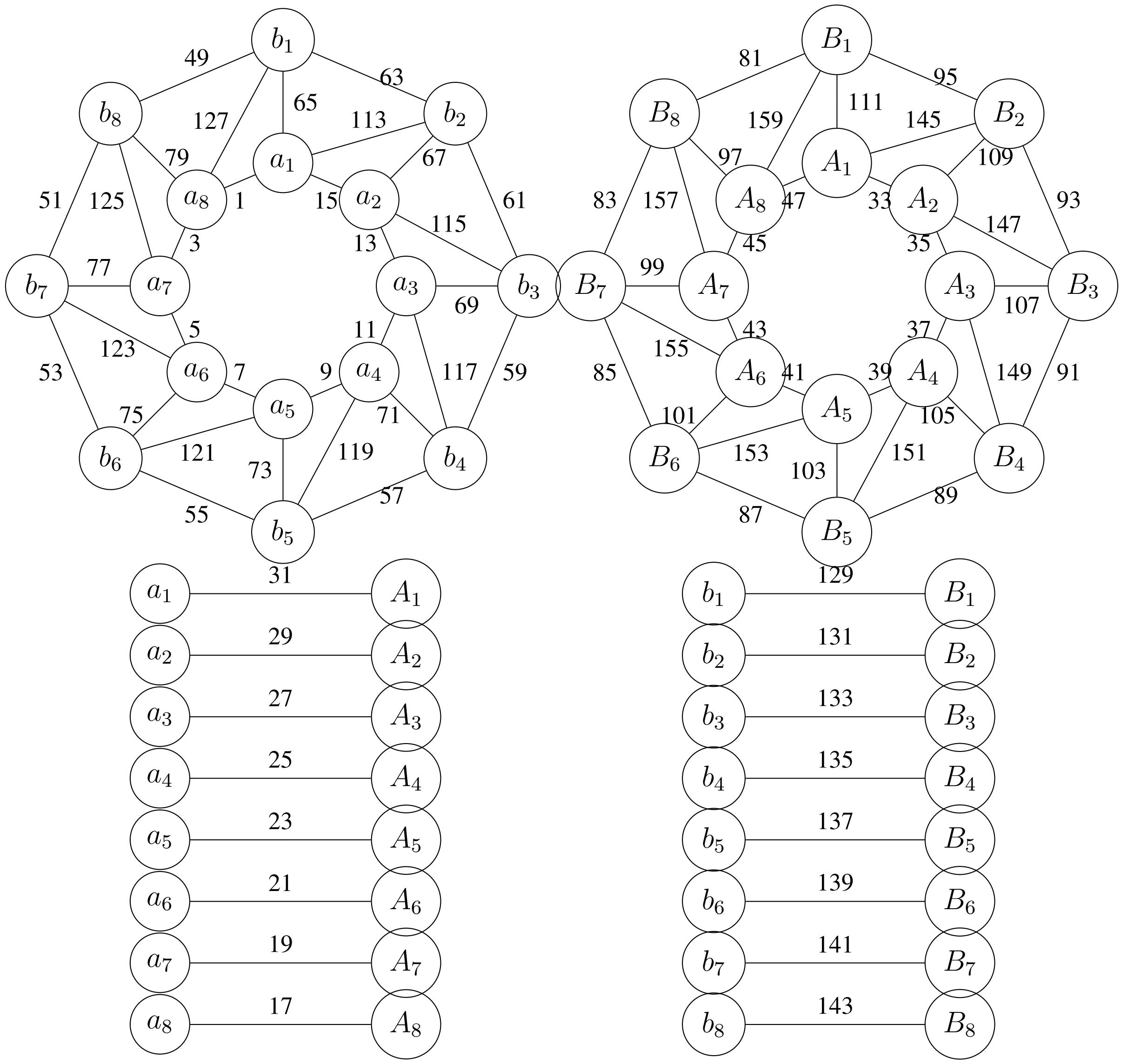

Figure 7. Edge Odd Graceful Labeling on Cartesian Product $P_2 \times A_8$

From the Figure 7., we obtain vertex weights mod 160 respect to labeling f_2 as follows:

$$\begin{array}{llll}
wt_{f_3}(a_1) = 65 & wt_{f_3}(b_1) = 113 & wt_{f_3}(A_1) = 47 & wt_{f_3}(B_1) = 95 \\
wt_{f_3}(a_2) = 79 & wt_{f_3}(b_2) = 115 & wt_{f_3}(A_2) = 33 & wt_{f_3}(B_2) = 93 \\
wt_{f_3}(a_3) = 77 & wt_{f_3}(b_3) = 117 & wt_{f_3}(A_3) = 35 & wt_{f_3}(B_3) = 91 \\
wt_{f_3}(a_4) = 75 & wt_{f_3}(b_4) = 119 & wt_{f_3}(A_4) = 37 & wt_{f_3}(B_4) = 89 \\
wt_{f_3}(a_5) = 73 & wt_{f_3}(b_5) = 121 & wt_{f_3}(A_5) = 39 & wt_{f_3}(B_5) = 87 \\
wt_{f_3}(a_6) = 71 & wt_{f_3}(b_6) = 123 & wt_{f_3}(A_6) = 41 & wt_{f_3}(B_6) = 85 \\
wt_{f_3}(a_7) = 69 & wt_{f_3}(b_7) = 125 & wt_{f_3}(A_7) = 43 & wt_{f_3}(B_7) = 83 \\
wt_{f_3}(a_8) = 67 & wt_{f_3}(b_8) = 127 & wt_{f_3}(A_8) = 45 & wt_{f_3}(B_8) = 81.
\end{array}$$

3.3. Cartesian Product of Path P_2 and Double Sun Flower DSF_n

Definition 1 *A sun flower graph of order $2n$, denoted by SF_n, is a graph that is isomorphic to a graph obtained by deleting edges of the outer cycle from antiprism graph A_n (see FIGURE 8.).*

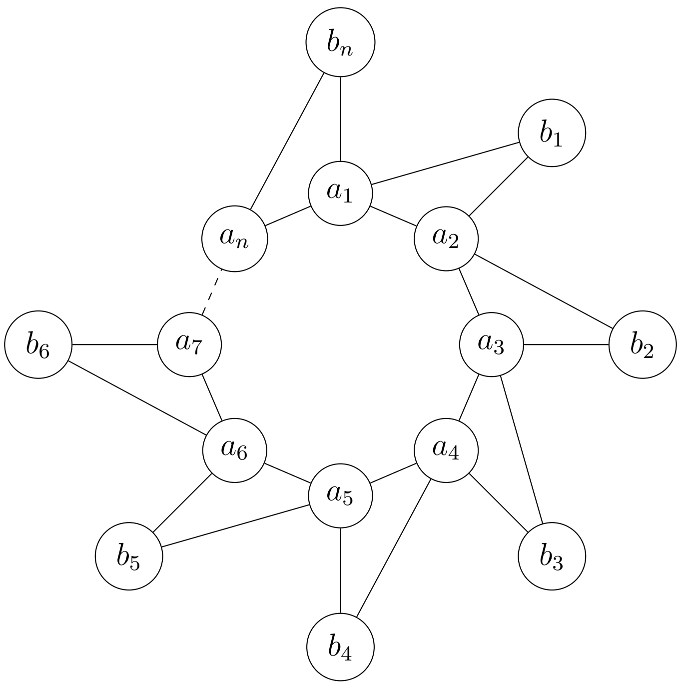

Figure 8. Sun Flower Graph SF_n

The vertex set and the edge set of SF_n of order $2n$, are, respectively

$$V(SF_n) = \{a_i, b_i | i = 1, 2, \ldots, n\}$$

and

$$E(SF_n) = \{a_i a_{(i+1) mod\ n}, a_i b_i, a_{(i+1) mod\ n} b_i | i = 1, 2, \ldots, n\}.$$

Thus we have $|V(SF_n)| = 2n$ and $|E(SF_n)| = 3n$.

Definition 2 *By a double sun flower graph of order $3n$, denoted by DSF_n, is a graph obtained from the graph SF_n (see FIGURE 8.) by inserting a new vertex c_i on each edges $a_i a_{i+1}$ and adding edges $b_i c_i$ for each i. (See FIGURE 9.)*

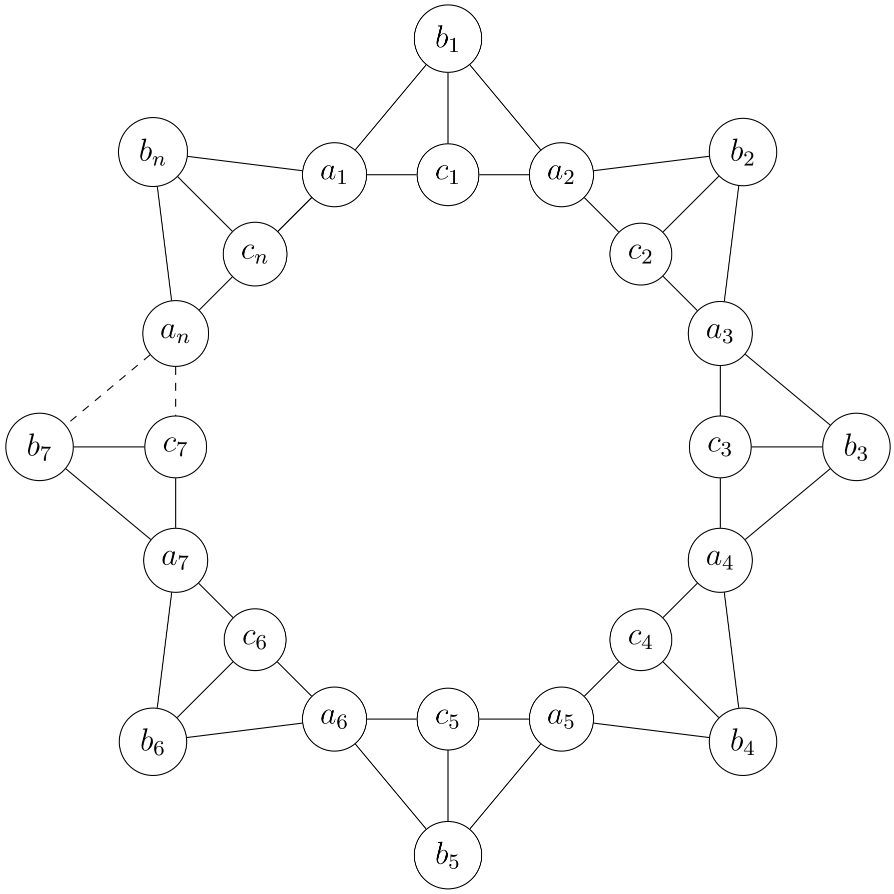

Figure 9. Double Sun Flower Graph DSF_n

Theorem 3 *Let $n \geq 3$ be an arbitrary positive integer. Then the graph $P_2 \times DSF_n$ is edge odd graceful.*

Proof. Let the vertex set and the edge set of $P_2 \times DSF_n$ be

$$V(P_2 \times DSF_n) = \{a_i, b_i, c_i, A_i, B_i, C_i | i = 1, 2, \ldots, n\}$$

and

$$E(P_2 \times DSF_n) = \{a_i b_i, a_i c_i, b_i c_i, b_i a_{(i+1)modn}, c_i a_{(i+1)modn} | i = 1, 2, \ldots, n\} \cup$$
$$\{A_i B_i, A_i C_i, B_i C_i | i = 1, 2, \ldots, n\} \cup$$
$$\{B_i A_{(i+1)modn}, C_i A_{(i+1)modn}, a_i A_i, b_i B_i, c_i C_i | i = 1, 2, \ldots, n\}.$$

It is clear that

$$|V(P_2 \times DSF_n)| = 6n$$

and

$$|E(P_2 \times DSF_n)| = 13n.$$

We construct an edge labeling $f_3 : E(P_2 \times DSF_n) \to \{1, 3, \ldots, 26n-1\}$ for two cases, whenever n is odd and n is even.

Case for n odd

We define the labeling as follows:

$$\begin{aligned}
f_3(a_ib_i) &= 20n - 2i + 1\\
f_3(a_ic_i) &= 16n - 2i + 1\\
f_3(b_ic_i) &= 12n + 2i - 1\\
f_3(b_ia_{(i+1)modn}) &= 10n + 2i - 1\\
f_3(c_ia_{(i+1)modn}) &= 18n - 2i + 1\\
f_3(A_iB_i) &= 10n - 2i + 1\\
f_3(A_iC_i) &= 6n - 2i + 1\\
f_3(B_iC_i) &= 22n + 2i - 1\\
f_3(B_iA_{(i+1)modn}) &= 2i - 1\\
f_3(C_iA_{(i+1)modn}) &= 8n - 2i + 1\\
f_3(a_iA_i) &= 20n + 2i - 1\\
f_3(b_iB_i) &= 2n + 2i - 1\\
f_3(c_iC_i) &= 26n - 2i + 1
\end{aligned}$$

for all $i = 1, 2, \ldots, n$. It is easy to see that

$$\begin{aligned}
\{f_3(a_ib_i)|i = 1, 2, \ldots, n\} &= \{18n + 1, 18n + 3, \ldots, 20n - 1\}\\
\{f_3(a_ic_i)|i = 1, 2, \ldots, n\} &= \{14n + 1, 14n + 3, \ldots, 16n - 1\}\\
\{f_3(b_ic_i)|i = 1, 2, \ldots, n\} &= \{12n + 1, 12n + 3, \ldots, 14n - 1\}\\
\{f_3(b_ia_{(i+1)modn})|i = 1, 2, \ldots, n\} &= \{10n + 1, 10n + 3, \ldots, 12n - 1\}\\
\{f_3(c_ia_{(i+1)modn})|i = 1, 2, \ldots, n\} &= \{16n + 1, 16n + 3, \ldots, 18n - 1\}\\
\{f_3(A_iB_i)|i = 1, 2, \ldots, n\} &= \{8n + 1, 8n + 3, \ldots, 10n - 1\}\\
\{f_3(A_iC_i)|i = 1, 2, \ldots, n\} &= \{4n + 1, 4n + 3, \ldots, 6n - 1\}\\
\{f_3(B_iC_i)|i = 1, 2, \ldots, n\} &= \{22n + 1, 22n + 3, \ldots, 24n - 1\}\\
\{f_3(B_iA_{(i+1)modn})|i = 1, 2, \ldots, n\} &= \{1, 3, \ldots, 2n - 1\}\\
\{f_3(C_iA_{(i+1)modn})|i = 1, 2, \ldots, n\} &= \{6n + 1, 6n + 3, \ldots, 8n - 1\}\\
\{f_3(a_iA_i)|i = 1, 2, \ldots, n\} &= \{20n + 1, 20n + 3, \ldots, 22n - 1\}\\
\{f_3(b_iB_i)|i = 1, 2, \ldots, n\} &= \{2n + 1, 2n + 3, \ldots, 4n - 1\}\\
\{f_3(c_iC_i)|i = 1, 2, \ldots, n\} &= \{24n + 1, 24n + 3, \ldots, 26n - 1\}.
\end{aligned}$$

Thus f_3 is a bijection. Moreover, we then have vertex weights $mod2k$, where $k = max\{|V(P_2 \times DSF_n)|, |E(P_2 \times DSF_n)|\} = max\{5n, 13n\} = 13n$ as follows:

$wt_{f_3}(a_1) = 6n - 1$	$wt_{f_3}(A_1) = 18n - 1$	
$wt_{f_3}(a_i) = 6n - 2i + 1$	$wt_{f_3}(A_i) = 18n - 2i + 1$	for $i = 2, 3, \dots, n$
$wt_{f_3}(b_i) = 18n + 4i - 2$	$wt_{f_3}(B_i) = 8n + 4i - 2$	for $i = 1, 2, \dots, n$
$wt_{f_3}(c_i) = 20n - 4i + 2$	$wt_{f_3}(C_i) = 10n - 4i + 2$	for $i = 1, 2, \dots, n$

so that

$$\begin{array}{lcl}
\{wt_{f_3}(a_1)\} & = & \{6n-1\} \\
\{wt_{f_3}(a_i)|i=2,3,\dots,n\} & = & \{4n+1, 4n+3, \dots, 6n-3\} \\
\{wt_{f_3}(b_i)|i=1,2,\dots,n\} & = & \{18n+2, 18n+6, \dots, 22n-1\} \\
\{wt_{f_3}(c_i)|i=1,2,\dots,n\} & = & \{16n+2, 16n+6, \dots, 20n-2\} \\
\{wt_{f_3}(A_1)\} & = & \{18n-1\} \\
\{wt_{f_3}(A_i)|i=2,3,\dots,n\} & = & \{16n+1, 16n+3, \dots, 18n-3\} \\
\{wt_{f_3}(B_i)|i=1,2,\dots,n\} & = & \{8n+2, 8n+6, \dots, 12n-2\} \\
\{wt_{f_3}(C_i)|i=1,2,\dots,n\} & = & \{6n+2, n+6, \dots, 10n-2\}.
\end{array}$$

It is easy to check that for arbitrary odd number $n \geq 3$, all the weights are different. Therefore f_3 is an edge odd graceful labeling. Thus, for $n \geq 3$ odd, $P_2 \times DSF_n$ is edge odd graceful.

Case for n even

For $n \geq 3$ even we define the labeling as follows:

$$\begin{array}{lcl}
f_3(a_ib_i) & = & 16n - 2i + 1 \\
f_3(a_ic_i) & = & 20n - 2i + 1 \\
f_3(b_ic_i) & = & 2i - 1 \\
f_3(b_ia_{(i+1)modn}) & = & 10n + 2i - 1 \\
f_3(c_ia_{(i+1)modn}) & = & 16n - 2i + 1 \\
f_3(A_iB_i) & = & 8n - 2i + 1 \\
f_3(A_iC_i) & = & 10n - 2i + 1 \\
f_3(B_iC_i) & = & 12n + 2i - 1 \\
f_3(B_iA_{(i+1)modn}) & = & 22n + 2i - 1 \\
f_3(C_iA_{(i+1)modn}) & = & 18n - 2i + 1 \\
f_3(a_iA_i) & = & 20n + 2i - 1 \\
f_3(b_iB_i) & = & 2n + 2i - 1 \\
f_3(c_iC_i) & = & 26n - 2i + 1
\end{array}$$

for all $i = 1, 2, \dots, n$. We obtain

$$
\begin{aligned}
\{f_3(a_ib_i)|i=1,2,\ldots,n\} &= \{14n-1,14n+1,\ldots,16n-1\}\\
\{f_3(a_ic_i)|i=1,2,\ldots,n\} &= \{18n+1,18n+3,\ldots,20n-1\}\\
\{f_3(b_ic_i)|i=1,2,\ldots,n\} &= \{1,3,\ldots,2n-1\}\\
\{f_3(b_ia_{(i+1)modn})|i=1,2,\ldots,n\} &= \{10n+1,10n+3,\ldots,12n-1\}\\
\{f_3(c_ia_{(i+1)modn})|i=1,2,\ldots,n\} &= \{14n+1,14n+3,\ldots,16n-1\}\\
\{f_3(A_iB_i)|i=1,2,\ldots,n\} &= \{6n+1,6n+3,\ldots,8n-1\}\\
\{f_3(A_iC_i)|i=1,2,\ldots,n\} &= \{8n+1,8n+3,\ldots,10n-1\}\\
\{f_3(B_iC_i)|i=1,2,\ldots,n\} &= \{12n+1,12n+3,\ldots,14n-1\}\\
\{f_3(B_iA_{(i+1)modn})|i=1,2,\ldots,n\} &= \{22n+1,22n+3,\ldots,24n-1\}\\
\{f_3(C_iA_{(i+1)modn})|i=1,2,\ldots,n\} &= \{16n+1,16n+3,\ldots,18n-1\}\\
\{f_3(a_iA_i)|i=1,2,\ldots,n\} &= \{20n+1,20n+3,\ldots,22n-1\}\\
\{f_3(b_iB_i)|i=1,2,\ldots,n\} &= \{2n+1,2n+3,\ldots,4n-1\}\\
\{f_3(c_iC_i)|i=1,2,\ldots,n\} &= \{24n+1,24n+3,\ldots,26n-1\}.
\end{aligned}
$$

We then have vertex weights $mod2k$, where $k = max\{|V(P_2 \times DSF_n)|, |E(P_2 \times DSF_n)|\} = max\{5n, 13n\} = 13n$ as follows:

$$
\begin{aligned}
&wt_{f_3}(a_1) = 20n-1 &&\\
&wt_{f_3}(a_i) = 20n-2i+1 && \text{for } i=2,3,\ldots,n\\
&wt_{f_3}(b_i) = 2n+4i-2 && \text{for } i=1,2,\ldots,n\\
&wt_{f_3}(c_i) = 26n-4i+2 && \text{for } i=1,2,\ldots,n.\\
&wt_{f_3}(A_1) = 26n-1 &&\\
&wt_{f_3}(A_i) = 26n-2i+1 && \text{for } i=2,3,\ldots,n\\
&wt_{f_3}(B_i) = 18n+4i-2 && \text{for } i=1,2,\ldots,n\\
&wt_{f_3}(C_i) = 14n-4i+2 && \text{for } i=1,2,\ldots,n.
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
\{wt_{f_3}(a_1)\} &= \{20n-1\}\\
\{wt_{f_3}(a_i)|i=2,3,\ldots,n\} &= \{18n+1,18n+3,\ldots,20n-3\}\\
\{wt_{f_3}(b_i)|i=1,2,\ldots,n\} &= \{2n+2,2n+6,\ldots,6n-2\}\\
\{wt_{f_3}(c_i)|i=1,2,\ldots,n\}\} &= \{22n+2,22n+6,\ldots,26n-2\}.\\
\{wt_{f_3}(A_1)\} &= \{26n-1\}\\
\{wt_{f_3}(A_i)|i=2,3,\ldots,n\} &= \{24n+1,24n+3,\ldots,26n-3\}\\
\{wt_{f_3}(B_i)|i=1,2,\ldots,n\} &= \{18n+2,18n+6,\ldots,22n-2\}\\
\{wt_{f_3}(C_i)|i=1,2,\ldots,n\} &= \{10n+2,10n+6,\ldots,14n-2\}.
\end{aligned}
$$

Therefore f_3 is an edge odd labeling for every even number $n \geq 3$ and hence $P_2 \times DSF_n$ is edge odd graceful for each even positive integer $n \geq 3$. Thus $P_2 \times DSF_n$ is edge odd graceful for arbitrary positive integer $n \geq 3$. □

Figure 10. shows an edge odd labeling on $P_2 \times DSF_3$. To make the figure look clear, we

draw the graph into several separate parts. Respect to the labeling f_3 for $P_2 \times DSF_3$ we have

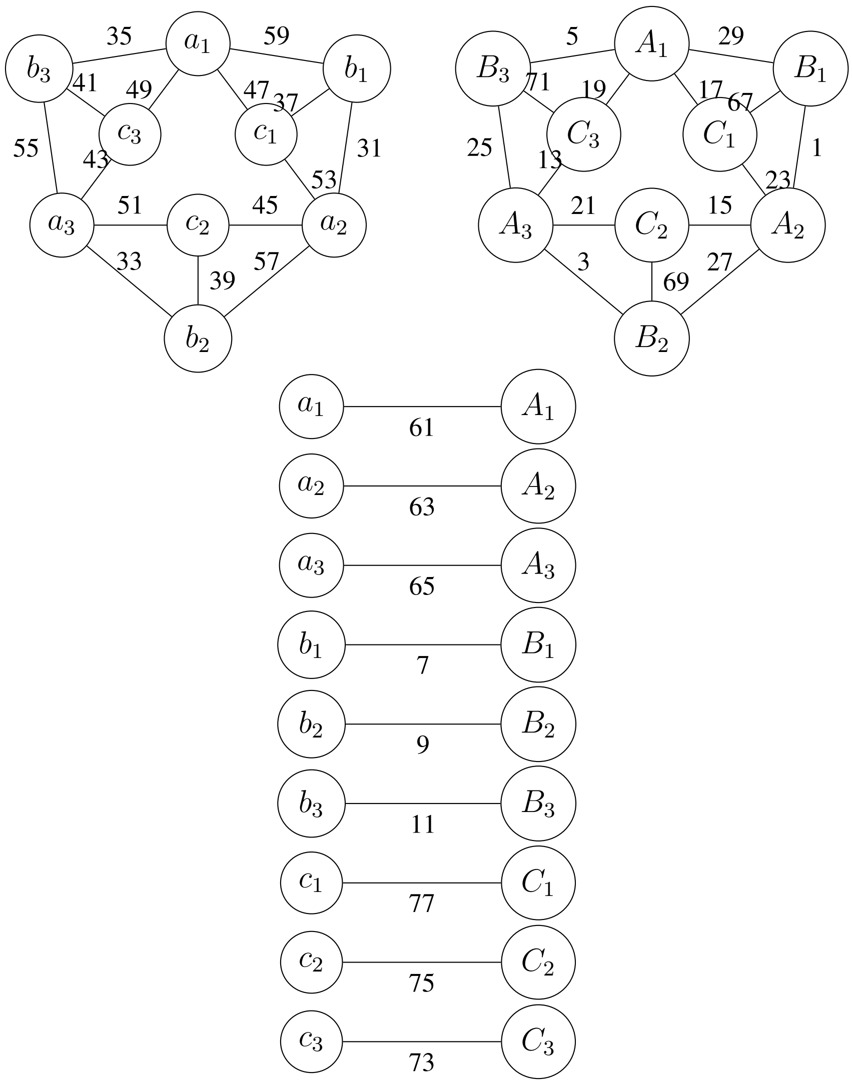

Figure 10. Edge Odd Graceful Labeling f_3 for $P_2 \times DSF_3$

the following vertex weights:

$$
\begin{array}{ll}
wt_{f_2}(a_1) = 17 & wt_{f_2}(A_1) = 53 \\
wt_{f_2}(a_2) = 15 & wt_{f_2}(A_2) = 51 \\
wt_{f_2}(a_3) = 13 & wt_{f_2}(A_3) = 49 \\
wt_{f_2}(b_1) = 56 & wt_{f_2}(B_1) = 26 \\
wt_{f_2}(b_2) = 60 & wt_{f_2}(B_2) = 30 \\
wt_{f_2}(b_3) = 64 & wt_{f_2}(B_3) = 34 \\
wt_{f_2}(c_1) = 58 & wt_{f_2}(C_1) = 28 \\
wt_{f_2}(c_2) = 54 & wt_{f_2}(C_2) = 24 \\
wt_{f_2}(c_3) = 50 & wt_{f_2}(C_3) = 20.
\end{array}
$$

ACKNOWLEDGEMENT

This work was done by the support from Universitas Gadjah Mada under Research Grant Year 2019 (Hibah Penelitian Dosen Dana Masyarakat Alokasi Fakultas Tahun 2019).

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