JOURNAL OF FUNDAMENTAL MATHEMATICS
AND APPLICATIONS (JFMA) VOL. 4 NO. 2 (NOV 2021)
Available online at www.jfma.math.fsm.undip.ac.id

QUADRATIC POLYNOMIAL OF POWER SUMS AND ALTERNATING POWER SUMS

Leomarich F. Casinillo

Department of Mathematics and Physics, Visayas State University, Visca, Baybay City, Leyte, Philippines
Email : leomarich_casinillo@yahoo.com

Abstract. This study aims to construct quadratic polynomials in n for power sums and alternating power sums of consecutive positive integers. In addition, this paper evaluated the said quadratic polynomials under odd and even terms of the series.
Keywords: Quadratic polynomial, power sums, alternating power sums

I. INTRODUCTION

In number theory and combinatorics, power sum series and alternating power sum series of consecutive positive integers are widely investigated concepts, which remains intriguing and give much attention to research [1, 2, 3, 4, 5, 6]. There are also some scientist and applied mathematicians uses the said series for mathematical real life applications [7, 8]. Let n and m be a positive integer. Then, it is well known, among others, that the sum of the p-th power of the first t positive integers

$$S_t^o(1,n,p) = \sum_{j=1}^{t=2n-1} j^p = 1^p + 2^p + 3^p + \cdots + t^p \tag{1}$$

and

$$S_t^e(1,n,p) = \sum_{j=1}^{t=2n} j^p = 1^p + 2^p + 3^p + \cdots + t^p \tag{2}$$

are closely connected to the Bernoulli polynomials $B_p(x)$ [9, 10, 11]. In equation (1) and (2), the series has odd and even terms, respectively. The next series (3) and (4), represents as power sum of consecutive positive integer with m^p as the first term with odd and even terms, respectively. Indeed, here it is

$$S_t^o(m,n,p) = \sum_{j=m}^{m+2n-2} j^p = m^p + (m+1)^p + (m+2)^p + \cdots + (m+2n-2)^p \tag{3}$$

where $t \equiv 1(mod\ 2)$, and

$$S_t^e(m,n,p) = \sum_{j=m}^{m+2n-1} j^p = m^p + (m+1)^p + (m+2)^p + \cdots + (m+2n-1)^p \tag{4}$$

where $t \equiv 0(mod\ 2)$. For the alternating power sum series of the first t positive integers, we have the following equations for $m = 1$,

$$A_t^o(1,n,p) = \sum_{j=1}^{t=2n-1} j^p = 1^p - 2^p + 3^p - \cdots + t^p \tag{5}$$

and also,

$$A_t^e(1,n,p) = \sum_{j=1}^{t=2n} j^p = 1^p - 2^p + 3^p - \cdots - t^p \tag{6}$$

In general, if m is even, we have

$$A_t^o(m,n,p) = \sum_{j=m}^{m+2n-2} (-1)^j j^p = m^p - (m+1)^p + \cdots + (m+2n-2)^p \tag{7}$$

where $t = 2n - 1$, and

$$A_t^e(m,n,p) = \sum_{j=m}^{m+2n-1} (-1)^j j^p = m^p - (m+1)^p + \cdots - (m+2n-1)^p \tag{8}$$

where $t = 2n$. Considering that m is odd, then, we obtain

$$A_t^o(m,n,p) = \sum_{j=m}^{m+2n-2} (-1)^{j+1} j^p = m^p - (m+1)^p + \cdots + (m+2n-2)^p \tag{9}$$

where $t = 2n - 1$, and lastly, we have

$$A_t^e(m,n,p) = \sum_{j=m}^{m+2n-1} (-1)^{j+1} j^p = m^p - (m+1)^p + \cdots - (m+2n-1)^p \tag{10}$$

where $t = 2n$. This paper investigates the construction process of quadratic polynomial in n which gives the explicit formula for power sums and alternating power sums.

II. RESULTS

The following theorem is quick from equation (3).

Theorem 2.1. *Let n and p be positive integers and let $t = 2n - 1$. If $p = 1$ and $n \geq 1$, or $p \geq 2$ and $n \leq 3$ then,*

$$S_t^o(1,n,p) = \left[\frac{1}{2}(4^p + 5^p - 2^p - 3^p)\right]n^2 + \left\{\frac{1}{2}[5(2^p) + 5(3^p) - 3(4^p) - 3(5^p)]\right\}n + [5^p + 4^p - 2(3^p) - 2(2^p) + 1]. \tag{11}$$

Proof: Let $t = 2n - 1$. Suppose that n and p be positive integers. Then, we assume that $S_t^o(1,n,p) = an^2 + bn + c$, and $a = \alpha_1(p), b = \alpha_2(p)$ and $c = \alpha_3(p)$ are parameters to be estimated which are functions of positive integer p. Simulating the series $S_t^o(1,n,p)$ for $n \in \{1,2,3\}$, then we obtain the following system of linear equations

$$\begin{cases} a + b + c = 1 \\ 4a + 2b + c = 1 + 2^p + 3^p \\ 9a + 3b + c = 1 + 2^p + 3^p + 4^p + 5^p \end{cases} \tag{12}$$

Reducing the system of equations in (12) to triangular form, then we have

$$\begin{cases} a + b + c = 1 \\ 3a + b = 2^p + 3^p \\ 2a = 4^p + 5^p - 2^p - 3^p \end{cases} \tag{13}$$

Applying the backward substitution in (13), it follows that

$$\begin{cases} a = \frac{1}{2}(4^p + 5^p - 2^p - 3^p) \\ b = \frac{1}{2}[5(2^p) + 5(3^p) - 3(4^p) - 3(5^p)] \\ c = 5^p + 4^p - 2(3^p) - 2(2^p) + 1 \end{cases} \tag{14}$$

Hence, this completes the proof of *Theorem* 1. □

It is worth noting that Theorem 3 shows that $S_t^o(1,n,p) = an^2 + bn + c$ works only for $p = 1$ and $n \geq 1$, or $p \geq 2$ and $n \leq 3$ whenever $t = 2n - 1$. In Table 1, it illustrates some quadratic polynomial in n for $1 \leq p \leq 6$.

Table 1. Quadratic polynomials of sum series for $p \in \{1,2,3,4,5,6\}$ under odd terms.

p	$S_t^o(1,n,p) = an^2 + bn + c,\ t = 2n - 1$	**Series**
1	$S_t^o(1,n,1) = 2n^2 - n$	$1^1 + 2^1 + 3^1 + \cdots + t^1$
2	$S_t^o(1,n,2) = 14n^2 - 29n + 16$	$1^2 + 2^2 + 3^2 + \cdots + t^2$
3	$S_t^o(1,n,3) = 77n^2 - 196n + 120$	$1^3 + 2^3 + 3^3 + \cdots + t^3$
4	$S_t^o(1,n,4) = 392n^2 - 1079n + 688$	$1^4 + 2^4 + 3^4 + \cdots + t^4$
5	$S_t^o(1,n,5) = 1937n^2 - 5536n + 3600$	$1^5 + 2^5 + 3^5 + \cdots + t^5$
6	$S_t^o(1,n,6) = 9464n^2 - 27599n + 18136$	$1^6 + 2^6 + 3^6 + \cdots + t^6$

The next result is immediate from the definition of the series $S_t^o(1,n,p)$ where $t \equiv 0(mod\ 2)$.

Theorem 2.2. *Let n and p be positive integers and let $t = 2n$. If $p = 1$ and $n \geq 1$, or $p \geq 2$ and $n \leq 3$ then,*

$$S_t^e(1,n,p) = \left[\frac{1}{2}(5^p + 6^p - 3^p - 4^p)\right]n^2 + \left\{\frac{1}{2}[5(3^p) + 5(4^p) - 3(5^p) - 3(6^p)]\right\}n + [6^p + 5^p - 2(4^p) - 2(3^p) + 2^p + 1]. \tag{15}$$

Proof: Supposing n and p are positive integers and we let $t = 2n - 1$. Again, we assume that $S_t^e(1,n,p) = an^2 + bn + c$, where $a = \beta_1(p), b = \beta_2(p)$ and $c = \beta_3(p)$ are parameters which are functions of positive integer p. So, we simulate the series $S_t^e(1,n,p)$ for $n \in \{1,2,3\}$, then we obtained the following system of linear equations

$$\begin{cases} a + b + c = 1 + 2^p \\ 4a + 2b + c = 1 + 2^p + 3^p + 4^p \\ 9a + 3b + c = 1 + 2^p + 3^p + 4^p + 5^p + 6^p \end{cases} \quad (16)$$

By the method of elimination, the system (16) follows as

$$\begin{cases} a + b + c = 1 + 2^p \\ 3a + b = 3^p + 4^p \\ 2a = 5^p + 6^p - 3^p - 4^p \end{cases} \quad (17)$$

So, by backward substitution in (17), it quickly follows that

$$\begin{cases} a = \frac{1}{2}(5^p + 6^p - 3^p - 4^p) \\ b = \frac{1}{2}[5(3^p) + 5(4^p) - 3(5^p) - 3(6^p)] \\ c = 6^p + 5^p - 2(4^p) - 2(3^p) + 2^p + 1 \end{cases} \quad (18)$$

Hence, the proof of Theorem 2 is complete. □

By simulation, Theorem 2 reveals that $S_t^e(1,n,p) = an^2 + bn + c$ is only applicable for $p = 1$ and $n \geq 1$, or $p \geq 2$ and $n \leq 3$ whenever $t = 2n$. Table 2 illustrates some quadratic polynomial in n for $1 \leq p \leq 6$

Table 2. Quadratic polynomials of sum series for $p \in \{1,2,3,4,5,6\}$ under even terms.

p	$S_t^e(1,n,p) = an^2 + bn + c,\ t = 2n$	Series
1	$S_t^e(1,n,1) = 2n^2 + n$	$1^1 + 2^1 + 3^1 + \cdots + t^1$
2	$S_t^e(1,n,2) = 18n^2 - 29n + 16$	$1^2 + 2^2 + 3^2 + \cdots + t^2$
3	$S_t^e(1,n,3) = 125n^2 - 284n + 168$	$1^3 + 2^3 + 3^3 + \cdots + t^3$
4	$S_t^e(1,n,4) = 792n^2 - 2039n + 1264$	$1^4 + 2^4 + 3^4 + \cdots + t^4$
5	$S_t^e(1,n,5) = 4817n^2 - 13184n + 8400$	$1^5 + 2^5 + 3^5 + \cdots + t^5$
6	$S_t^e(1,n,6) = 28728n^2 - 81359n + 52696$	$1^6 + 2^6 + 3^6 + \cdots + t^6$

The following theorem shows the explicit formula of the series $A_t^o(1,n,p)$ whenever $t \equiv 1(mod\ 2)$. The series is positive for all positive integer n and p.

Theorem 2.3. *Let* n *and* p *be positive integers and let* $t = 2n - 1$. *If* $p \leq 2$ *and* $n \geq 1$, *or* $p \geq 3$ *and* $n \leq 3$ *then,*

$$A_t^o(1,n,p) = \left[\frac{1}{2}(5^p + 2^p - 4^p - 3^p)\right]n^2 + \left\{\frac{1}{2}[3(4^p) + 5(3^p) - 5(2^p) - 3(5^p)]\right\}n + [5^p + 2(2^p) - 4^p - 2(3^p) + 1] > 0. \quad (19)$$

Proof: Since $t \equiv 1 (mod\ 2)$, then, there are $\frac{t+1}{2}$ positive terms and $\frac{t-1}{2}$ negative terms in the series. Clearly, the sum of positive terms is greater than the absolute value of the sum of negative terms which follows that $A_t^o(1, n, p) > 0$ for all positive integer p. Now, assume that $A_t^o(1, n, p) = an^2 + bn + c$, where $a = \gamma_1(p), b = \gamma_2(p)$ and $c = \gamma_3(p)$ are parameters which are function of positive integer p. Simulate the series $A_t^o(1, n, p)$ under $n \in \{1, 2, 3\}$, then we obtained the following system equations below

$$\begin{cases} a + b + c = 1 \\ 4a + 2b + c = 1 - 2^p + 3^p \\ 9a + 3b + c = 1 - 2^p + 3^p - 4^p + 5^p \end{cases} \quad (20)$$

Applying the method of elimination in (20), we have,

$$\begin{cases} a + b + c = 1 \\ 3a + b = -2^p + 3^p \\ 2a = 2^p - 3^p - 4^p + 5^p \end{cases} \quad (21)$$

Finally, by backward substitution, it follows that

$$\begin{cases} a = \frac{1}{2}(5^p + 2^p - 4^p - 3^p) \\ b = \frac{1}{2}[3(4^p) + 5(3^p) - 5(2^p) - 3(5^p)] \\ c = 5^p + 2(2^p) - 4^p - 2(3^p) + 1 \end{cases} \quad (22)$$

Thus, substituting the solution to equation $A_t^o(1, n, p) = an^2 + bn + c$, then this completes the proof. □

Theorem 3 reveals that $A_t^o(1, n, p) = an^2 + bn + c > 0$ works only for $p \le 2$ and $n \ge 1$, or $p \ge 3$ and $n \le 3$ whenever $t = 2n - 1$. Table 3 shows some quadratic polynomial in n for $1 \le p \le 6$.

Table 3. Quadratic polynomials of alternating sign series for $p \in \{1, 2, 3, 4, 5, 6\}$ under odd terms.

p	$A_t^o(1, n, p) = an^2 + bn + c,\ t = 2n - 1$	**Series**
1	$A_t^o(1, n, 1) = n$	$1^1 - 2^1 + 3^1 - \cdots + t^1$
2	$A_t^o(1, n, 2) = 2n^2 - n$	$1^2 - 2^2 + 3^2 - \cdots + t^2$
3	$A_t^o(1, n, 3) = 21n^2 - 44n + 24$	$1^3 - 2^3 + 3^3 - \cdots + t^3$
4	$A_t^o(1, n, 4) = 152n^2 - 391n + 240$	$1^4 - 2^4 + 3^4 - \cdots + t^4$
5	$A_t^o(1, n, 5) = 945n^2 - 2624n + 1680$	$1^5 - 2^5 + 3^5 - \cdots + t^5$
6	$A_t^o(1, n, 5) = 5432n^2 - 15631n + 10200$	$1^6 - 2^6 + 3^6 - \cdots + t^6$

The next *Theorem* provides the explicit formula of the series $A_t^e(1, n, p)$ whenever $t \equiv 0 (mod\ 2)$. The series is strictly negative for all positive integer n and p.

Theorem 2.4. *Let n and p be positive integers and let $t = 2n$. If $p \le 2$ and $n \ge 1$, or $p \ge 3$ and $n \le 3$ then,*

$$A_t^e(1,n,p) = \left[\frac{1}{2}(5^p - 6^p - 3^p + 4^p)\right]n^2 + \left\{\frac{1}{2}[5(3^p) - 5(4^p) - 3(5^p) + 3(6^p)]\right\}n$$
$$+[5^p - (6^p) - 2(3^p) + 2(4^p) - 2^p + 1] < 0. \quad (23)$$

Proof: Let n and p be positive integers. Since $t \equiv 1(mod\ 2)$, then, there are $\frac{t}{2}$ positive terms and $\frac{t}{2}$ negative terms in the series. Now, the sum of positive terms is lesser than the absolute value of the sum of negative terms which clearly follows that $A_t^o(1,n,p) < 0$ for all positive integer p. Again we assume that $A_t^o(1,n,p) = an^2 + bn + c$, where $a = \delta_1(p), b = \delta_2(p)$ and $c = \delta_3(p)$ are parameters which are functions of positive integer p. We now simulate the series $A_t^e(1,n,p)$ under $n \in \{1,2,3\}$, then we obtained the following system

$$\begin{cases} a + b + c = 1 - 2^p \\ 4a + 2b + c = 1 - 2^p + 3^p - 4^p \\ 9a + 3b + c = 1 - 2^p + 3^p - 4^p + 5^p - 6^p \end{cases} \quad (24)$$

Applying the method of elimination, we have

$$\begin{cases} a + b + c = 1 - 2^p \\ 3a + b = 3^p - 4^p \\ 2a = 5^p - 6^p - 3^p + 4^p \end{cases} \quad (25)$$

Finally, by backward substitution, it follows that

$$\begin{cases} a = \frac{1}{2}(5^p - 6^p - 3^p + 4^p) \\ b = \frac{1}{2}[5(3^p) - 5(4^p) - 3(5^p) + 3(6^p)] \\ c = 5^p - (6^p) - 2(3^p) + 2(4^p) - 2^p + 1 \end{cases} \quad (26)$$

Hence, the proof of Theorem 4 is complete. □

The Theorem 4 above only covers $p \le 2$ and $n \ge 1$, or $p \ge 3$ and $n \le 3$ where $t = 2n$. It is worth noting that the theorem clearly implies that $A_t^e(1,n,p) < 0$. In Table 4, we consider $1 \le p \le 6$ as the power of the terms in the alternating sum series with even terms.

Table 4. Quadratic polynomials of alternating sign series for $p \in \{1,2,3,4,5,6\}$ under even terms.

p	$A_t^e(1,n,p) = an^2 + bn + c,\ t = 2n$	Series
1	$A_t^e(1,n,1) = -n$	$1^1 - 2^1 + 3^1 - \cdots - t^1$
2	$A_t^e(1,n,2) = -2n^2 - n$	$1^2 - 2^2 + 3^2 - \cdots - t^2$
3	$A_t^e(1,n,3) = -27n^2 + 44n - 24$	$1^3 - 2^3 + 3^3 - \cdots - t^3$
4	$A_t^e(1,n,4) = -248n^2 + 569n - 336$	$1^4 - 2^4 + 3^4 - \cdots - t^4$
5	$A_t^e(1,n,5) = -1935n^2 + 5024n - 3120$	$1^5 - 2^5 + 3^5 - \cdots - t^5$
6	$A_t^e(1,n,6) = -13832n^2 + 38129n - 24360$	$1^6 - 2^6 + 3^6 - \cdots - t^6$

III. CONCLUSION

In this study, we had developed quadratic polynomials for power sums $S_t^o(1,n,p)$ and $S_t^e(1,n,p)$, and alternating power sums $A_t^o(1,n,p)$ and $A_t^e(1,n,p)$, where n and p are positive integers, and t is the number of terms in the series. For power sums, the quadratic polynomials only works for $p = 1$ and $n \geq 1$, or $p \geq 2$ and $n \leq 3$. On the other hand, for alternating power sums, the quadratic polynomials only works for $p \leq 2$ and $n \geq 1$, or $p \geq 3$ and $n \leq 3$. Additionally, it is concluded that $A_t^o(1,n,p) > 0$ and $A_t^e(1,n,p) < 0$.

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